HOMOMORPHISMS OF CERTAIN ALGEBRAS OF MEASURES

IRVING GLICKSBERG

The problem of determining all isomorphisms between the L_1 algebras of a pair of locally compact groups G and H has been considered by J. G. Wendel [16, 17] and H. Helson [7] (in the abelian case); these authors showed in particular that all norm-decreasing isomorphisms arise essentially from isomorphisms between the groups (and are isometries). In the abelian case a device suggested by Helson leads to much more, and we shall determine all norm-decreasing *homomorphisms* of certain algebras of measures (similar to L_1) on G into the algebra of measures on H(cf. 2.1 below.).

Let $M(G)$ denote the Banach algebra of all finite, complex, regular Borel measures on G, with convolution as multiplication. $L_1(G)$ forms a subalgebra of $M(G)$, in fact an ideal. Because of this, knowledge of the norm-decreasing homomorphisms of L_1 algebras into algebras of measures on another group leads to the determination of all norm-decreasing isomorphisms between $M(G)$ and $M(H)$; indeed when G and H are abelian we shall show that for each norm-decreasing isomorphism of a (not necessarily closed) subalgebra of $M(G)$ which contains $L_1(G)$ with a similar subalgebra of $M(H)$ there is an isomorphism γ of G onto H and a fixed character $\hat{g}$ of G for which $T\mu$ is just the measure $\hat{g}\mu$ transported to H via γ (whence $TL_1(G) = L_1(H)$ and T is an isometry). This is exactly the abelian Helson-Wendel result extended to superalgebras of L_1; in the non-commutative situation we can only obtain the analogous result for compact groups.

Aside from familiar facts about harmonic analysis (as given in [10, 15]) our main tools will be the following results obtained in [6] for a compact group G:

(1) *each multiplicative subgroup of non-negative elements of the unit ball of $M(G)$, other than the trivial subgroup $\{0\}$, consists of translates of Haar measure of a fixed normal subgroup of G* [6, 2.4];

(2) *each non-zero idempotent in the unit ball of $M(G)$ is Haar measure of a subgroup multiplied by a multiplicative character of this subgroup* [6, 4.3].

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NOTATION. As usual $C_0(G)$ will denote the continuous complex

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functions on G vanishing at infinity; $M(G)$ is of course $C_0(G)^*$. The space of all continuous bounded complex functions on G will be denoted by $C(G)$.

When G is abelian, $G^\wedge$ will denote its character group with generic element $\hat{g}$; the respective identities of G and $G^\wedge$ will be g_0 and $\hat{g}_0$. In general measures on G will be denote by the letter μ and those on H by ν with $\mu_g\,(\nu_h)$ the mass 1 at g (h). It will be convenient to use μ for the measure and also for the corresponding integral, writing $\mu(f) = \int f(g)\mu(dg)$ where integration is always over the entire group. For notational ease we shall take the Fourier-Stieltjes transform $\hat{\mu}$ of μ to be defined by $\hat{\mu}(\hat{g}) = \int (g, \hat{g})\mu(dg)(=\mu(\hat{g}))$; in particular for absolutely continuous measures, inversion will involve the familiar conjugation.

On occasion we shall need to multiply a measure μ by a function f: $f\mu$ will denote the measure we might define by $f\mu(dx) = f(x)\mu(dx)$. Finally it should perhaps be stated explicitly that the term "subalgebra" should only be taken in the algebraic sense, and all references to norms on subalgebras of $M(G)$ are to the norm of $M(G)$.

1. Preliminaries. If T is an isomorphism of $L_1(G)$ onto $L_1(H)$, and G and H are abelian then one has a dual homeomorphism τ of $H^\wedge$ onto $G^\wedge$ for which $(T\mu)^\wedge = \hat{\mu} \circ \tau$. This fact from the Gelfand theory formed the starting point of Helson's investigation [7], which proceeded to show τ had algebraic properties as well when T is norm-decreasing. Helson observed [7, §2] that τ could be extended to map almost periodic functions in a linear norm-decreasing fashion, but found no application for his observation, which will be fundamental for our abelian results.

Our first result yields the algebraic content of the norm-decreasing character of somewhat more general maps. Here and elsewhere $\|\ \|_\infty$ will denote the usual supremum norm for functions, and 0 the function identically zero.

THEOREM 1.1. *Let G and H be a pair of abelian topological groups, with $G^\wedge$ and $H^\wedge$ their (algebraic) groups of continuous characters. Let τ be any map of $H^\wedge$ into $G^\wedge \cup \{0\}$ with $\tau\hat{h}_0 = \hat{g}_0$. If*

$$\left\|\sum_{i=1}^{n} a_i \tau\hat{h}_i\right\|_\infty \le \left\|\sum_{i=1}^{n} a_i\hat{h}_i\right\|_\infty \tag{1.11}$$

for any trigonometric polynomial $\sum_{i=1}^{n} a_i\hat{h}_i$ on H, then $\tau^{-1}\,G^\wedge$ is a subgroup of $H^\wedge$ and the restriction of τ to this subgroup an algebraic homomorphism.[1]

[1] Actually we could take $G^\wedge$ and $H^\wedge$ to be any groups of (multiplicative) characters on a pair of (not necessarily abelian) groups G and H. One need only replace G^* and H^* (below) by the duals of the (discrete) groups $G^\wedge, H^\wedge$ (into which G and H map onto dense subsets).

COROLLARY 1.2. *If* τ: $H^\wedge \to G^\wedge \cup \{0\}$ *satisfies* (1.11) *and* $\tau\hat{h}_1 \in G^\wedge$ *then* $\sigma : \hat{h} \to \tau(\hat{h}_1)^{-1}\tau(\hat{h}\hat{h}_1)$ *is multiplicative on the subgroup* $\hat{h}_1^{-1}\tau^{-1}(G^\wedge)$ *of* $H^\wedge$, *and otherwise vanishes.*

COROLLARY 1.3. *If* $\tau : H^\wedge \to G^\wedge$ *satisfies* (1.11) *then* $\sigma : \hat{h} \to (\tau\hat{h}_0)^{-1}\tau\hat{h}$ *is a homomorphism of* $H^\wedge$ *into* $G^\wedge$. *Conversely if* σ *is a homomorphism* (1.11) *holds. Finally identical equality obtains in* (1.11) *iff* τ *is one-to-one as well.*

Proofs. In (1.11) we are of course demanding that the obvious linear extension of τ mapping trigonometric polynomials on H into those on G be norm-decreasing, and thus we have a norm-decreasing extension of this map taking $\mathfrak{A}(H)$, the almost periodic functions on H, into $\mathfrak{A}(G)$. Letting H^* and G^* be the almost periodic compactifications[2] of H and G we then have a norm-decreasing map T of $C(H^*)$ into $C(G^*)$ with $TH^{*\wedge} \subset G^{*\wedge} \cup \{0\}$. As a consequence the norm-decreasing adjoint map T^* of $C(G^*)^* = M(G)$ into $C(H^*)^* = M(H^*)$ is multiplicative, for $T^*(\mu_1 * \mu_2)(\hat{h}) = \mu_1 * \mu_2(T\hat{h}) = \mu_1(T\hat{h})\mu_2(T\hat{h}) = T^*\mu_1(\hat{h})T^*\mu_2(\hat{h})$ since $T\hat{h}$ is either 0 or a character. Hence $(T^*(\mu_1 * \mu_2))^\wedge = (T^*\mu_1)^\wedge(T^*\mu_2)^\wedge$ and from the one-to-one nature of $^\wedge$ we obtain $T^*(\mu_1 * \mu_2) = T^*\mu_1 * T^*\mu_2$.

Moreover from $T\hat{h}_0 = \tau\hat{h}_0 = \hat{g}_0$ we see that T^* preserves non-negativity; for $\mu \geq 0$ and $||\mu|| = 1$ imply $1 = \mu(\hat{g}_0) = \mu(T\hat{h}_0) = T^*\mu(\hat{h}_0) \leq ||T^*\mu|| \leq ||\mu|| = 1$ so that $T^*\mu(1) = 1 = ||T^*\mu||$, and therefore $T^*\mu \geq 0$.

Consequently T^* maps the multiplicative subgroup $\{\mu_g : g \in G^*\}$ of the unit ball of $M(G^*)$ into a subgroup of the unit ball of $M(H^*)$ which consists of non-negative measures. Thus by [**6**, 2.4] (cf. introduction (1)) the image consists of translates of Haar measure ν of some subgroup K of H^*, and we can write $T^*\mu_g = \nu^{\gamma(g)}$ where $\nu^{\gamma(g)}$ is the translate of ν to the coset $\gamma(g) \in H^*/K$. For $\hat{h}$ in $K^\perp$(=the subgroup of $H^{*\wedge} = H^\wedge$ of all characters identically 1 on K, hence constant on cosets mod K) we have $(\gamma(g), \hat{h}) = \nu^{\gamma(g)}(\hat{h}) = T^*\mu_g(\hat{h}) = \mu_g(T\hat{h}) = (g, \tau\hat{h})$ for all g in $G \subset G^*$. But as usual this implies τ is multiplicative on the subgroup $K^\perp$ of $H^\wedge$ since, for $\hat{h}_1, \hat{h}_2$ in $K^\perp$, $(g, \tau(\hat{h}_1\hat{h}_2)) = (\gamma(g), \hat{h}_1\hat{h}_2) = (\gamma(g), \hat{h}_1)(\gamma(g), \hat{h}_2) = (g, \tau\hat{h}_1)(g, \tau\hat{h}_2) = (g, \tau\hat{h}_1\tau\hat{h}_2)$ for all g in G. On the other hand for $\hat{h} \notin K^\perp$ we have $0 = \nu^{\gamma(g)}(\hat{h}) = T^*\mu_g(\hat{h}) = \mu_g(T\hat{h}) = (g, \tau\hat{h})$ for all g in G, and thus $\tau\hat{h} = 0$; consequently $\tau^{-1}G^\wedge$ is precisely the subgroup $K^\perp$ of $H^\wedge$, and our proof of Theorem 1.1 is complete.

We might remark that the converse of 1.1 can be obtained in somewhat the fashion of the corresponding assertion of 1.3 (below), and

[2] It will be convenient to view G as a dense subset of G^* and $\mathfrak{A}(G)$ as the restrictions to G of the elements of $C(G^*)$ [**10**, **15**]. Similarly we consider the elements of $G^\wedge$ as the restrictions to G of the elements of $G^{*\wedge}$.

equality obtains identically in (1.11) iff $\tau H^\frown \subset G^\frown$ and, as in 1.3, τ is one-to-one. Since we shall have no use for these facts proofs will be omitted.

The proof of Corollary 1.2 follows immediately from noting that

$$\left\|(\tau\hat{h}_1)^{-1}\sum_{i=1}^n a_i\tau\hat{h}_i\right\|_\infty = \left\|\sum_{i=1}^n a_i\tau\hat{h}_i\right\|_\infty$$

(so that (1.11) holds for σ) while $\sigma(\hat{h}_0) = \hat{g}_0$. Evidently σ is independent of the particular choice of $\hat{h}_1$.

The direct portion of Corollary 1.3 is a consequence of 1.2, taking $\hat{h}_1 = \hat{h}_0$. For the converse part we note that if σ is a homomorphism then interpreting it as a map of $H^{*\frown}$ into $G^{*\frown}$ we have a dual homomorphism γ of G^* into H^*, and

$$\left(\sum_{i=1}^n a_i\sigma\hat{h}_i\right)(g) = \left(\sum_{i=1}^n a_i\hat{h}_i\right)(\gamma(g)) .$$

Consequently (since we may consider G and H as dense subsets of G^* and H^*) we have

$$\begin{aligned}(1.12)\qquad \left\|\sum_{i=1}^n a_i\sigma\hat{h}_i\right\|_\infty &= \sup_{G^*}\left|\sum_{i=1}^n a_i\sigma\hat{h}_i(g)\right| = \sup_{G^*}\left|\sum_{i=1}^n a_i\hat{h}_i(\gamma(g))\right| \\ &= \sup_{\gamma(G^*)}\left|\sum_{i=1}^n a_i\hat{h}\,(h)\right| \le \sup_{H^*}\left|\sum_{i=1}^n a_i\hat{h}_i(h)\right| = \left\|\sum_{i=1}^n a_i\hat{h}_i\right\|_\infty ,\end{aligned}$$

and (1.11) holds. Clearly identical equality obtains if $\gamma(G^*) = H^*$. On the other hand since γ is continuous and G^* compact, $\gamma(G^*)$ is a compact subgroup of H^*, and if $\gamma(G^*) \neq H^*$ some non-zero f in $C(H^*)$ vanishes on $\gamma(G^*)$; since f can be approximated uniformly by trigonometric polynomials equality in (1.12) cannot always obtain. Thus identical equality is equivalent to $\gamma(G^*) = H^*$, or dually, to the one-to-oneness of σ, hence of τ.

We shall return to some reformulations and analogues of these results in §6.

2. Homomorphisms. In order to utilize the device suggested by Helson we need not restrict our attention to Banach algebras. We need only insist that our subalgebra A of $M(G)$ have $G^\frown \cup \{0\}$ as its space of multiplicative functionals and be large enough to determine the norm of each trigonometric polynomial on G. Unless something to the contrary is stated G and H will represent locally compact abelian groups throughout this section.

It will be convenient to extend the definition of the Fourier-Stieltjes transform $\hat{\mu}$ of μ in $M(G)$ by setting $\hat{\mu}(0) = \mu(0) = 0$, and regard

$G^\wedge \cup \{0\}$ as the one point compactification of $G^\wedge$. Consider the following conditions on a subalgebra A of $M(G)$:

(2.01) *For each trigonometric polynomial* $\sum_{i=1}^n a_i\hat{g}_i$ *on* G

$$\left\|\sum_{i=1}^n a_i\hat{g}_i\right\|_\infty = \sup\left\{\left|\mu\left(\sum_{i=1}^n a_i\hat{g}_i\right)\right| : \mu \in A, \|\mu\| \leq 1\right\} ;$$

(2.02) *The set of maps* $\mu \to \hat{\mu}(\hat{g}), \hat{g} \in G^\wedge \cup \{0\}$, *corresponds in a one-to-one fashion to the set of all multiplicative linear functionals on* A, *and* $A^\wedge \subset C(G^\wedge \cup \{0\})$.

When both conditions hold $A^\wedge$ *contains*[3] *sufficiently many functions to determine the topology of* $G^\wedge \cup \{0\}$; for (2.01) implies $A^\wedge$ separates any pair of elements of the compact space $G^\wedge \cup \{0\}$, and thus each $\hat{g}_1$ in $G^\wedge \cup \{0\}$ has a base of neighborhoods of the form $\{\hat{g} : |\hat{\mu}_i(\hat{g}) - \hat{\mu}_i(\hat{g}_1)| < \varepsilon, i = 1, 2, \cdots, n\}$, where $\mu_i \in A$. $A = L_1(G)$ clearly satisfies these condition, and will of course be the most important example.

THEOREM 2.1 *Let A satisfy* (2.01) *and* (2.02). *Then if T is a non-zero norm-decreasing homomorphism of A into $M(H)$ there is a compact subgroup H_0 of H, a continuous* (*not necessarily open*) *homomorphism γ of G into H/H_0, and characters $\hat{g}$ of G and $\hat{h}$ of H for which*

$$T\mu(f) = \mu(\hat{g}[S(\hat{h}f)\circ\gamma]), \qquad f \in C_0(H)\,, \tag{2.11}$$

where S denotes the map of $C_0(H)$ onto $C_0(H/H_0)$ defined by $Sf(hH_0) = \int_{H_0} f(hh')\nu(dh')$ (*where ν is Haar measure on H_0*); *alternatively*

$$T\mu = \hat{h}S^*\Gamma\hat{g}\mu \tag{2.12}$$

where Γ is the homomorphism of $M(G)$ into $M(H/H_0)$ defined by setting $\Gamma\mu(f) = \mu(f\circ\gamma), f \in C_0(H/H_0)$, *and S^* is the adjoint of S mapping $M(H/H_0)$ into $M(H)$. Conversely each such quadruple $H_0, \gamma, \hat{g}, \hat{h}$ defines a non-zero norm-decreasing T via* (2.11) *or* (2.12).

Proof. For each $\hat{h}$ in $H^\wedge$, $\mu \to (T\mu)^\wedge(\hat{h})$ defines a multiplicative functional on A, and thus we obtain a unique $\tau\hat{h}$ in $G^\wedge \cup \{0\}$ for which $(T\mu)^\wedge(\hat{h}) = \hat{\mu}(\tau\hat{h})$. Since the elements of $A^\wedge$ suffice to define the topology of $G^\wedge \cup \{0\}$ and the functions $(T\mu)^\wedge$ are continuous on $H^\wedge$ one clearly has τ: $H^\wedge \to G^\wedge \cup \{0\}$ continuous.

On the other hand τ satisfies (1.11) as a consequence of (2.01):

$$\left\|\sum_{i=1}^n a_i\tau\hat{h}_i\right\|_\infty = \sup_{\|\mu\|\leq 1}\left|\mu\left(\sum_{i=1}^n a_i\tau\hat{h}_i\right)\right| \tag{2.13}$$

[3] Of course this holds when $A^\wedge$ is only a separating subalgebra of $C(G^\wedge \cup \{0\})$; but then we can only assert that $G^\wedge$ forms a subspace of the space of multiplicative functionals on A (taken in the w^* topology).

$$= \sup_{\|\mu\| \le 1} \left| T\mu\left(\sum_{i=1}^{n} a_i \hat{h}_i\right)\right|$$
$$\le \sup_{\|T\mu\| \le 1} \left| T\mu\left(\sum_{i=1}^{n} a_i \hat{h}_i\right)\right| \le \left\|\sum_{i=1}^{n} a_i \hat{h}_i\right\|_\infty .$$

Thus in order to apply Corollary 1.2 we need only verify that $\tau\hat{h}_1 \in G^\wedge$ for some $\hat{h}_1$ in $H^\wedge$; but such an $\hat{h}_1$ exists since otherwise $\tau H^\wedge = 0$, $(TA)^\wedge(H^\wedge) = 0$ and thus $TA = 0$ by the one-to-oneness of the Fourier-Stieltjes transformation. Consequently $\sigma : \hat{h} \to (\tau\hat{h}_1)^{-1}\tau(\hat{h}\hat{h}_1)$ is multiplicative on the subgroup $K = \sigma^{-1}(G^\wedge) = \hat{h}_1^{-1}\tau^{-1}(G^\wedge)$ of $H^\wedge$, and of course vanishes elsewhere. As we have seen τ, and thus σ, is continuous on $H^\wedge$ so that $\sigma^{-1}\{0\}$ is closed and $K = \sigma^{-1}(G^\wedge)$ is open. Therefore K is an open and closed subgroup of $H^\wedge$, whence $H^\wedge/K$ is discrete, and the dual $H_0 = K^\perp$ of $H^\wedge/K$ is a compact subgroup of H.

Dual to the continuous homomorphism $\sigma|K : K \to G^\wedge$ we have a continuous homomorphism γ of G into $K^\wedge = H/K^\perp = H/H_0$, and thus for $\hat{h}$ in K and g in G, $(g, \sigma\hat{h}) = (\gamma(g), \hat{h}) = \nu^{\gamma(g)}(\hat{h})$, where $\nu^{\gamma(g)}$ is again the translate to the coset $\gamma(g)$ of Haar measure ν on H_0. Moreover the formula

$$(g, \sigma\hat{h}) = \nu^{\gamma(g)}(\hat{h}) \tag{2.25}$$

clearly also holds when $\hat{h} \notin K = H_0^\perp$, since both sides are then zero. Combining (2.25) with $\sigma(\hat{h})\tau(\hat{h}_1) = \tau(\hat{h}\hat{h}_1)$, or $\sigma(\hat{h}\hat{h}_1^{-1})\tau(\hat{h}_1) = \tau(\hat{h})$, we make the following computation, with $F \in L_1(H^\wedge)$:

$$\begin{aligned}
T\mu(\bar{\hat{F}}) &= \int \bar{\hat{F}}(h) T\mu(dh) = \int \overline{F(\hat{h})}(T\mu)^\wedge(\hat{h})d\hat{h} \\
&= \int \overline{F(\hat{h})}\hat{\mu}(\tau\hat{h})d\hat{h} = \iint \overline{F(\hat{h})}(g, \tau\hat{h})\mu(dg)d\hat{h} \\
&= \iint \overline{F(\hat{h})}(g, \sigma(\hat{h}\hat{h}_1^{-1}))(g, \tau\hat{h}_1)d\hat{h}\mu(dg) \\
&= \iint \overline{F(\hat{h})}\nu^{\gamma(g)}(\hat{h}\hat{h}_1^{-1})(g, \tau\hat{h}_1)d\hat{h}\mu(dg) \\
&= \iiint \overline{F(\hat{h})}(h, \hat{h}\hat{h}_1^{-1})(g, \tau\hat{h}_1)\nu^{\gamma(g)}(dh)d\hat{h}\mu(dg) \\
&= \iint \left(\int \overline{F(\hat{h})}(h, \hat{h})d\hat{h}\right)(h, \hat{h}_1^{-1})\nu^{\gamma(g)}(dh)(g, \tau\hat{h}_1)\mu(dg) \\
&= \iint \overline{\hat{F}(h)}(h, \hat{h}_1^{-1})\nu^{\gamma(g)}(dh)(g, \tau\hat{h}_1)\mu(dg) \\
&= \int S(\hat{h}_1^{-1}\bar{\hat{F}})(\gamma(g))(g, \tau\hat{h}_1)\mu(dg) ,
\end{aligned}$$

or, setting $\hat{h} = \hat{h}_1^{-1}$ and $\hat{g} = \tau\hat{h}_1$

$$T\mu(\bar{\hat{F}}) = \mu(\hat{g} \cdot [S(\hat{h}\bar{\hat{F}}) \circ \gamma]) \; ; \tag{2.26}$$

since $L_1(H^\frown)^\frown$ is dense in $C_0(H)$ and both sides of (2.11) are continuous in f, (2.11) follows. The alternative form (2.12) follows when we make the obvious notational transfer.

Conversely given $H_0, \gamma, \hat{g}$ and $\hat{h}$, and thus S^* and Γ, the right side of (2.12) clearly defines a norm-decreasing homomorphism of $M(G) \to M(H)$, as the composition of four norm-decreasing homomorphisms. To see that $T \neq 0$ we need only verify that (2.11) remain valid for $f = \hat{h}' \in H^\frown$; for then

$$T\mu(\hat{h}') = \mu(\hat{g}[S(\hat{h}\hat{h}') \circ \gamma])$$

while $S(\hat{h}\hat{h}') \in (H/H_0)^\frown$ if $\hat{h}\hat{h}' \in H_0^\perp$, and then $S(\hat{h}\hat{h}') \circ \gamma = \hat{g}' \in G^\frown$. Consequently $TA(\hat{h}') = A(\hat{g}\hat{g}') = A^\frown(\hat{g}\hat{g}') \neq \{0\}$ for an appropriate $\hat{h}'$, by (2.02).

But that (2.11) remains valid for $f = \hat{h}' \in H^\frown$ follows from the same sort of computation as the preceding; with $F \in L_1(H^\frown)$ one obtains

$$\begin{aligned}
\int \overline{F(\hat{h}')}(T\mu)(\hat{h}')d\hat{h}' &= T\mu(\bar{\hat{F}}) = \int (g, \hat{g})S(\hat{h}\bar{\hat{F}})(\gamma(g))\mu(dg) \\
&= \iint (g, \hat{g})(h, \hat{h})\overline{\hat{F}(h)}\nu^{\gamma(g)}(dh)\mu(dg) \\
&= \int \overline{F(\hat{h}')}\Big(\iint (g, \hat{g})(h, \hat{h})(h, \hat{h}')\nu^{\gamma(g)}(dh)\mu(dg)\Big)d\hat{h}' \\
&= \int \overline{F(\hat{h}')}\Big(\int (g, \hat{g})[S(\hat{h}\hat{h}')(\gamma(g))]\mu(dg)\Big)d\hat{h}' \\
&= \int \overline{F(\hat{h}')}\mu(\hat{g}[S(\hat{h}\hat{h}') \circ \gamma])d\hat{h}
\end{aligned}$$

whence $(T\mu)(\hat{h}') = \mu(\hat{g}[S(\hat{h}\hat{h}') \circ \gamma])$ for almost all $\hat{h}'$. But the second expression vanishes for $\hat{h}'$ in the open complement of $\hat{h}^{-1}H_0^\perp$ so that (by continuity) the first also vanishes there. On the other hand for $\hat{h}'$ in $\hat{h}^{-1}H_0^\perp$ (also open) we have $\mu(\hat{g}[S(\hat{h}\hat{h}') \circ \gamma])$ continuous as a function of $\hat{h}'$ since $S(\hat{h}\hat{h}') \circ \gamma = \sigma(\hat{h}\hat{h}')$ where σ is the continuous homomorphism of $H_0^\perp = (H/H_0)^\frown \to G^\frown$ dual to γ. Consequently both expressions are continuous functions of $\hat{h}'$ on $\hat{h}^{-1}H_0^\perp$ as well, and thus coincide on this open set.

2.2 REMARK. When the subgroup H_0 is trivial (i.e. $= \{h_0\}$) one may write 2.12 in the more concise form $T\mu = \Gamma\hat{g}\mu$; for clearly we have $T\mu = \hat{h}\Gamma\hat{g}\mu$ so that $T\mu(f) = \mu(\hat{g}[(\hat{h}f) \circ \gamma]) = \mu(\hat{g}(\hat{h} \circ \gamma)(f \circ \gamma))$ and we may replace $\hat{g}$ by $\hat{g}(\hat{h} \circ \gamma) \in G^\frown$. This situation will of course occur if each $\hat{h}$ in $H^\frown$ produces a non-zero functional on TA, i.e. when $\tau H^\frown \subset G^\frown$; for then $H_0^\perp = K = H^\frown$.

2.3 REMARK. If A is an ideal of a larger subalgebra A_0 of $M(G)$ and A satisfies (2.01) and (2.02) there is the possibility of applying Theorem 2.1 to certain norm-decreasing homomorphisms T on A_0. For provided $TA \neq \{0\}$, we may apply the result to the pair A and $T|A$ to obtain $T|A = T_1|A$ where T_1 represents the homomorphism (given by the right side of (2.12)) of all of $M(G)$ into $M(H)$; consequently (since A is an ideal in A_0) for $\mu \in A$, $\mu' \in A_0$,

$$T\mu' * T\mu = T(\mu' * \mu) = T_1(\mu' * \mu) = T_1\mu' * T_1\mu = T_1\mu' * T\mu$$

and $(T\mu' - T_1\mu') * T\mu = 0$. Hence $T\mu' - T_1\mu'$ annihilates TA, and we need only know that TA has no non-zero annihilators in $M(H)$ (not TA_0) to conclude that $T\mu' = T_1\mu'$ for all μ' in A_0. As a particular case

COROLLARY 2.31. *Let A satisfy* (2.01) *and* (2.02) *and let A_0 be a larger subalgebra of $M(G)$ in which A forms an ideal. If T is a norm-decreasing isomorphism of A_0 onto $M(H)$, then T is determined as in Theorem* 2.1, *indeed as in* 2.2 *since $H_0 = \{h_0\}$.*

Since $\mu * A = 0$ implies $\hat{\mu}A^\frown = 0$ while $A^\frown(\hat{g}) \neq \{0\}$ for each $\hat{g}$ in $\hat{G}$ by (2.02), A has no non-zero annihilators in A_0. Thus since T is an isomorphism, TA has no non-zero annihilators in $TA_0 = M(H)$, and $T\mu = \hat{h}S^*\Gamma\hat{g}\mu$. But if ν denotes Haar measure of H_0 one clearly has $(\hat{h}\nu) * T\mu = T\mu$ so that we must have $\hat{h}\nu$ the identity of $M(H)$, hence $H_0 = \{h_0\}$.

2.4. The following example shows how completely wrong Theorem 2.1 is for arbitrary large subalgebras of $M(G)$ in general; it was suggested to the author by K. de Leeuw. Let G be any non-discrete locally compact abelian group and, for μ in $M(G)$, let $\mu = \mu^p + \mu^c$ be the Lebesgue decomposition of μ into discrete and continuous parts, i.e., μ^p is a countable linear combination of point masses (converging in norm) and μ^c vanishes on all one point sets. Since the continuous measures form an ideal and $\mu_1^p * \mu_2^p$ is still discrete, $\mu \to \mu^p$ is a norm-decreasing homomorphism of $M(G) \to M(G)$, or indeed of $M(G)$ onto $M(G^d)(G^d = G$ in the discrete topology); clearly the map is not induced by any continuous $\gamma : G \to G^d$.

2.5. The restriction that T be norm-decreasing in Theorem 2.1 can be replaced by apparently weaker conditions in certain cases. The following result has a much simpler proof when $A = L_1(G)$.

THEOREM 2.5. *Let A be a subalgebra of $M(G)$ satisfying* (2.02) *which is spanned by its non-negative elements and has sufficiently many of these to determine the non-negative almost periodic functions, i.e.,*

(2.51) $f \in \mathfrak{A}(G)$ and $\mu(f) \geq 0$ for all $\mu \geq 0$ in A imply $f \geq 0$.
If T is any non-zero homomorphism of A into $M(H)$ which preserves order ($\mu \geq 0 \Rightarrow T\mu \geq 0$) *then T is norm-decreasing. If A also satisfies* (2.01) *then* $T\mu = S^*\Gamma\mu$, $\mu \in A$.

Proof. As in Theorem 2.1 we obtain $\tau : H^\wedge \to G^\wedge \cup \{0\}$ with $T\mu(\hat{h}) = \mu(\tau\hat{h})$. The functional $\mu \to T\mu(\hat{h}_0)$ cannot be zero for then $0 = T\mu(1) = ||T\mu||$ for $\mu \geq 0$, whence $TA = 0$ since the non-negative elements span A. For $\mu \geq 0$ in A, $\mu(\tau\hat{h}_0) = T\mu(\hat{h}_0) \geq 0$ so that $\tau\hat{h}_0 \geq 0$ by (2.51), and thus $\tau\hat{h}_0 = \hat{g}_0$. Consequently for $\mu \geq 0$, $||\mu|| = \mu(\hat{g}_0) = \mu(\tau\hat{h}_0) = T\mu(\hat{h}_0) = ||T\mu||$[4].

Let τ_0 denote the linear extension of τ mapping trigonometric polynomials. If p is a non-negative trigonometric polynomial on H and $\mu \geq 0$ is in A then $\mu(\tau_0 p) = T\mu(p) \geq 0$, so that $\tau_0 p \geq 0$ by (2.51). Thus τ_0 preserves the order of real valued trigonometric polynomials, and since $\tau\hat{h}_0 = \hat{g}_0$, $-1 \leqq p \leqq 1$ implies $-1 \leqq \tau_0 p \leqq 1$. But for any trigonometric polynomial $p = \sum_{i=1}^n a_i\hat{h}_i$, if $p^* = \sum_{i=1}^n \bar{a}_i\hat{h}_i^{-1}$ then $(p + p^*)/2$ and $(p - p^*)/2i$ are real valued, with values bounded by $-||p||_\infty$, $||p||_\infty$. Hence $||\tau_0(p + p^*)/2||_\infty \leqq ||p||_\infty$, $||\tau_0(p - p^*)/2||_\infty = ||\tau_0(p - p^*)/2i||_\infty \leqq ||p||_\infty$, and therefore $||\tau_0 p|| \leqq 2||p||_\infty$.

Consequently τ_0 extends to a bounded map of $\mathfrak{A}(H)$ into $\mathfrak{A}(G)$, which we may view as a map of $C(H^*)$ into $C(G^*)$; calling the extension τ_0 we have $T\mu(f) = \mu(\tau_0 f)$, $\mu \in A, f \in C(H^*)$, since this held for trigonometric polynomials. Moreover this identity implies τ_0 (as extended) preserves order by (2.51), so the adjoint $\tau_0^* : M(G^*) \to M(H^*)$ must also preserve order. As before we conclude from $\tau_0\hat{h}_0 = \hat{g}_0$ that $||\tau_0^*\mu|| = ||\mu||$ for $\mu \geq 0$ in $M(G^*)$. Therefore τ_0^* maps the point masses on G^* into the unit ball of $M(H^*)$, and thus their w^* closed convex circled hull into the same set. Since the hull coincides with the unit ball of $M(G^*)$, $||\tau_0^*|| = ||\tau_0|| \leqq 1$, and, for μ in A,

$$\sup_{||f||_\infty \leqq 1} |T\mu(f)| = \sup_{||f||_\infty \leqq 1} |\mu(\tau_0 f)| \leqq \sup_{||\tau_0 f||_\infty \leqq 1} |\mu(\tau_0 f)| \leqq ||\mu||$$

where f varies in $\mathfrak{A}(H)$. But the norm of $T\mu$ as a functional on $\mathfrak{A}(H)$ coincides with its norm as a measure ([4], [6, §5]), whence $||T\mu|| \leqq ||\mu||$, and T is norm-decreasing.

For the final statement in 2.5 we need only note that since $\tau\hat{h}_0 = \hat{g}_0$,

[4] At this point the proof for $A = L_1$ is essentially complete; for T is clearly norm-decreasing on simple functions (rather, on the corresponding measures) and these are dense.

and since our present τ coincides with that obtained in the proof of Theorem 2.1, we may take $\hat{h}_1 = \hat{h}_0$ in deriving (2.26), so that $\hat{h} = \hat{h}_0$, $\hat{g} = \hat{g}_0$ in (2.12), completing our proof.

It should perhaps be noted that portions of the above proof can be used to obtain an analogue of Theorem 1.1 in which (1.11) is replaced by "$\sum_{i=1}^n a_i \tau \hat{h}_i \geq 0$ if $\sum_{i=1}^n a_i \hat{h}_i \geq 0$"; for clearly our argument shows this condition implies (1.11).

If the group H has a connected dual we can replace "norm-decreasing" in Theorem 2.1 by "bounded".

THEOREM 2.6. *Let A be a subalgebra of $M(G)$ satisfying* (2.01) *and* (2.02), *and suppose $H^\wedge$ is connected. If T is any bounded non-zero homomorphism of A into $M(H)$, then T is norm-decreasing; consequently there is a homomorphism $\gamma : G \to H$ and a $\hat{g}$ in $G^\wedge$ for which $T\mu = \Gamma \hat{g} \mu$, $\mu \in A$. In particular if A is a closed subalgebra, all non-zero homomorphisms of A into $M(H)$ arise in this fashion.*[5]

Proof. As in the proof of Theorem 2.1 we obtain a continuous map $\tau : H^\wedge \to G^\wedge \cup \{0\}$ with $\tau^{-1} G^\wedge \neq \phi$; further, the linear extension of τ mapping trigonometric polynomials is bounded by a computation analogous to (2.13), and we may view this as extending to a bounded map $\tau_0 : C(H^*) \to C(G^*)$. Again $\tau_0^* : M(G^*) \to M(H^*)$ is multiplicative (as in 1.1), for $\tau_0^* \mu(\hat{h}) = \mu(\tau_0 \hat{h}) = \mu(\tau \hat{h})$, or $(\tau_0^* \mu)^\wedge = \hat{\mu} \circ \tau$, $\mu \in M(G^*)$.

Now (for any locally compact abelian G) if we define $\tilde{\mu} \in M(G^*)$ corresponding to $\mu \in M(G)$ by setting $\tilde{\mu}(f) = \int_G f(g) \mu(dg)$, $f \in C(G^*)$ (so that $\tilde{\mu}$ represents the restriction of the integral corresponding to μ to almost periodic functions) then $\mu \to \tilde{\mu}$ is an isometric isomorphism of $M(G)$ into $M(G^*)$ ([4], or [6, §5]), and, as functions on the *set* $G^\wedge$, $\hat{\tilde{\mu}} = \hat{\mu}$. Moreover as a consequence of a theorem of Bochner-Schoenberg-Eberlein [4], $M(G)^\sim$ consists of just those μ in $M(G^*)$ with $\hat{\mu}$ continuous on the *space* $G^\wedge$. Thus, for μ in $M(G)$, since τ is continuous and $(\tau_0^* \tilde{\mu})^\wedge = \hat{\tilde{\mu}} \circ \tau = \hat{\mu} \circ \tau$, we have $(\tau_0^* \tilde{\mu})^\wedge$ the transform of some measure $\sigma\mu$ in $M(H)$, i.e., $\tau_0^* \tilde{\mu} = (\sigma\mu)^\sim$. Clearly σ is a multiplicative map of $M(G)$ into $M(H)$. Since $\tau^{-1} G^\wedge \neq \phi$ for any $\hat{h}$ therein we have $|\sigma\mu_g(\hat{h})| = |\tau_0^* \tilde{\mu}_g(\hat{h})| = |\tilde{\mu}_g(\tau \hat{h})| = |\mu_g(\tau \hat{h})| = 1$ for all g in G, whence $\sigma\mu_g \neq 0$. Consequently if E denotes the set of all point masses on G, σE forms a bounded non-zero subgroup of $M(H)$ so that ($H^\wedge$ being connected) by a theorem of Beurling and Helson [3, §5] σE consists of unimodular multiples of point masses on H. Thus E maps into the unit ball of $M(H)$ under σ,

[5] Such homomorphisms being automatically bounded since A is a Banach algebra and $M(H)$ is semisimple.

or equivalently $E^{\sim}$ maps into the unit ball of $M(H^*)$ under τ_0^*. But $E^{\sim}$ is w^* dense in the set of point masses on G^*, and thus τ_0^* carries all point masses on G^* into the unit ball of $M(H^*)$. As in the proof of Theorem 2.5 this implies $\|T\mu\| \leq \|\mu\|$, $\mu \in A$. The final assertions of 2.6 now follow from 2.1 and 2.2, since the connectedness of $H^\wedge$ precludes the existence of any non-trivial compact subgroup H_0 of H. A consequence of our proof is

COROLLARY 2.61. *Let $H^\wedge$ be connected, and let $\tau : H^\wedge \to G^\wedge \cup \{0\}$ be any non-zero continuous map for which*

$$\left\|\sum_{i=1}^{n} a_i \tau \hat{h}_i\right\|_\infty \leq M \left\|\sum_{i=1}^{n} a_i \hat{h}_i\right\|_\infty$$

for all trigonometric polynomials $\sum_{i=1}^{n} a_i \hat{h}_i$ on H. Then M can be replaced by 1, $\tau H^\wedge \subset G^\wedge$, and $\hat{h} \to (\tau\hat{h}_0)^{-1}\tau\hat{h}$ is a homomorphism.[6]

For the map again extends to a bounded map τ_0 of $C(H^*)$ into $C(G^*)$ with $\|\tau_0\| = \|\tau_0^*\| \leq 1$ so that M can be replaced by 1. Since a translate of $\tau^{-1}G^\wedge$ provides us with an open subgroup of $H^\wedge$ by 1.1, $\tau^{-1}G^\wedge = H^\wedge$ and we need only apply 1.3.

2.7. A result of Leibenson [9], improved by Kahane [8], can be stated as follows: the only maps τ of the circle group T^1 into itself for which $f \circ \tau$ has an absolutely convergent series whenever f does are of the form $\tau(t) = t_1 \cdot t^n$, where $t_1 \in T^1$ and n is an integer. The following corollary of 2.6 yields a stronger assertion as a special case ($G^\wedge = H^\wedge = T^1$, $A = L_1(G)$); the result is of course essentially a dual formulation of 2.6.

COROLLARY 2.71. *Let A be a closed subalgebra of $M(G)$ satisfying* (2.01) *and* (2.02), *and let $H^\wedge$ be connected.*[7] *Then any map τ of $H^\wedge$ into $G^\wedge$ for which*

$$f \in A^\wedge \text{ implies } f \circ \tau \in M(H)^\wedge$$

must be of the form

$$\tau(\hat{h}) = \hat{g} \cdot \sigma(\hat{h})$$

where $\hat{g} \in G^\wedge$ and σ is a continuous homomorphism of $H^\wedge$ into $G^\wedge$.

[6] Note that continuity cannot be dropped from our hypothesis: for a map of $R^\wedge$ which merely interchanges two elements produces a bounded map of trigonometric polynomials on R.

[7] In this and our subsequent results involving a connected dual (viz: parts of 3.5, 4.2, 4.3, and 5.1) (2.01) can always be replaced by the requirement that $\|p\|_\infty \leq K \sup\{|\mu(p)| : \mu \in A, \|\mu\| \leq 1\}$ for all trigonometric polynomials p.

Proof. Let $T\mu$ be that element of $M(H)$ for which $(T\mu)^\wedge = \hat{\mu} \circ \tau$, $\mu \in A$. Clearly T is an algebraic homomorphism of A into $M(H)$, which must be bounded since A is a Banach algebra and $M(H)$ is semisimple. Moreover T is non-zero, since otherwise $A^\wedge(\tau H^\wedge) = 0$, contradicting (2.02). Thus 2.6 applies to yield a continuous homomorphism $\gamma : G \rightarrow H$ and a $\hat{g}$ in $G^\wedge$ with $T\mu = \Gamma\hat{g}\mu$, $\mu \in A$, whence as before

$$\hat{\mu}(\tau\hat{h}) = (T\mu)^\wedge(\hat{h}) = T\mu(\hat{h}) = \Gamma\hat{g}\mu(\hat{h}) = \mu(\hat{g}(\hat{h} \circ \gamma)) = \hat{\mu}(\hat{g}(\hat{h} \circ \gamma))$$

for all μ in $\hat{A}, h$ in $H^\wedge$. Consequently $\tau(\hat{h}) = \hat{g}(\hat{h} \circ \gamma) = \hat{g}\sigma(\hat{h})$ where $\sigma : H^\wedge \rightarrow G^\wedge$ is the continuous homomorphism dual to γ.

It should be noted that we cannot obtain the type of boundedness required in 2.6 by simply assuming A is a Banach algebra under *some* norm.

An analogous result, in which connectedness is replaced by more stringent requirements on τ, is a consequence of 2.5 and Bochner's theorem. We shall omit its most general statement, taking our algebra A to be $L_1(G)$ so that no specific hypotheses concerning the algebra appear.

COROLLARY. 2.72. *Let τ be a map of $H^\wedge$ into $G^\wedge$ for which $\varphi \circ \tau$ is positive definite on $H^\wedge$ whenever φ is a positive definite element of $C_0(G^\wedge)$. Then τ is a continuous (but not necessarily open) homomorphism.*

Proof. Since the Fourier-Stieltjes transform of a measure is a linear combination of four positive definite functions we may define $T\mu$ as before for μ in $A = L_1(G)$ to obtain a non-zero homomorphism of $L_1(G)$ into $M(H)$. Moreover, $\mu \geq 0, \mu \in L_1(G)$ imply $\hat{\mu}$ is a positive definite element of $C_0(G^\wedge)$, and thus $(T\mu)^\wedge = \hat{\mu} \circ \tau$ is positive definite. Thus (by Bochner's theorem again) $T\mu \geq 0$, and we may apply 2.5 to obtain $T\mu = S^*\Gamma\mu, \mu \in L_1(G)$, in the notation of Theorem 2.1. But for $\hat{h} \notin H_0^\perp$ we have $L_1(G)^\wedge(\tau\hat{h}) = (TL_1(G))^\wedge(\hat{h}) = (S^*\Gamma L_1(G))^\wedge(\hat{h}) = 0$; hence from $\tau H^\wedge \subset G^\wedge$ we conclude that $H_0^\perp = H^\wedge$ and H_0 is trivial, $T\mu = \Gamma\mu$ and therefore $T\mu(\hat{h}) = \mu(\tau\hat{h}) = \mu(\hat{h} \circ \gamma)$, so that τ appears as the dual to γ, completing our proof.

The same proof (except for the final step) applies if one takes τ only to be a non-trivial map of $H^\wedge$ into $G^\wedge \cup \{0\}$ (i.e., with $\tau^{-1}G^\wedge \neq \phi$); one obtains the fact that $T\mu = S^*\Gamma\mu, \mu \in L_1(G)$, and concludes that τ is a continuous homomorphism on the open subgroup $\tau^{-1}G^\wedge$ of $H^\wedge$ (in order to consider $\varphi \circ \tau$ as defined on all of $H^\wedge$ one should include 0 in the domain of φ, with $\varphi(0) = 0$).

2.8. It is tempting to try the same approach in the non-commutative situation, replacing characters by finite dimensional matricial representations; apparently only in case H is compact can we obtain any consequences without a deeper investigation.

For any map σ of functions and matrix $U = (u_{ij})$ of functions let σU represent the matrix $(\sigma(u_{ij}))$. Then if U is any bounded continuous finite dimensional matricial representation of H, $\nu \to \nu(U)$ is a bounded representation of $M(H)$. Moreover if $T : L_1(G) \to M(H)$ is any bounded homomorphism, then $\mu \to T\mu(U)$ is a bounded representation of $L_1(G)$ and, as is well known, must be of the form $\mu \to \mu(\tilde{U})$, where $\tilde{U}$ is a continuous bounded matricial representation[8] of G. Viewing $C(H)$ as a subspace of $M(H)^*$, the adjoint T^* maps $C(H)$ into $L_1(G)^* = L_\infty(G)$, and we may clearly identify $\tilde{U}$ and $T^*U = (T^*u_{ij})$ as identical matrices of elements of $L_\infty(G)$. Consequently we can take T^*u_{ij} as a continuous function, indeed an almost periodic function, on G.

Now if H is compact the Peter-Weyl theorem assures us that we can view T^* as mapping $C(H)$ into $\mathfrak{A}(G)$; moreover this map τ is clearly norm-decreasing if T is. Each μ in $M(G)$ provides us with a functional $\tilde{\mu}$ on $\mathfrak{A}(G)$, and since $\tau^* : \mathfrak{A}(G)^* \to M(H)$ is norm-decreasing, $||\tau^*\tilde{\mu}|| \le ||\tilde{\mu}|| \le ||\mu||$ so that $\sigma : \mu \to \tau^*\tilde{\mu}$ is a norm-decreasing map of $M(G)$ into $M(H)$. But σ is automatically multiplicative: for

$$\begin{aligned}\sigma(\mu * \mu')(U) = (\mu * \mu')^{\sim}(\tau U) = \mu * \mu'(\tilde{U}) = \mu(\tilde{U})\mu'(\tilde{U}) \\ = \sigma\mu(U)\sigma\mu'(U) = (\sigma\mu * \sigma\mu')(U)\end{aligned}$$

for all U, so that $\sigma(\mu * \mu') = \sigma(\mu) * \sigma(\mu')$ by the Peter-Weyl theorem. Thus $E = \{\sigma\mu_g : g \in G\}$ forms a multiplicative group in the unit ball of $M(H)$.

Unfortunately the results of [6] do not determine all groups in the ball of $M(H)$ in the non-abelian case, but only those consisting of non-negative measures. E will be such a group if T (and therefore T^*, τ, τ^* and σ) preserves order; moreover we then have $\mu \to T\mu(1)$ a non-zero representation of $L_1(G)$ if $T \ne 0$ (otherwise $0 = T\mu(1) = ||T\mu||$ for all $\mu \ge 0$, hence for all μ). Since $\mu \to T\mu(1)$ also preserves order, $T^*1 = 1$. As a consequence T is automatically norm-decreasing (cf. footnote 4), and $E \ne \{0\}$ since $\sigma\mu_g(1) = \mu_g(\tau 1) = \mu_g(T^*1) = 1$. We thus have E a set of translates of Haar measure of a normal subgroup H_0 of H, and can write as before $\sigma\mu_g = \nu^{\gamma(g)}$, $\gamma(g) \in H/H_0$.

But the map $g \to \tilde{\mu}_g$ of G into $\mathfrak{A}(G)^*$ (taken in the w^* topology) is continuous, so that $g \to \tau^*\tilde{\mu}_g = \nu^{\gamma(g)}$ is w^* continuous, and one can easily conclude that γ is a continuous homomorphism of G into H/H_0. Moreover since $g \to \tilde{\mu}_g$ is w^* continuous we can represent $\tilde{\mu}$ as the w^* convergent vector valued integral $\int \tilde{\mu}_g \mu(dg)$, $\mu \in M(G)$. Applying τ^* we

[8] That is, a homomorphism into a group of (possibly singular) matrices.

obtain $\tau^*\tilde{\mu} = \int \tau^*\tilde{\mu}_g \mu(dg) = \int \nu^{\gamma(g)}\mu(dg)$ so that $\tau^*\tilde{\mu}(f) = \int \nu^{\gamma(g)}(f)\mu(dg) = \mu(Sf \circ \gamma), f \in C(H)$, in our earlier notation. Finally we have $\tau^*\tilde{\mu} = T\mu$, $\mu \in L_1(G)$: for $\tau^*\tilde{\mu}(U) = \mu(\tau U) = \mu(T^*U) = T\mu(U)$, all U. Hence we may write $T = S^*\Gamma$.

Actually if T is any non-zero norm-decreasing homomorphism what we really need to know is that some one-dimensional representation of H induces a non-zero representation of $L_1(G)$. For then we have multiplicative characters χ' and χ of H and G respectively for which $T\mu(\chi') = \mu(\chi)$; consequently $\chi' T\chi^{-1}\mu(1) = T\chi^{-1}\mu(\chi') = \chi^{-1}\mu(\chi) = \mu(1)$ and the norm-decreasing map $T_0 : \mu \to \chi' T\chi^{-1}\mu$ has $T_0^* 1 = 1$, whence it is easily seen to preserve order (as in 1.1). Thus $T\mu = (\chi')^{-1}S^*\Gamma\chi\mu$.

THEOREM 2.9. *Let G be any locally compact group, H any compact group. Then any non-zero order-preserving homomorphism $T : L_1(G) \to M(H)$ is of the form $S^*\Gamma$. If T is merely norm-decreasing and $T^*\chi'$ is a non-zero element of $L_\infty(G)$ for some multiplicative character χ' of H, then $T\mu = \chi'' S^*\Gamma\chi\mu$, where χ'', χ are multiplicative characters of H and G respectively; indeed $\chi'' = (\chi')^{-1}, \chi = T^*\chi'$.*

3. **Isomorphisms.** An almost immediate consequence of Corollary 2.31 is the fact that isometric isomorphisms between $M(G)$ and $M(H)$ arise in the same simple fashion as in the case of L_1 algebras. Actually we have a stronger result.

THEOREM 3.1. *Let G and H be locally compact abelian groups, and let A be a subalgebra of $M(G)$ containing $L_1(G)$, B a similar subalgebra of $M(H)$. Then for any isomorphism T of A onto B which is norm-decreasing on $L_1(G)$ there is an isomorphism γ of G onto H and a character $\hat{g}$ of G for which*

$$T\mu(f) = \mu(\hat{g}(f \circ \gamma)), \quad f \in C_0(H), \mu \in A .$$

Thus T is an isometry and $T_1(G) = L_1(H)$.

Before proceeding to the proof of Theorem 3.1 we might note that $L_1(G)$ can be replaced in our hypothesis by any subalgebra of $M(G)$ satisfying (2.01) and (2.02) which is an ideal in A.

Proof of Theorem 3.1. Applying Theorem 2.1 to the restriction of T to $L_1(G)$ we obtain characters $\hat{g}_1$ and $\hat{h}_1$, and operators S^* and Γ for which $T\mu = \hat{h}_1 S^*\Gamma\hat{g}_1\mu$, $\mu \in L_1(G)$. Consider the norm-decreasing isomorphism $T_0 = \hat{h}_1^{-1}T\hat{g}_1^{-1}$ of $A_0 = \hat{g}_1 A$ onto $B_0 = \hat{h}_1^{-1}B$. A_0 contains $L_1(G)$, and B_0 contains $L_1(H)$, while $T_0\mu = S^*\Gamma\mu$ for μ in $L_1(G)$. Evidently $\tilde{\nu} * T_0\mu = T_0\mu$, $\mu \in L_1(G)$, where $\tilde{\nu}$ is Haar measure on H_0. Since $\tilde{\nu}$ is an idempotent, $\mu \to \tilde{\nu} * T_0\mu$ is a homomorphism of A_0 into $M(H)$ which is one-to-one on $L_1(G)$. Consequently it is one-to-one on all of A_0: for $\tilde{\nu} * T_0\mu = 0$ implies $\tilde{\nu} * T_0(\mu * \mu') = 0$, $\mu' \in L_1(G)$, whence $\mu * \mu' = 0$ by the one-to-oneness on $L_1(G)$, and $\mu = 0$. But if $H_0 \neq \{h_0\}$ we have $H_0^{\perp}$ a proper open and closed subgroup of $H^{\wedge}$ so that we can find a ν in $L_1(H)$, $\nu \neq 0$, with $\hat{\nu}(H_0^{\perp}) = 0$, by the regularity of $L_1(H)$. Since $\hat{\tilde{\nu}}$ is the characteristic function of $H_0^{\perp}$, $(\tilde{\nu} * \nu)^{\wedge} = \hat{\tilde{\nu}}\hat{\nu} = 0$, and $\tilde{\nu} * \nu = 0$; on the other hand $\nu = T_0\mu$, $\mu \in A_0$, $\mu \neq 0$, so that $\tilde{\nu} * \nu \neq 0$ by the one-to-oneness of $\mu \rightarrow \tilde{\nu} * T_0\mu$, and we conclude that $H_0 = \{h_0\}$. Thus γ appears as a continuous homomorphism of G into H, and we may now write $T_0\mu = \Gamma\mu$, $\mu \in L_1(G)$.

As a consequence[9] we have $(T_0\mu)^{\wedge}(\hat{h}) = (\Gamma\mu)^{\wedge}(\hat{h}) = \mu(\hat{h} \circ \gamma) = \hat{\mu}(\hat{h} \circ \gamma)$, $\mu \in L_1(G)$, with $\hat{h} \circ \gamma \in G^{\wedge}$, so $T_0\mu \rightarrow (T_0\mu)^{\wedge}(\hat{h})$ is a non-zero functional on $T_0L_1(G)$. Repeating a previous computation, we have, for μ in A_0 and μ' in $L_1(G)$

$$T_0\mu * T_0\mu' = T_0(\mu * \mu') = \Gamma(\mu * \mu') = \Gamma\mu * \Gamma\mu' = \Gamma\mu * T_0\mu' ,$$

$L_1(G)$ being an ideal, so that $(T_0\mu - \Gamma\mu) * T_0L_1(G) = 0$. Thus for each $\hat{h}$, $(T_0\mu - \Gamma\mu)^{\wedge}(\hat{h}) = 0$ whence $T_0\mu = \Gamma\mu$, $\mu \in A_0$. Consequently $T\mu(f) = \hat{h}\Gamma\hat{g}\mu(f) = \Gamma\hat{g}\mu(\hat{h}f) = \hat{g}\mu(\hat{h} \circ \gamma \cdot f \circ \gamma) = \mu(\hat{g}_1(f \circ \gamma))$ for μ in A, and it remains to show γ is an isomorphism of G onto H.

First $\gamma(G)$ is dense in H; for otherwise we have a non-zero f in $C_0(H)$ with $f \circ \gamma = 0$, while $\nu(f) \neq 0$ for some ν in $L_1(H)$, $\nu = T_0\mu$, whence $0 \neq \nu(f) = T_0\mu(f) = \mu(f \circ \gamma) = 0$. Moreover γ is one-to-one since if $\gamma(g_1) = \gamma(g_2)$ then $\mu = \mu_{g_1} - \mu_{g_2}$ has $\Gamma\mu = 0$ (for $\mu(f \circ \gamma) = f(\gamma(g_1)) - f(\gamma(g_2)) = 0$). But then for μ' in $L_1(G)$ we have $\mu * \mu' \in L_1(G)$ and $T_0(\mu * \mu') = \Gamma(\mu * \mu') = \Gamma\mu * \Gamma\mu' = 0$ whence $\mu * \mu' = 0$ for all μ' in $L_1(G)$, and clearly $\mu = 0$, $g_1 = g_2$. Indeed the argument shows Γ is one-to-one on $M(G)$.

Consequently it is sufficient to show γ^{-1} is continuous on $\gamma(G)$; for then γ is a homeomorphism, $\gamma(G)$ is therefore locally compact and, being dense in H, must coincide with H as is well known. Suppose then that the net $h_\delta = \gamma(g_\delta) \rightarrow h_0 = \gamma(g_0)$. Clearly $\Gamma\mu_{g_\delta} = \nu_{h_\delta}$. For μ in A_0 with $T_0\mu$ in $L_1(H)$ we have $\nu_{h_\delta} * T_0\mu \in L_1(H) \subset T_0A_0 = \Gamma A_0$; clearly $\Gamma(\mu_{g_\delta} * \mu) = \nu_{h_\delta} * \Gamma\mu = \nu_{h_\delta} * T_0\mu$ so that $\mu_{g_\delta} * \mu \in A_0$ since Γ is one-to-one on $M(G)$, and further $T_0(\mu_{g_\delta} * \mu) = \nu_{h_\delta} * T_0\mu$. But $\|T_0(\mu_{g_\delta} * \mu) - T_0\mu\| = \|\nu_{h_\delta} * T_0\mu - T_0\mu\| \rightarrow 0$, $T_0\mu$ being in $L_1(H)$, and since $T_0^{-1}|L_1(H)$ is automatically continuous, $\|\mu_{g_\delta} * \mu - \mu\| \rightarrow 0$. As a consequence $\{g_\delta: \delta \geq \delta_0\}$ is contained in some compact $K \subset G$ for some δ_0; otherwise a cofinal subnet tends to infinity and $\overline{\lim}\, \|\mu_{g_\delta} * \mu - \mu\| = 2\|\mu\|$ for each such μ. If g is any cluster point of $\{g_\delta\}$ in K then, for each $\hat{g}$, $\mu_g * \mu(\hat{g})$ is a cluster

[9] This follows as in the final part of the proof of 2.1.

point of $\{\mu_{g_\delta} * \mu(\hat{g})\}$, which of course converges to $\mu(\hat{g})$ since $\|\mu_{g_\delta} * \mu - \mu\| \to 0$. Thus $\mu_g * \mu = \mu$ and $T_0\mu = \Gamma\mu = \Gamma(\mu_g * \mu) = \nu_{\gamma(g)} * \Gamma\mu = \nu_{\gamma(g)} * T_0\mu$; since $T_0\mu$ is an arbitrary element of $L_1(H)$ we clearly have $\gamma(g) = h_0$ and $g = g_0$. Consequently $\{g_\delta\}$ converges to g_0 by the compactness of K, and γ^{-1} is continuous.

Finally we have $\Gamma L_1(G) = L_1(H)$ since strong continuity of the map $g \to \mu_g * \mu$ is equivalent to strong continuity of $h \to \nu_h * \Gamma\mu$, and L_1 consists of just those measures for which strong continuity holds, by a theorem of Plessner. Consequently $TL_1(G) = L_1(H)$ and our proof is complete.

Applying 2.5 and 2.6 to $T|L_1(G)$, we obtain

COROLLARY 3.11. *Let T be any isomorphism of A onto B for which $\mu \geq 0$, $\mu \in L_1(G)$ imply $T\mu \geq 0$. Then T is an isometry Γ induced by an isomorphism γ of G onto H.*

COROLLARY 3.12. *If $H^\wedge$ is connected any isomorphism T of A onto B is on isometry determined as in* 3.1.

THEOREM 3.2. *When G and H are arbitrary compact groups, the conclusions drawn in Theorem* 3.1 *and Corollary* 3.11 *continue to hold.*

Proof. Consider first the situation indicated by 3.1, and let μ^0, ν^0 be the Haar measures on G and H. Then $T\mu^0$ is a non-zero idempotent in the unit ball of $M(H)$, and thus, by the result (2) cited in the introduction, of the form $\chi_1\nu$ where ν is Haar measure of a subgroup of H, and χ_1 is a multiplicative character of this subgroup.

But $A * \mu^0 = K\mu^0$, K the complex field, so $B * (\chi_1\nu) = K(\chi_1\nu)$. Taking $M(H) = C(H)^*$ in the w^* topology, the linear map $\nu' \to \nu' * (\chi_1\nu)$ of $M(H)$ into itself is of course continuous, and clearly is of norm $\leqq 1$. In particular the unit ball of B maps into $D \cdot (\chi_1\nu)$, where D is the unit disc $\{z : |z| \leqq 1\}$ in K. Since each ν_h is w^* adherent to the unit ball of $L_1(H) \subset B$, we obtain $\nu_h * (\chi_1\nu) \in D(\chi_1\nu)$ for each h in H, and the carrier of ν must be translation invariant. Consequently $\nu = \nu^0$ and χ_1 appears as a character of the full group H.

Thus $\mu \to T\mu(\chi_1^{-1})$ is a non-trivial one-dimensional representation of $L_1(G)$: for $T\mu^0(\chi_1^{-1}) = \chi_1\nu^0(\chi_1^{-1}) = 1$. As in 2.8 we obtain a multiplicative character χ of G for which $T\mu(\chi_1^{-1}) = \mu(\chi)$; since $\mu^0(\chi) = 1$, $\chi = 1$, and by 2.9 we have $T\mu = \chi_1 S^*\Gamma\mu$, $\mu \in L_1(G)$. Setting $T_0\mu = \chi_1^{-1}T\mu$ we obtain an isomorphism of A onto $B_0 = \chi_1^{-1}B$, with $T_0\mu = S^*\Gamma\mu$, $\mu \in L_1(G)$, and, in particular, $T_0\mu^0 = \chi_1^{-1}T\mu^0 = \nu^0$.

As in 3.1, γ must be one-to-one; otherwise we have a $\mu \neq 0$ in $M(G)$ with $\Gamma\mu = 0$ so that $T_0(\mu * \mu') = S^*\Gamma(\mu * \mu') = S^*\Gamma\mu * S^*\Gamma\mu' = 0$

for all μ' in $L_1(G)$, and[10] $\mu * L_1(G) = 0$, $\mu = 0$. Moreover if the compact image $\gamma(G)$ of G in H/H_0 were not all of H/H_0 we should have an f in $C(H/H_0)$ with $f \neq 0, f \geq 0, f \circ \gamma = 0$; thus if ρ denotes the canonical map of H onto H/H_0,

$$0 < \nu^0(f \circ \rho) = T_0\mu^0(f \circ \rho) = S^*\Gamma\mu^0(f \circ \rho) = \Gamma\mu^0(f) = \mu^0(f \circ \gamma) = 0 ;$$

consequently γ maps G onto H/H_0, and therefore is a homeomorphism and isomorphism between these groups. But now Γ appears as an isometry mapping $M(G)$ onto $M(H/H_0)$, and since S^* is easily seen to be an isometry, $T_1 = T_0 | L_1(G) = S^*\Gamma | L_1(G)$ is isometric. This combines with $T_0\mu^0 = \nu^0$ to show T_1 and T_1^{-1} preserve order: for

$$\begin{aligned}\mu \geq 0 \Longleftrightarrow \mu * \mu^0 = \|\mu\|\mu^0 \Longleftrightarrow T_1\mu * \nu^0 = \|\mu\|\nu^0 = \|T_1\mu\|\nu^0 \\ \Longleftrightarrow T_1\mu \geq 0 .\end{aligned}$$

Consequently T_1 maps $\{\mu : 0 \leq \mu \leq \mu^0\}$ onto $\{\nu : 0 \leqq \nu \leqq \nu^0\}$, or, more generally, the algebra $L_\infty(G)\mu^0 = \{f \cdot \mu^0 : f \in L_\infty(G)\}$ onto $L_\infty(H) \cdot \nu^0$. As an isometry T_1 thus maps closure onto closure, or $L_1(G)$ onto $L_1(H)$, and we are forced to conclude that H_0 is trivial since its Haar measure acts as an identity on $T_1L_1(G) = L_1(H)$. Hence γ is an isomorphism of G onto H, and $T_1\mu = \Gamma\mu$. As before we conclude that $T_0\mu = \Gamma\mu$, $\mu \in A$: for with $\mu' \in L_1(G)$, $T_0\mu * T_0\mu' = T_0(\mu * \mu') = \Gamma(\mu * \mu') = \Gamma\mu * \Gamma\mu' = \Gamma\mu * T_0\mu'$ and $(T_0\mu - \Gamma\mu) * L_1(H) = 0$. Thus we have $T\mu = \chi_1\Gamma\mu = \Gamma\chi\mu$, as in 2.2, or $T\mu(f) = \mu(\chi \cdot (f \circ \gamma))$, proving the analogue of 3.1. The analogue of 3.11 follows since our T must then be norm-decreasing on $L_1(G)$ as in 2.8.

3.3. Returning to the abelian case, the results of Šreider [14] for $G = R$ indicate that $G^\wedge$ forms a smaller part of the maximal ideal space of $M(G)$ than one might initially presume. As one would suspect from the one-to-one nature of the Fourier-Stieltjes transformation however, $G^\wedge$ would seem still to occupy a rather dominant rôle in the Gelfand representation of $M(G)$; this view is certainly reinforced by 3.1 since it shows the norm-decreasing automorphisms of $M(G)$ can only induce self-homeomorphisms of the maximal ideal space which leave $G^\wedge$ invariant, and indeed preserve its algebraic structure.

3.4. A variant of the proof of Theorem 3.1 yields the form of all norm-decreasing isomorphisms of $L_1(G)$ onto a closed subalgebra of $L_1(H)$; when $G^\wedge$ is connected this yields the answer to the question: what (proper, closed) subalgebras isomorphic to $L_1(G)$ can $L_1(G)$ contain? Clearly if G_1 is a proper open subgroup isomorphic to G then $L_1(G)$

[10] If $\mu * L_1(G) = 0$ then for $f, F \in C(G)$ we have $0 = \int\int f(g_1g_2)\mu(dg_1)F(g_2^{-1})\mu^0(dg_2) = \int f * F(g_1)\mu(dg_1)$, whence $\mu = 0$ since such convolutions $f * F$ are dense in $C(G)$.

provides such a subalgebra; when $G^\frown$ is connected these are the only candidates.[11]

THEOREM 3.5. *Let A be a closed ideal in $M(G)$ satisfying* (2.01) *and* (2.02) *and let T be an isomorphism of A onto a closed subalgebra B of $L_1(H)$. If T is norm-decreasing then $T\mu = \hat{h}S^*\Gamma\hat{g}\mu$ (as in 2.1) where γ is an isomorphism of G onto an open subgroup of H/H_0. In particular if $H^\frown$ is connected any isomorphism of A onto B is of the form $\mu \to \Gamma\hat{g}\mu$, where γ maps G isomorphically onto an open subgroup of H.*

Proof. By 2.6 the second assertion follows from the first. By 2.1 we have $T\mu = \hat{h}S^*\Gamma\hat{g}\mu$; as before we can eliminate $\hat{g}, \hat{h}$, and may as well assume $T\mu = S^*\Gamma\mu$. Since A is an ideal in $M(G)$ and $\mu * A = 0$ implies $\mu = 0$ by (2.02), we conclude exactly as in 3.1 that γ is one-to-one.

Moreover if γ^{-1} isn't continuous on $\gamma(G) \subset H/H_0$, for some neighborhood U of g_0 we have $\gamma^{-1}(VH_0) \cap U' \neq \phi$ for each neighborhood V of h_0; let $g_V \in \gamma^{-1}(VH_0) \cap U'$ and let $\gamma(g_V) = h_V H_0$ where $h_V \in V$. Since

$$T(\mu * \mu_{g_V}) = S^*\Gamma(\mu * \mu_{g_V}) = S^*\Gamma\mu * \nu^{\gamma(g_V)} = S^*\Gamma\mu * \nu_{h_V} = T\mu * \nu_{h_V}$$

(where $\nu^{\gamma(g)}$ is the translate to $\gamma(g)$ of Haar measure on H_0 as before) we conclude from the strong convergence of $T\mu * \nu_{h_V}$ to $T\mu$ and the automatic continuity of T^{-1} (B being closed) that $||\mu * \mu_{g_V} - \mu|| \to 0$. Noting that $\mu * \mu_g = \mu$ for all μ in A implies $g = g_0$ by (2.02), our previous argument yields the fact that $g_V \to g_0$, contradicting $g_V \in U'$. Thus γ is a (topological) isomorphism, $\gamma(G)$ is locally compact and therefore closed, and we need only show $\gamma(G)$ open to complete our proof.

Let H_1 be the inverse image of $\gamma(G)$ under the canonical homomorphism of H onto H/H_0, a closed subgroup of H. If $f \in C_0(H)$ vanishes on H_1 we clearly have $S^*\Gamma\mu(f) = 0$ so that the regular Borel measure $S^*\Gamma\mu$ vanishes on all Borel subsets of the complement of H_1. Since $S^*\Gamma\mu$ is a non-zero element of $L_1(H)$ for some μ, H_1 clearly contains some compact set C of positive Haar measure, and thus must be open[12]; hence $\gamma(G)$ is open and our proof complete.

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[11] It is of course not the answer otherwise. For example let A_n be the algebra of integrable f on the circle T^1 with $f(e^{i\theta}) = f(e^{i(\theta+2\pi/n)})$; then setting $g(t) = f(t^{1/n})$ yields a well defined element g of $L_1(T_1)$, and $f \to g$ is easily seen to be an isomorphism of A_n with $L_1(T^1)$.

[12] For we can find a Baire subset E of H containing C with $E \backslash C$ of (Haar) measure zero; then the Borel measurable function $h \to \varphi_E(h)\varphi_E{}^{-1}(h^{-1}h') - \varphi_C(h)\varphi_C{}^{-1}(h^{-1}h')$ (for h' fixed, φ_E the characteristic function) differs from zero only on a subset of $(E\backslash C) \cup (h'E\backslash h'C)$ so that $\varphi_E * \varphi_E{}^{-1}(h') = \int \varphi_C(h)\varphi_C{}^{-1}(h^{-1}h')dh$. As usual the fact that $\varphi_E * \varphi_E{}^{-1} \neq 0$ on a neighborhood U of h_0 yields for $h' \in U$ a h in C with $h' \in hC^{-1} \subset H_1$ whence $U \subset H_2$ and H_1 is open.

4. Some other isomorphisms. The rôle of condition (2.02) in §2 was confined to providing us with a map τ of character groups dual to a given homomorphism of our algebra A. In certain situations such a τ arises naturally in the absence of (2.02) and provided (2.01) holds, our approach may again be applicable. For example suppose A is a closed subalgebra of $M(G)$ satisfying (2.01) for which $G^\wedge$ forms a subspace of the maximal ideal space $\mathfrak{M}$ of A; further suppose $G^\wedge$ is connected. Then any endomorphism T of A for which the dual map $\tau : \mathfrak{M} \rightarrow \mathfrak{M}$ sends $G^\wedge$ into itself necessarily has $\tau(\hat{g}) = \hat{g}_1 \cdot \sigma(\hat{g}), \hat{g} \in G^\wedge$, where σ is an endomorphism of $G^\wedge$. For T is necessarily bounded so that τ induces a bounded map of $\mathfrak{A}(G)$ into itself (by an analogue of (2.13), using (2.01)), and Corollary 2.61 applies. Consequently T is itself determined as before; similarly if $G^\wedge$ is not connected but T is also norm-decreasing, or order-preserving while (2.51) obtains, we can apply Corollary 1.3 or the remark following 2.5 to the same end.

Exactly such a situation arises in connection with the Arens-Singer theory of generalized analytic functions [1], in particular in Arens' subsequent generalization of the conformal mappings of the disc [2]. There (among other things) Arens is interested in the automorphisms[13] of a certain closed subalgebra A_1 of $L_1(G)$, G locally compact abelian; one has a fixed closed subset G_+ of G satisfying [1, §2]

(4.01) *G_+ is a subsemigroup of G, i.e., $x, y \in G_+$ imply $xy \in G_+$,*

(4.02) *the interior of G_+ is dense in G_+ and generates G;*

$A_1 = L_1(G_+)$ is then the set of all elements of $L_1(G)$ vanishing off G_+. As Arens and Singer showed, $L_1(G_+)$ has $G^\wedge$ as the Šilov boundary of its maximal ideal space; consequently (by a well known property of the Šilov boundary) any automorphism T of $L_1(G_+)$ induces a self-homeomorphism τ of its maximal ideal space which maps $G^\wedge$ onto itself. Moreover the fact that G_+ generates G shows (2.01) and (2.51) hold for $L_1(G_+)$. For the closure G_+^- of G_+ in G^* is a generating subsemigroup of G^*, while any closed subsemigroup of a compact group is a subgroup [5, 11]. Thus $G_+^- = G^*$, and G_+, as well as its interior, is dense in G^*; since point masses concentrated at interior points can clearly be approximated by elements of the unit ball of $L_1(G_+)$ in the weak topology defined by almost periodic functions, we obtain (2.01) and (2.51).

Consequently if $G^\wedge$ is connected we have $\tau(\hat{g}) = \hat{g}_1 \cdot \sigma(\hat{g}), \hat{g} \in G^\wedge$, by 2.61, where σ is an automorphism of $G^\wedge$. Writing elements of $L_1(G)$ as functions rather than measures, we thus have $(Tf)^\wedge(\hat{g}) = \hat{f}(\hat{g}_1\sigma(\hat{g})) = (\hat{g}_1 f)^\wedge(\sigma(\hat{g})) = k[(\hat{g}_1 f) \circ \gamma]^\wedge(\hat{g})$, where γ^{-1} is the automorphism of G dual to σ, and $k > 0$ compensates for the change in Haar measure produced by γ (of course $k = 1$ if G is discrete). Clearly $Tf = k(\hat{g}_1 f) \circ \gamma$ says $\gamma G_+ = G_+$

[13] These are not Arens' full set of generalized conformal mappings, which correspond to the automorphisms of his algebra A_0.

and we have additional information about τ. Thus in the classical case of the Arens-Singer theory (where G is the group Z of integers, G_+ the non-negative integers, and $L_1(G_+)^\frown$ may be viewed[14] as the algebra of analytic functions with absolutely convergent Taylor series on the disc $|z| \leqq 1$ (the maximal ideal space of $L_1(G_+)$)) γ must be the identity, so that τ reduces to a rotation on $|z| = 1$, hence[15] is a rotation of $|z| \leq 1$. In other words *the only self-homeomorphisms τ of the disc which* (via $F \to F \circ \tau$) *map the set of analytic functions with absolutely convergent series on the disc onto itself are rotations.*

Again in the case $G = Z \times Z$, $G_+ = \{(m, n) : m, n \geq 0\}$, where $L_1(G_+)$ can be viewed as the algebra of analytic functions of two complex variables with power series absolutely convergent on $|z| \leq 1, |w| \leqq 1$, there are clearly only two candidates for γ ($(m, n) \to (n, m)$ and the identity), and thus the general automorphism is of the form

$$\sum a_{mn} z^m w^n \to \sum a_{mn} c^m z^m d^n w^n$$

or

$$\sum a_{mn} z^m w^n \to \sum a_{mn} c^m w^m d^n z^n$$

where c and d are fixed unimodular constants; in other words each automorphism is induced by separate rotations of each disc $|z| \leq 1$, $|w| \leq 1$, plus a possible interchange of variables. Clearly this extends to n complex variables.

Generalizing our setting slightly we have

THEOREM 4.1. *Let G and H be locally compact abelian groups with closed subsemigroups G_+ and H_+ satisfying* (4.02), *and let $L_1(G_+)$, $L_1(H_+)$ be defined as above. Then if either group has a connected dual an isomorphism T of $L_1(G_+)$ onto $L_1(H_+)$ is an isometry of the form $Tf = k(\hat{g}f) \circ \gamma$, where k is a positive constant and γ an isomorphism of H onto G with $\gamma H_+ = G_+$. Without connectedness the same applies to order-preserving or norm-decreasing isomorphisms.*

4.2. Clearly most of what we have said applies equally well to any closed algebra satisfying (2.01) for which $G^\frown$ yields the Šilov boundary. And any closed subalgebra A of $L_1(G)$, with $A^\frown$ a translation invariant

[14] More precisely $L_1(G_+)^\frown$ is the set of restrictions to $|z| = 1$ of these functions (since $^\frown$ still is the Fourier transformation and not the full Gelfand representation, cf. [**10**, p. 72]).

[15] For τ is analytic as the function representing the characteristic function of $\{1\}$ under the full Gelfand representation. Alternatively we could note that knowledge of τ on the Šilov boundary determines τ among all automorphism-inducing self-homeomorphisms of $\mathfrak{M}$; since here rotation of the full disc is clearly such a homeomorphism it coincides with τ on the full disc.

set of functions on $G^\wedge$ which separate the elements of $G^\wedge \cup \{0\}$, has $G^\wedge$ the Šilov boundary ∂ of its maximal ideal space. For $G^\wedge$ forms a subspace of the maximal ideal space (cf. footnote 3), while if $\mu \to \mu^0$ is the Gelfand representation of A, $\|\mu^0\|_\infty = \lim \|\mu^{(n)}\|^{1/n} = \|\hat{\mu}\|_\infty = |\hat{\mu}(\hat{g})|$ for some $\hat{g}$ in $G^\wedge$, for each μ in A, and $\partial \subset G^\wedge$. But since $A^\wedge$ is translation invariant we clearly have ∂ a translation invariant subset of $G^\wedge$, and $\partial = G^\wedge$ (this is precisely the argument of [1]).

Consequently we obtain as before

THEOREM 4.3. *Let A be a closed subalgebra of $L_1(G)$ which is closed under multiplication by elements of $G^\wedge$, B a similar subalgebra of $L_1(H)$, and suppose A satisfies* (2.01) *while B merely has $B^\wedge$ a separating set of functions on $H^\wedge \cup \{0\}$. Then if $H^\wedge$ is connected any isomorphism T of A onto B is an isometry of the form $T\mu = \Gamma\hat{g}\mu$ (notation as in* 2.1), *where γ is an isomorphism of G onto H. Without connectedness the same applies to norm-decreasing (or, if A satisfies* 2.51, *order-preserving) isomorphisms.*

Here γ is the isomorphism dual to the isomorphism σ we obtain from 2.61, etc., rather than its inverse, which is the γ of 4.1.

5. When G is discrete a general theorem of Šilov [13] shows that $L_1(G)$ is the direct sum of a pair of ideals if and only if $G^\wedge$ is disconnected. When $G^\wedge$ is connected $L_1(G)$ may still be the vector space direct sum of a closed ideal and a closed subalgebra, and Theorem 2.6 then reveals the exact situation.

THEOREM 5.1. *Let $G^\wedge$ be connected, and $L_1(G) = A \oplus I$ where A is a (non-zero) closed subalgebra and I a (non-zero) closed ideal. Then G is the direct product of a discrete subgroup G_1 and an open subgroup G_2 for which $A = L_1(G_2)$ and $I = \{\mu : \mu \in L_1(G), \hat{\mu}(\hat{g}_1 G_1^\perp) = 0\}$, where $\hat{g}_1 \in G_2^\perp$. Conversely any such decomposition of G and character $\hat{g}_1$ orthogonal to G_2 yields a decomposition of $L_1(G)$ of the type described.*

Proof. Let T be the projection of $L_1(G)$ onto A, a nonzero homomorphism. By 2.6, $T\mu = \Gamma\hat{g}_1\mu$ where Γ is induced by a continuous endomorphism γ of G. Let σ be the endomorphism of $G^\wedge$ dual to γ, so that $\hat{g} \circ \gamma = \sigma(\hat{g})$ and $T\hat{\mu}(\hat{g}) = \hat{\mu}(\hat{g}_1(\hat{g} \circ \gamma)) = \hat{\mu}(\hat{g}_1\sigma(\hat{g}))$. Since $T^2 = T$, $\hat{\mu}(\hat{g}_1\sigma(\hat{g})) = T^2\hat{\mu}(\hat{g}) = T\hat{\mu}(\hat{g}_1\sigma(\hat{g})) = \hat{\mu}(\hat{g}_1\sigma(\hat{g}_1\sigma(\hat{g})))$. Consequently $\sigma(\hat{g}) = \sigma(\hat{g}_1\sigma(\hat{g})) = \sigma(\hat{g}_1)\sigma(\sigma(\hat{g}))$ whence (setting $\hat{g} = \hat{g}_0$) $\hat{g}_0 = \sigma(\hat{g}_1) = \hat{g}_1 \circ \gamma$ and $\sigma \circ \sigma = \sigma$. Dually $\gamma \circ \gamma = \gamma$, and thus the algebraic subgroup $G_2 = \gamma(G)$ of G, on which γ acts as an identity map, is closed (for $\gamma(g_\delta) \to g$ implies $\gamma(\gamma(g_\delta)) = \gamma(g_\delta) \to \gamma(g)$ and $\to g$ whence $g = \gamma(g) \in G_2$). Moreover the fact that $\hat{g}_0 = \hat{g}_1 \circ \gamma$ says $\hat{g}_1 \in G_2^\perp$.

But G_2 is open as well. For $\Gamma\mu$ is a non-zero element of $L_1(G)$ for some μ in $L_1(G)$, while $\Gamma\mu(f) = \mu(f \circ \gamma) = 0$ for $f \in C_0(G)$ vanishing on $G_2 = \gamma(G)$, so that the regular Borel measure $\Gamma\mu$ vanishes on all Borel sets in the complement of G_2; thus G_2 contains some compact subset C of positive Haar measure, and must be open (cf. footnote 12).

Set $G_1 = \{g\gamma(g)^{-1}: g \in G\}$, clearly an algebraic subgroup of G. Then $g = (g\gamma(g)^{-1}) \cdot \gamma(g)$ yields a direct product decomposition of G, $G = G_1 \otimes G_2$: for $g \in G_1 \cap G_2$ implies $g = g'\gamma(g')^{-1} = \gamma(g) = \gamma(g')\gamma(g')^{-1} = g_0$. Since G_2 is open, G_1 is clearly discrete, and evidently γ is the projection of G onto G_2 corresponding to our decomposition.

Let $\mu^{g_1G_2}$ be the restriction of the measure μ in $L_1(G)$ to g_1G_2, so that $\mu = \sum_{g_1 \in G_1} \mu^{g_1G_2}$ and

$$\Gamma\mu(f) = \mu(f \circ \gamma) = \sum_{g_1 \in G_1} \int_{g_1G_2} f(\gamma(g))\mu(dg), \quad f \in C_0(G) .$$

Since

$$\mu_{g_1^{-1}} * \mu^{g_1G_2}(f) = \int f(g_1^{-1}g)\mu^{g_1G_2}(dg) = \int_{g_1G_2} f(g_1^{-1}g)\mu(dg)$$

and $g_1^{-1}g = \gamma(g)$ for $g \in g_1G_2$ we have $\Gamma\mu = \sum_{g_1 \in G_1} \mu_{g_1^{-1}} * \mu^{g_1G_2}$. But this clearly implies Γ, and therefore T, maps $L_1(G)$ into $L_1(G_2)$; indeed it shows Γ and (since $\hat{g}_1 \in G_2^{\perp}$) T leave elements of $L_1(G_2)$ fixed so that $A = TL_1(G) = L_1(G_2)$. On the other hand I, being the kernel of T, consists of just those μ in $L_1(G)$ with $\Gamma\hat{g}_1\mu = 0$, i.e. with $\hat{\mu}(\hat{g}_1\sigma(\hat{g})) = 0, \hat{g} \in G^{\wedge}$. Thus $\mu \in I$ if and only if $\hat{\mu}(\hat{g}_1\sigma(G^{\wedge})) = 0$ or $\hat{\mu}(\hat{g}_1G_1^{\perp}) = 0$ since σ, as the dual to the projection γ of $G_1 \otimes G_2$ onto G_2, is the projection of $G^{\wedge} = G_2^{\perp} \otimes G_1^{\perp}$ onto $G_1^{\perp}$.

Conversely given $G = G_1 \otimes G_2, \hat{g}_1 \in G_2^{\perp}$ one need only set $T\mu = \sum_{g_1 \in G_1}\mu_{g_1^{-1}} * (\hat{g}_1\mu^{g_1G_2})$ to obtain a projection of $L_1(G)$ onto $L_1(G_2)$; writing $\hat{g}_1'\hat{g}_2'$ (with $\hat{g}_1' \in G_2^{\perp}, \hat{g}_2' \in G_1^{\perp}$) as the generic element of $G^{\wedge}$ an easy computation shows $T\mu(\hat{g}_1'\hat{g}_2') = \mu(\hat{g}_1\hat{g}_2')$ so that T is clearly multiplicative and I, as described, is its kernel.

If $G^{\wedge}$ is disconnected our present tools can only be applied to those decompositions for which $\|T\| = 1$ (that other cases occur can be seen from the results of [12] for G the circle group); one can then obtain an analogous result, somewhat complicated by the fact that γ appears as a homomorphism of G into G/G_0, G_0 compact, and indeed the decomposition of L_1 arises from a decomposition of G/G_0, $G/G_0 = G_1/G_0 \otimes G_2/G_0$, and A appears as $S^*L_1(G_2/G_0)$.

6. Some reformulations. When G and H are compact abelian groups Corollary 1.3 has an interesting reformulation; our final section will be devoted to this result and some analogues.

THEOREM 6.1. *Let G and H be compact abelian groups and let T*

be any norm-decreasing linear map of the Banach space $C(H)$ *into* $C(G)$ *for which* $TH^{\wedge} \subset G^{\wedge}$. *Then there is a homomorphism* γ *of* G *into* H *for which* $Tf = (T\hat{h}_0) \cdot f \circ \gamma, f \in C(H)$. *In particular if* $T\hat{h}_0 = \hat{g}_0$ *then* T *is a Banach algebra homomorphism when* $C(G)$ *and* $C(H)$ *are equipped with ordinary multiplication.*

Further the range of T *is dense iff* γ *is one-to-one, and then* $TH^{\wedge} = G^{\wedge}$ *and* T *is onto, while* T *is an isometry iff* $\gamma(G) = H$.

Although we could obtain a proof by noting that T is merely the linear extension of $\tau = T|H^{\wedge}$ we obtain in the proof of Theorem 1.1, an appeal to Corollary 1.3 is more direct. Clearly τ satisfies the hypothesis of 1.3, and thus $\sigma : \hat{h} \to (\tau\hat{h}_0)^{-1}\tau\hat{h}$ is a homomorphism of $H^{\wedge}$ into $G^{\wedge}$. Since $H^{\wedge}$ and $G^{\wedge}$ are discrete, and σ thus continuous, we have a continuous dual homomorphism $\gamma : (g, \sigma(\hat{h})) = (\gamma(g), \hat{h})$. Thus

$$\Big(\sum_{i=1}^{n} a_i\hat{h}_i\Big)(\gamma(g)) = \Big(\sum_{i=1}^{n} a_i\sigma(\hat{h}_i)\Big)(g) = (g, (\tau\hat{h}_0)^{-1})\Big(T\sum_{i=1}^{n} a_i\hat{h}_i\Big)(g),$$

or

$$T\sum_{i=1}^{n} a_i\hat{h}_i = \tau\hat{h}_0\Big[\Big(\sum_{i=1}^{n} a_i\hat{h}_i\Big) \circ \gamma\Big].$$

Since trigonometric polynomials are dense $Tf = \tau\hat{h}_0 \cdot (f \circ \gamma), f \in C(H)$.

For the final statements, we clearly need only consider the case $T\hat{h}_0 = \hat{g}_0$. Note that if γ is not one-to-one then $Tf = f \circ \gamma$ says the range of T consists of functions constant on the cosets of the non-trivial kernel of γ, and thus the range cannot be dense in $C(G)$. On the other hand if γ is one-to-one, then (by compactness) it is an isomorphism of G with a subgroup $\gamma(G)$ of H. Thus for any character χ of $\gamma(G)$ we have a character $\hat{g}$ of G for which $\chi \circ \gamma = \hat{g}$, and since $\chi = \hat{h}|\gamma(G)$ for some $\hat{h}$ in $H^{\wedge}$ we obtain $\hat{h} \circ \gamma = \chi \circ \gamma = \hat{g}$, whence $G^{\wedge} = TH^{\wedge}$. Further if $F \in C(G)$ then any continuous extension f of $F \circ \gamma^{-1} \in C(\gamma(G))$ to all of H (available by Urysohn's lemma) yields $f \circ \gamma = F$, and T is onto. Lastly, if $\gamma(G)$ is proper we have an non-zero $f \in C(H)$ vanishing on $\gamma(G)$ so that $Tf = f \circ \gamma = 0$, and T is not even one-to-one, while if $\gamma(G) = H$ then T is clearly an isometry.

In one case specific mention of characters as such can be eliminated, yielding the weaker result: if T is a linear norm-decreasing one-to-one map of $C(H)$ into $C(G)$ taking the positive definite functions in the ball of $C(H)$, $P_0(H)$, onto $P_0(G)$, then $f \to (T\hat{h}_0)^{-1}Tf$ is multiplicative. For with one-to-oneness the set of extreme points $H^{\wedge} \cup \{0\}$ of $P_0(H)$ maps onto those of $P_0(G), G^{\wedge} \cup \{0\}$.

In this form we have an indication that a similar result can be obtained for the L_1 algebras of locally compact abelian groups.

THEOREM 6.2. *Let* G *and* H *be locally compact abelian groups and* $P_1(G), P_1(H)$ *be the integrable positive definite functions. If* T *is a* *linear isometry of the Banach space* $L_1(G)$ *onto* $L_1(H)$ *with* $TP_1(G) = P_1(H)$ *then* T *is an algebra isomorphism.*

Before proceeding to a proof of 6.2 we should perhaps note an abstract version. Recall that an extreme positive (extendable) functional on a commutative Banach * algebra is a * preserving multiplicative functional. Then

THEOREM 6.3. *Let* A *and* B *be commutative Banach * algebras with (without) identities, and suppose* B *is semisimple and symmetric. Let* T *be a linear isometry of the Banach space* A *into* B *for which the adjoint map* T^* *takes the positive (extendable) functionals on* B *onto those on* A. *Then* T *is a * isomorphism of the algebras* A *into* B.

Proof. Let $P(A)$, $P(B)$ be the set of positive (extendable) functionals of norm 1 on A, B. We know T^*, being an isometry, maps $P(A)$ onto $P(B)$. Since it is one-to-one T^* must map the set $P(A)^e$ of extreme points of $P(A)$ onto $P(B)^e$. But these sets consist of * preserving multiplicative functionals, and since each multiplicative functional on B is * preserving by hypothesis, and thus an extreme positive (extendable) functional, T^* provides us with a map of $\mathfrak{M}_B$, the maximal ideal space of B, into $\mathfrak{M}_A$. Consequently (with $\hat{}$ now the Gelfand representation), $(Taa')\hat{}(M) = (aa')\hat{}(T^*M) = \hat{a}(T^*M)\hat{a}'(T^*M) = (Ta)\hat{}(M) \cdot (Ta')\hat{}(M)$. Since B is semisimple, $Taa' = Ta \cdot Ta'$, and we need only verify $Ta^* = (Ta)^*$. But since M and T^*M are * preserving for M in $\mathfrak{M}_B$, $(Ta^*)\hat{}(M) = \hat{a}^*(T^*M) = \bar{\hat{a}}(T^*M) = \overline{(Ta)\hat{}(M)} = (Ta)^{*\hat{}}(M)$, so $Ta^* = (Ta)^*$ also follows from the semisimplicity of B.

The proof of Theorem 6.2 now follows quite simply, for, as is well known, the positive (extendable) functionals on $L_1(G)$ form the polar cone of $P_1(G)$. Thus the adjoint of T satisfies the requirements of 6.3 when $A = L_1(G)$, $B = L_1(H)$, and T is an algebra isomorphism.

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