SOME APPLICATIONS OF EXPANSION CONSTANTS

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1. For any metric space X (with distance function d) the *expansion constant* $E(X)$ of X is the greatest lower bound of real numbers μ which possess the following property ($S(x;\rho) = S_X(x;\rho) = \{y \in X; d(x, y) \leq \rho\}$ denotes the closed cell with center x and radius ρ):

For any family $\{S(x_\alpha;\rho_\alpha); \alpha \in A\}$ of pairwise intersecting cells in X,

$$\bigcap_{\alpha\in A} S(x_\alpha;\mu\rho_\alpha) \neq \phi .$$

If for every such family $\bigcap_{\alpha\in A} S(x_\alpha; E(X)\rho_\alpha) \neq \phi$, $E(X)$ is called *exact*.

The expansion constants of Minkowski spaces have been studied in [5]. In the present paper we deal (in § 2) with an application of the expansion constants to a problem on projections in Banach spaces; as corollaries we obtain Nachbin's [10] geometric characterization of Banach spaces with the Hahn-Banach extension property (§ 2) and Bohnenblust's [3] result on projections in Minkowski spaces, as well as some results which we believe to be new (§ 4). In § 3 we discuss the relation of expansion constants to a property of retractions in metric spaces, especially those convex in Menger's sense; as a corollary we obtain Aronszajn-Panitchpakdi's [2] characterization of spaces with the unlimited uniform extension property. Section 4 contains additional remarks and examples.

2. In order to apply expansion constants to projections in Banach spaces, it is convenient to introduce the notion of projection constants.

DEFINITION 1. For any normed space X the *projection constant* $p(X)$ is the greatest lower bound of real numbers μ which possess the following property: For any normed space Y which contains X as a subspace of deficiency 1, there exists a projection P of Y onto X such that $\|P\| \leq \mu$. If for any such Y there exists a projection of norm less than or equal to $p(X)$, the projection constant $p(X)$ is called *exact*.

(The projection constant $p(X)$ should not be confused with the projection constant $\mathscr{P}(X)$ studied in [6].)

We show now that if X is a normed space then $E(X)$ actually coincides with $p(X)$.

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THEOREM 1. *For any normed space X we have $p(X) = E(X)$; moreover, if one of the constants is exact, so is the other.*

Proof. If X is not complete, then $p(X) = \infty = E(X)$. The first part follows immediately from the remark that, if Y is any subspace of the completion of X containing X as a subspace of deficiency 1, there exists no projection of Y onto X. On the other hand, $E(X) = \infty$ for any metric space X which is not complete. Indeed, if $\{x_n; n = 1, 2, \cdots\}$ is a Cauchy sequence in X which is not convergent, let $\rho_n = 2\lim_{k\to\infty} d(x_n, x_k)$, for $n = 1, 2, \cdots$. Then the cells $\{S(x_n; \rho_n); n = 1, 2, \cdots\}$ are mutually intersecting, but $\bigcap_{n=1}^{\infty} S(x_n; \mu\rho_n) = \phi$ for any μ, which implies $E(X) = \infty$. (We shall see in § 4 that if X is a complete metric space then $E(X) \leq 2$.)

Thus we may assume that X is complete. We shall first prove that $E(X) \leq p(X)$. To that effect, let Y be the linear sum of X (considered as a vector space) and a point $y_0 \notin X$. For any given family $\{S(x_\alpha; \rho_\alpha); \alpha \in A\}$ of mutually intersecting cells (of X) we shall define a norm in Y, such that Y becomes a Banach space containing X as a subspace, and that for any projection P of Y onto X we have:

$$\bigcap_{\alpha \in A} S(x_\alpha; \|P\| \cdot \rho_\alpha) \neq \phi\ . \tag{2.1}$$

If $\inf \rho_\alpha = 0$ any norm on Y establishes (2.1) since $\bigcap_{\alpha\in A} S(x_\alpha; \rho_\alpha) \neq \phi$. We prove this relation in the following way: If for some $\beta \in A$ we have $\rho_\beta = 0$, then $x_\beta \in S(x_\alpha; \rho_\alpha)$ for all $\alpha \in A$. On the other hand, if for a sequence of indices $\alpha_n \in A$ we have $\lim \rho_{\alpha_n} = 0$, then (since $d(x_\alpha, x_\beta) \leq \rho_\alpha + \rho_\beta$) $\{x_{\alpha_n}\}$ is a Cauchy sequence. Since X is complete there exists $x_0 = \lim_n x_{\alpha_n}$. We claim that $x_0 \in \bigcap_{\alpha\in A} S(x_\alpha, \rho_\alpha)$. Indeed, for any $\alpha \in A$ and any $\varepsilon > 0$, let n be such that $\rho_{\alpha_n} < 1/2\varepsilon$ and $d(x_{\alpha_n}, x_0) < 1/2\varepsilon$; then $d(x_\alpha, x_0) \leq d(x_\alpha, x_{\alpha_n}) + d(x_{\alpha_n}, x_0) < \rho_\alpha + \varepsilon$. Since ε is arbitrary, it follows that $x_0 \in S(x_\alpha; \rho_\alpha)$ for any $\alpha \in A$, as required.

Thus we are left with the case $\inf \rho_\alpha > 0$. Let $y_\alpha = (x_\alpha + y_0)/\rho_\alpha$ for each $\alpha \in A$ and $K = \{y_\alpha; \alpha \in A\}$. If S is the unit cell of X, we denote by T the closure (in the product topology of $Y = X \times Ry_0$) of the convex hull of the set $S \cup K \cup (-K) \subset Y$. Since T is a centrally symmetric convex body in Y it defines a norm (according to which T is the unit cell). We claim that $T \cap X = S$, i.e. that X (as a Banach space) is a subspace of Y. Obviously, this will be established if we show that the intersection of X with a segment connecting a point of K with a point of $-K$ belongs to S. Now, an elementary computation shows that $[y_\alpha, -y_\beta] \cap X$ is the point $(x_\alpha - x_\beta)/(\rho_\alpha + \rho_\beta)$ whose norm is less than or equal to 1, since $\|x_\alpha - x_\beta\| \leq \rho_\alpha + \rho_\beta$ is a consequence of the assumption that the members of $\{S(x_\alpha; \rho_\alpha)\}$ are pairwise intersecting. Now let P be any projection of Y onto X and let $x_0 = -P(y_0)$. Then

$$P(y_\alpha) = P\Big(\frac{x_\alpha + y_0}{\rho_\alpha}\Big) = \frac{x_\alpha - x_0}{\rho_\alpha} \text{ and therefore}$$

$$\Big|\Big|\frac{x_\alpha - x_0}{\rho_\alpha}\Big|\Big| \leq ||P|| \cdot ||y_\alpha|| \leq ||P|| \text{ for each } \alpha \in A\,.$$

In other words, $x_0 \in S(x_\alpha; ||P||\,\rho_\alpha)$ for each $\alpha \in A$ which implies (2.1) and thus establishes $E(X) \leq p(X)$.

In order to derive the converse inequality $p(X) \leq E(X)$ let Y be any Banach space containing X as a maximal subspace, and let $y_0 \in Y, y_0 \notin X$. The triangle inequality implies that $S_X(x, ||x - y_0||) \cap S_X(x', ||x' - y_0||) \neq \phi$ for any $x, x' \in X$. Let μ be such that $\bigcap_{x \in X} S_X(x; \mu ||x - y_0|| \neq \phi$, and denote by x_0 any point of that intersection. Thus $||x - x_0|| \leq \mu ||x - y_0||$ for any $x \in X$. We define a projection P of Y onto X by $P(x + \lambda y_0) = x + \lambda x_0$, and we shall show that $||P(x + \lambda y_0)|| \leq \mu ||x + \lambda y_0||$, i.e., that $||P|| \leq \mu$. Obviously, we may assume $\lambda \neq 0$ and then, by the definition of x_0, we have $||P(x + \lambda y_0)|| = ||x + \lambda x_0|| = |\lambda| \cdot || - x/\lambda - x_0|| \leq \mu |\lambda| \cdot || - x/\lambda - y_0|| = \mu ||x + \lambda y_0||$.

This completes the proof of our last assertion, and thus also the proof of Theorem 1.

The connection between projection constants and extensions of linear transformations may be found using the following lemma:

If X and Y are any normed spaces, if Z contains Y as a subspace of deficiency 1, *and if f is any linear transformation from Y to X, then there exist a normed space W and a linear transformation F from Z to W such that*:

(i) *W contains X as a subspace of deficiency* 1;

(ii) *F coincides with f on Y*;

(iii) $||F|| = ||f||$.

We omit the simple proof of this lemma since a more general extension theorem of this type has been proved by Sobczyk [13, Theorem 4.1].

Using the above lemma, the following corollary results immediately from Theorem 1:

For any Banach spaces X, Y and Z, with Y a maximal subspace of Z, any linear transformation f from Y to X, and any $\mu > p(X) = E(X)$, there exists a linear transformation F from Z to X, coinciding with f on Y, such that $||F|| \leq ||\mu|| f||$; if $p(X)$ is exact, there exists such an F even for $\mu = p(X)$.

A standard application of Zorn's lemma or transfinite induction yields therefore:

The following two properties of a normed space X are equivalent:

(i) *$E(X) = 1$ and is exact*;

(ii) *For any normed spaces Y and Z, with $Y \subset Z$, and any linear* *transformation f from Y to X, there exists a linear transformation F from Z to X such that F coincides with f on Y and $\|F\| = \|f\|$.*

Since the "binary intersection property" of Nachbin [10] is equivalent to "$E(X)=1$ and is exact," the last statement is precisely Nachbin's characterization of spaces with the Hahn-Banach extension property [10, Theorem 1].

3. In the case of metric spaces, expansion constants may be used to obtain information on retraction properties, in close analogy to the procedure applied in § 2 to projections in normed spaces.

A *retraction* r of a metric space Y onto a metric space $X \subset Y$ is a (continuous) mapping of Y onto X such that $r(x) = x$ for each $x \in X$.

DEFINITION 2. The norm $\|r\|$ of a retraction r of Y onto $X \subset Y$ is the greatest lower bound of numbers μ such that $d(r(y_1), r(y_2)) \leq \mu d(y_1, y_2)$ for all $y_1, y_2 \in Y$. The *retraction constant* $r(X)$ of a metric space X is the greatest lower bound of numbers μ with the property: For any metrix space Y which contains X any only one point not in X, there exists a retraction of Y onto X with norm less than or equal to μ. If $r(X) = \min \mu$, the retraction constant $r(X)$ is called *exact*.

Obviously $r(X) = \infty$ if X is not complete, and it is easily shown that for complete spaces $r(X) \leq 2$.

Since metrically convex spaces have special properties with respect to retractions, we recall their definition (essentially that of Menger [9]):

A metric space X is called *metrically convex* if for any pair of points $x', x'' \in X$ and any λ, $0 < \lambda < 1$, there exists a point $y \in X$ such that $d(x', y) = \lambda d(x', x'')$ and $d(x'', y) = (1 - \lambda) d(x', x'')$.

In analogy to Theorem 1 we have:

THEOREM 2. (i) *For any metric space X, $E(X) \leq r(X)$.*

(ii) *For any metrically convex metric space X, $E(X) = r(X)$; moreover, if one of the constants is exact, so is the other.*

Proof. (i) Since for uncomplete spaces both constants are infinite, we will assume that X is complete. Let $\{S(x_\alpha; \rho_\alpha); \alpha \in A\}$ be any family of mutually intersecting cells in X. We shall define a space $Y = X \cup \{y_0\}$ (with distance function D) such that for any retraction r of Y onto X we have

$$\bigcap_{\alpha \in A} S(x_\alpha; \|r\| \cdot \rho_\alpha) \neq \phi \, . \tag{3.1}$$

This will prove part (i) of the Theorem. Now, if $\inf \rho_\alpha = 0$ the reasoning used in the proof of Theorem 1 shows that any metrization of Y is appropriate. Thus there remains the case $\inf \rho_\alpha > 0$. Then, let

$D(x', x'') = d(x', x'')$ for all $x', x'' \in X$, and let $D(x, y_0)$ be the greatest lower bound of those numbers μ for which $S(x; \mu)$ contains $S(x_\alpha; \rho_\alpha)$ for some $\alpha \in A$. (This metric on Y was used also in [2]). Since $D(x, y_0) > 0$ follows obviously from $\inf \rho_\alpha > 0$, in order to establish that D is indeed a distance function on Y we have only to prove the triangle inequality for triples of points containing y_0, i.e. the relations

$$D(x', x'') \leq D(x', y_0) + D(y_0, x'') \tag{3.2}$$

and

$$D(x', y_0) \leq D(x', x'') + D(x'', y_0) \tag{3.3}$$

for all $x', x'' \in X$. To that effect let $\varepsilon > 0$ be given; then $S(x'; D(x', y_0) + \varepsilon) \supset S(x_\alpha; \rho_{\alpha'})$ and $S(x''; D(x'', y_0) + \varepsilon) \supset S(x_{\alpha''}; \rho_{\alpha''})$ for suitable $\alpha', \alpha'' \in A$. Since any two of the cells $S(x_\alpha; \rho_\alpha)$ have common points, there exists a $z \in X$ such that $d(x', z) \leq D(x', y_0) + \varepsilon$ and $d(x'', z) \leq D(x'', y_0) + \varepsilon$. Then $D(x', x'') = d(x', x'') \leq d(x', z) + d(x'', z) \leq D(x', y_0) + D(x'', y_0) + 2\varepsilon$. Since ε was arbitrary, (3.2) results. On the other hand, since $S(x'; d(x', x'') + D(x'', y_0) + \varepsilon) \supset S(x''; D(x'', y_0) + \varepsilon) \supset S(x_{\alpha''}; \rho_{\alpha''})$, we have $D(x', y_0) \leq d(x', x'') + D(x'', y_0) + \varepsilon$ for any $\varepsilon > 0$, which establishes (3.3).

Now, let r be a retraction of Y onto X, and let $x_0 = r(y_0)$. Then, for any $\alpha \in A$ we have $d(x_\alpha, x_0) \leq \| r \| D(x_\alpha, y_0) \leq \| r \| \cdot \rho_\alpha$, which is equivalent to $x_0 \in \bigcap_{\alpha \in A} S(x_\alpha; \| r \| \rho_\alpha)$. Thus (3.1) holds, and the proof of (i) is complete.

The proof of (ii) is now easy. Let $Y = X \cup \{y_0\}$ and let D be the distance function of Y. We consider the family of cells in X defined by $\{S(x; D(x, y_0)); x \in X\}$. The triangle inequality which is satisfied by D, and the metric convexity of X imply that these cells are mutually intersecting. Let μ be a number such that $\bigcap_{x \in X} S(x; \mu D(x, y_0)) \neq \phi$, and let x_0 be any point of this intersection. Then the retraction r of Y onto X defined by $r(y_0) = x_0$ obviously satisfies $\| r \| \leq \mu$. This completes the proof of Theorem 2.

REMARKS. (i) If X is not metrically convex, $E(X) < r(X)$ is possible. The simplest example to this effect is that of a space X consisting of precisely two points. Then $E(X) = 1$ and $r(X) = 2$.

(ii) Let X, Y and $Z = Y \cup \{z_0\}$ be any metric spaces, and f a uniformly continuous transformation from Y to X, with subadditive modulus of continuity $\delta(\varepsilon)$ (see, e.g., [2]). It is easily established that there exists a metric space $X^* = X \cup \{x^*\}$, whose distance function coincides on X with the distance function of X, such that there exists an extension F of f, with domain Z and range in X^*, which is uniformly continuous with the modulus $\delta(\varepsilon)$. Therefore, using transfinite induction or Zorn's lemma, we obtain the following corollary of Theorem 2, which is equivalent to Theorem 2, § 3 of [2]:

For any metrically convex metric space X the following properties are equivalent:

(a) $r(X) = 1$ *and is exact*;

(b) *For any metric spaces Y and Z, with $Z \supset Y$, and any uniformly continuous transformation f from Y to X with subadditive modulus of continuity $\delta(\varepsilon)$, there exists a uniformly continuous transformation F from Z to X, coinciding with f on Y and having $\delta(\varepsilon)$ as modulus of continuity.*

4. Some properties of expansion constants $E(X)$ for finite dimensional Banach spaces X have been established in [5]. As a consequence of Theorem 1 of the present paper, these results yield the following information on the projection constants $p(X)$:

(i) If X is an n-dimensional Minkowski space then $1 \leq p(X) \leq 2n/(n+1)$. (This was first established by Bohnenblust [3].)

(ii) If E^n denotes the n-dimensional Euclidean space then $p(E^n) = \sqrt{2n/(n+1)}$.

The characterization of those n-dimensional Minkowski spaces X for which $E(X) = 2n/(n+1)$, given in Theorem 2 of [5], yields immediately a characterization of spaces X for which the upper bound is attained in (i).

As observed by Bohnenblust [3], $p(X) \leq 2$ for any Banach space X. By Theorem 1 this is a corollary of the following more general proposition:

$E(X) \leq 2$ *for any complete metric space* X.

Proof. Let $\{S(x_\alpha; \rho_\alpha); \alpha \in A\}$ be any family of mutually intersecting cells in X. The reasoning used in the proof of Theorem 1 shows that if $\inf \rho_\alpha = 0$ then $\bigcap_{\alpha \in A} S(x_\alpha; \rho_\alpha) \neq \phi$. Thus we may assume $\inf \rho_\alpha = \rho_0 > 0$. Given any $\varepsilon > 0$ let $\beta \in A$ be such that $\rho_\beta \leq (1+\varepsilon)\rho_0$. Since then $d(x_\alpha, x_\beta) \leq \rho_\alpha + \rho_\beta \leq \rho_\alpha + (1+\varepsilon)\rho_0 \leq (2+\varepsilon)\rho_\alpha$, we have $x_\beta \in \bigcap_{\alpha \in A} S(x_\alpha; (2+\varepsilon)\rho_\alpha)$, which proves our assertion.

The notion of expansion constants gives us a convenient method of obtaining information on the exactness of projection and retraction constants.

DEFINITION 3. A metric space X is said to have the *finite intersection property* if each family of cells $\{S(x_\alpha; \rho_\alpha); \alpha \in A\}$ of X satisfies the condition: Whenever every finite subfamily has a non-void intersection, then $\bigcap_{\alpha \in A} S(x_\alpha; \rho_\alpha) \neq \phi$.

Obviously, compact spaces and spaces with compact cells have the finite intersection property. As a consequence of the w^*-compactness of the unit cell of any adjoint Banach space ([1], [4]), adjoint Banach spaces (and thus especially reflexive, unitary, or finite dimensional Banach spaces) also have the finite intersection property.

For spaces with the finite intersection property we have:

THEOREM 3. *If X has the finite intersection property then $E(X)$, and therefore $r(X)$ (and $p(X)$ if X is a Banach space), are exact.*

Proof. Let $\{S(x_\alpha; \rho_\alpha); \alpha \in A\}$ be any family of mutually intersecting cells in X. By the definition of the expansion constant, any finite subfamily of the family $\{S_{\alpha,\varepsilon} = S(x_\alpha; (E(X) + \varepsilon)\rho_\alpha); \alpha \in A, \varepsilon > 0\}$ has a non-void intersection. Since X has the finite intersection property this implies that $\bigcap_{\alpha \in A, \varepsilon > 0} S_{\alpha,\varepsilon} \neq \phi$. But $S(x_\alpha; E(X) \cdot \rho_\alpha) = \bigcap_{\varepsilon > 0} S_{\alpha,\varepsilon}$ for each $\alpha \in A$ and therefore $\bigcap_{\alpha \in A} S(x_\alpha; E(X)\rho_\alpha) \neq \phi$ as claimed.

REMARK. We know of no Banach space which has the finite intersection property but is not an adjoint space; indeed, it seems reasonable to conjecture that only adjoint spaces have this property. On the other hand, a wider class of Banach spaces has exact projection and expansion constants. E.g., it is well known (Sobczyk [12]) that $p(c_0) = 2$ and is exact (it is not difficult to show directly that $E(c_0) = 2$ and is exact) but it is easily seen that c_0 does not have the finite intersection property (not even for families of cells having the same radius).

Another question, raised by Bohnenblust [3], is whether there exists a projection of norm ≤ 2 from any Banach space onto each of its maximal closed subspaces. A negative answer to Bohnenblust's problem follows immediately from the following example.

EXAMPLE. *Let X be the subspace of l defined by*

$$X = \left\{x = (x_1, x_2, \cdots) \in l; \sum_{n=1}^{\infty} \frac{n}{n+1} x_n = 0\right\}.$$

Then $E(X) = 2$ but $E(X)$ is not exact.

For reasons of convenience we shall, instead of X, consider its translate $H = \{x; \sum_{n=1}^{\infty} n/(n+1)x_n = 1\} \subset l$. Since X and H are isometric metric spaces, this is permissible. Now let $\{e_n; n = 1, 2, \cdots\}$ denote the usual basis of l, and S its unit cell $S = \{x \in l; \|x\| = \sum_{n=1}^{\infty} |x_n| \leq 1\}$. Obviously $n + 1/(n)e_n \in H$ for $n = 1, 2, \cdots$, but $S \cap H = \phi$ although $\operatorname{dist}(S, H) = 0$. (This last property was Klee's reason for introducing H in [8].) We consider in l the family of cells

$$\left\{S_n = S\left(\frac{n+1}{n} e_n; \frac{n+1}{n}\right) = \left\{x \in l; \left\|x - \frac{n+1}{n} e_n\right\| \leq \frac{n+1}{n}\right\}; n = 1, 2, \cdots .\right\}.$$

Then $S_n^* = S_n \cap H$ is a family of cells in H which are mutually intersecting since

$$\frac{(n+1)(k+1)}{n(k+1)+k(n+1)}(e_n+e_k) \in S_n^* \cap S_k^* .$$

We shall show that

$$H \cap \left[\bigcap_{n=1}^{\infty} S\left(\frac{n+1}{n}e_n; 2\frac{n+1}{2}\right)\right] = \phi ,$$

which will obviously prove our assertion. Let $K = \bigcap_{n=1}^{\infty} S(1+n^{-1})e_n; 2(n+1)/n)$; then, since $H \cap S = \phi$, it is sufficient to prove that $K \subset S$. Assuming that there exists an $x \in K$ such that $\|x\| = 1 + \varepsilon > 1$, we have by the definition of K:

$$0 \leq \left\| x - \frac{n+1}{n}e_n \right\| = \left| \frac{n+1}{n} - x_n \right| + \sum_{i \neq n} |x_i| \leq 2\frac{n+1}{n} .$$

for each n. Now, if for some n we have $x_n \geq 0$, it follows that either $x_n > (n+1)/n > 1$ or $-x_n + \sum_{i \neq n} |x_i| \leq (n+1)/n$. Both cases are possible only for a finite number of indices n; in the first case this is obvious, while in the second it follows from the fact that it implies $\|x\| - (n+1)/n \leq 2x_n$, i.e. $\varepsilon \leq (1/n) + 2x_n$. On the other hand, for those n for which $x_n \leq 0$ we have

$$\frac{n+1}{n} - x_n + \sum_{i \neq n} |x_i| \leq 2\frac{n+1}{n} ,$$

or

$$\|x\| \leq \frac{n+1}{n}, \text{ i.e. } \varepsilon \leq \frac{1}{n} ,$$

which is again possible only for a finite number of indices n. Thus $K \subset S$, which completes the proof.

REMARKS. (i) Since adjoint Banach spaces have exact expansion constants, the space X of the above example is not an adjoint space, although it is a maximal closed subspace of an adjoint space. It would be interesting to know whether every non-reflexive Banach space has a closed maximal subspace which is not an adjoint space.

(ii) Jung's constant $J(X)$ has been defined [5] in the same way as $E(X)$, with the additional condition that all the radii ρ_α be equal. The space X of the last example shows that it is possible to have $J(X) = E(X)$ with $J(X)$ exact and $E(X)$ not exact.

(iii) Theorem 4 of § 3 of [2] implies:

If X is a bounded, metrically convex metric space and $E(X) = 1$, then $E(X)$ is exact.

Although the condition of boundedness is perhaps redundant, the following example shows that it is impossible to drop the condition that X be metrically convex.

EXAMPLE. *Let $X = \{x_n ; n = 1, 2, \cdots\}$ with $d(x_n, x_x) = 1 + 1/n + 1/k$ for $n \neq k$. Then $E(X) = 1$ but $E(X)$ is not exact.*

Indeed, it is easily verified that $E(X) = 1$. On the other hand, the cells $S_n = S(x_n, 1 + 2/n) = \{x_k ; k \geq n\}$ for $n = 1, 2, \cdots$, are not only mutually intersecting, but we even have $S_k \supset S_n$ for $k \leq n$. But obviously $\bigcap_{n=1}^{\infty} S_n = \phi$. (Complete metric spaces containing descending sequences of cells with empty intersections have been considered by Sierpiński [**11**]; see also Harrop-Weston [**7**].)

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