EXTENSIONS OF BANACH ALGEBRAS

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1. Introduction. We are concerned with propositions of four types (1.1–1.4) about a commutative Banach algebra A and its various commutative Banach algebra extensions B.

1.1 TPr. *If* $\{B_i : i \in I\}$ *is a family of extensions of* A, *then there is an extension* B *of* A *and topological isomorphism* $\{f_i : i \in I\}$ *where* $f_i(B_i) \subset B$ *and* $f_i(a) = a$ *for* $a \in A$.

Let us call [*normally*] *solvable over* A a system Σ of polynomials over A (or more generally, multiple power series elements) such that there is an extension B of A in which there is a system of elements [whose norms do not exceed 1 and] whose substitution into Σ reduces each member equal to 0.

1.2 Sol. *Let* $\{\Sigma_i : i \in I\}$ *be a family of solvable systems such that no indeterminate occurs in more than one system. Then* $\Sigma = \bigcup \Sigma_i$ *is solvable.*

A system $\mathscr{J}$ of ideals is *removable* if in some extension, each ideal J of $\mathscr{J}$ generates the ideal (1).

1.3 RId. *Let* $\{J_i : i \in I\}$ *be a family of removable ideals. Then it is a removable system.*

An element $c \in A$ is called [*normally*] *subregular* if it has an inverse [of norm $\leqq 1$] in some extension.

1.4 Inv. *Let* $\{c_i : i \in I\}$ *be a family of subregular elements. Then, in some extension, each* c_i *has an inverse.*

Our findings on such propositions is that **TPr** is false, and that **Inv** is true if I is finite, but false if a natural norm restriction is brought in. By the *finite* form of 1.1–1.4 we mean that in which I is finite. By the *normal* form we mean the statements obtained if in (1.1) the f_i are required to be isometries, if 'solvable' in (1.2) is replaced by 'normally solvable', and 'subregular' in (1.4) by 'normally subregular'.

This gives four forms of propostions of each type:

(1.5)	*normal*	(*no qualification*)
	finite normal	*finite* .

For each type (1.1–1.4), there are rather obvious implications in (1.5), namely to the right, and downward. (To see this, one need only observe that c is subregular if and only if λc is normally subregular for some $\lambda \in \boldsymbol{C}$, etc.). For each form (1.5) there are implications among the

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types:

(1.6)
$$\begin{array}{ccc} \mathbf{TPr} & \Rightarrow & \mathbf{Sol} \\ & \swarrow\!\!\swarrow & \Downarrow \\ \mathbf{RId} & & \mathbf{Inv}\,. \end{array}$$

(For example, the diagonal rests on this observation: if J is removable then $1 - j_1x_1 - \cdots - j_nx_n$ is solvable for some $j_1, \cdots, j_n \in J$; and solving the latter removes the former.) We present our *results on these sixteen conceivable propositions* in this diagrammatic way. In each quadrant of (1.5) imagine a cluster of four symbols as in (1.6). Affix a dagger if the proposition is false, a star if true, and a reference to the crucial theorem. Unsettled cases have a question mark.

(1.7)

	†	†	†	?
	?	†	?	?
(1.7)				
	†	†	† (5.2)	?
	?	† (3.2)	?	* [1, 3.8]

Besides this there is a small positive result (7.1) which is a special case of **RId**.

Further results not included in the scheme (1.7) are as follows.

The *cortex* (class of non-removable maximal ideals) is sometimes greater than the Shilov boundary. This is based an a class of algebras of Shilov, whose theory we have felt obliged to sketch (sec. 4)

For completeness we have considered also the case where A has the *sup*-norm (that is, $\|a\| = \sup |\xi(a)|$, ξ ranging over all complex valued homomorphisms of A.) There **Sol** holds (6.1): There is one extension which normally solves all normally solvable systems. Necessary and sufficient conditions for **TPr** are given (5.3)

For some subalgebras A of the l_1-algebra B of a discrete abelian group, B provides inverses of norm 1 for all normally subregular elements (3.5, 3.6).

Section 2 provides more careful definitions of extension, and shows that when **Sol** can be proved, then the solving algebra can always be taken as a quotient-algebra of a power-series algebra.

2. Analytic extension. In order to save space we shall list here properties of a Banach algebra which we shall usually, if not always, require.

(2.11) *It is a Banach space.*

(2.12) *It is a linear algebra over the complex numbers* $\boldsymbol{C}$, *with unit* 1.

(2.13) $\|ab\| < \|a\|\,\|b\|, \quad \|1\| = 1$.

(2.14) *It is commutative.*

Let A be such an algebra. Let I be a set (to be used as indices). We want to define the commutative Banach algebra $A(X)$ *generated by the family* $X = \{x_i : i \in I\}$ and A. Because a norm has to be defined, we need some details. First we define the free commutative semi-group $S(X)$ with unit generated by X. $S(X)$ is the set of all functions from I to $\{0, 1, 2, \cdots\}$ which vanish at all but finitely many places. The operation is addition. We write it multiplicatively, and use the notation

$$x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n} \tag{2.2}$$

for the element which has the value k_j at i_j $(j = 1, \cdots, n)$ and is 0 otherwise. The function which is 0 everywhere is written as 1. A change in the order of the factors in (2.2) does not produce a different element, of course. Now $A(X)$ is the set of functions f from $S(X)$ to A such that

$$\|f\| = S(X)\,\|f(\quad)\| < \infty \ . \tag{2.3}$$

We may let

$$a x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}$$

stand for the element of $A(X)$ which has the value $a \in A$ at the element 2.2 of $S(X)$, and the value 0 elsewhere. We write a for $a1$ $(1 \in S(X))$. Then each f has the form

$$f = \sum_{j=1}^{\infty} a_j \xi_j \tag{2.4}$$

where each ξ_j has the form (2.2), and

$$\|f\| = \sum_{j=1}^{\infty} \|a_j\| \ . \tag{2.5}$$

Clearly the element of $A(X)$ can be added and multiplied, being functions with values in A. The algebra $A(X)$ is easily seen to satisfy the conditions 2.11–2.14. It clearly "contains" the algebra $A[X]$ of *polynomials* in the indeterminates with coefficients in A.

If I_0 is a subset of I, and X_0 is the corresponding system of indeterminates, then $A(X_0)$ can be canonically embedded in $A(X)$. The algebra $A(X)$ is not very interesting in itself. For example, its space of multiplicative linear functionals of the form

$$\Delta(A) \times D^I$$

where $\Delta(A)$ is the corresponding space for A (compare [2, 4.1])

A *Banach algebra extension* of A is an isometric isomorphism of A onto a subalgebra A_1 of a Banach algebra B where the unit of A_1 is that of B. When possible we abbreviate this by saying that B is an extension of A, and pretend that $A \subset B$.

A system $\Sigma = \{\gamma_k : k \in K\}$ of elements of $A(X)$ is called *normally solvable over* A if there is a Banach algebra extension B satisfying 2.11–2.14 and if for each $i \in I$ there is an element $b_i \in B$ with $||b_i|| \leqq 1$ such that if b_i be substituted for x_i in γ_k, then 0 results for each k. (If X contains any x_i not appearing in any γ_k, the corresponding b_i need not be expressly exhibited. It may be chosen as $0 \in B$.)

For an example, see (2.9) below.

A natural attempt to "solve Σ normally" is to form the closed ideal J generated by Σ in $A(X)$, and form

$$A_\Sigma = A(X) \bmod J \tag{2.6}$$

The norm in A_Σ is the canonical one for residue-class algebras [5, p. 14]. The main theorem of this section (2.8) is that this construction is always successful when Σ *is* normally solvable. (Obviously, if the construction is successful, Σ must be normally solvable.)

The only possible obstacle to this approach is that, whereas $A(X)$ is a Banach algebra extension of A, A_Σ might not be, because norms of elements in $A \subset A(X)$ might be diminished when A_Σ is formed (compare [1, pp. 537–8; 2, p. 204.])

2.7 PROPOSITION. *A_Σ is a Banach algebra extension of A and is normally solvable if, for each finite collection of polynomials $p_i, \cdots, p_n \in A(X)$ and indices $j_1, \cdots, j_n$, and each element $a \in A$, the inequality*

$$||a|| \leqq ||a - p_1\gamma_{j_1} - \cdots - p_n\gamma_{j_n}|| \tag{2.71}$$

holds.

The norm on the right is the one mentioned in (2.5). The proof of (2.7) may be omitted. It suffices to deal only with polynomials p_k in (2.7) because there are dense in $A(X)$.

It would be a waste of effort to have X contain any elements not involved in Σ, in essaying to verify (2.71).

The converse (2.7) is valid and we thus arrive at the following.

2.8 THEOREM. *Σ is normally solvable if and only if* (2.71) *holds for all $a, p_1, \cdots, p_n$ as specified in* (2.7).

To see the "only if", suppose B normally solves the system Σ, containing elements $\{b_i\}_{i \in I_0}$ where I_0 are the indices of the elements actually appearing in Σ, such that $\gamma_j(b) = 0$ for all j. Then we can set up a homomorphism

$$h : A(X) \to B$$

wherein $h(x_i) = b_i$, $i \in I_0$, $h(x_i) = 0$, $i \notin I_0$, and $h(a) = a$ for $a \in A$ (regarded as a subalgebra of $A(X)$ as well as of B.) Clearly $||h(f)|| \leqq ||f||$. The ideal J is contained in the kernel of h, because $h(\gamma_j) = 0$ for all $\gamma_j \in \Sigma$. We thus arrive at a homomorphism h^* of bound at most 1 [5, 7D] of A_Σ into B. Therefore the natural image $a + J$ of an element a from A has a norm (in A_Σ) not less than the norm of its image in B. The latter is a itself, so that $||a|| \leqq ||a + J||$. This implies (2.71), so that (2.8) is shown.

The necessary and sufficient condition given by (2.8) can in special cases be replaced by a simpler one.

2.9 THEOREM. *Let c, $d \in A$, and let n be a positive integer.*

Then

$$c = dx^n \qquad\qquad ||x|| \leqq 1$$

can be solved in some extension algebra if and only if, for every $a \in A$,

$$||ca|| \leqq ||da|| \,.$$

The proof, which resembles [1, sec. 3], is simple and may be omitted.

An illustration of the two-way utilization of (2.8) is the following.

2.91 THEOREM. *Let $c \in A$, and let $\mu > 0$. Then*

$$c = e^z \qquad\qquad ||z|| \leqq \mu$$

can be solved in some extension algebra if and only if for each $\nu > \mu$, and positive integer N, there is an extension in which for some n

$$c = \left(1 + \frac{y}{n}\right)^n \qquad\qquad n \geqq N,\ ||y|| < \nu \,,$$

Proof. It is evidently a matter of showing that $c - e^{\mu x}$ is normally solvable precisely when $c - \{1 + (\mu x\delta/n)\}^n$ is normally solvable for infinitely many n, whenever $\delta > 1$. The former can be solved in the latter circumstances because the class of normally solvable *elements* of $A(x)$ is closed, by (2.8). Conversely, if the latter is normally solved with x in some extension algebra B, and $n > \mu\delta$, then (letting $\lambda = \mu/n$) take $y = \lambda^{-1} \log(1 + \lambda x\delta)$, and obtain $e^{\mu y} = c$, with $||y|| \leqq -\lambda^{-1} \log(1 - \lambda\delta)$.

3. The union of normally solvable systems. In (1.2) we included the condition that the solvable families whose union is to be formed involve distinct collections of indeterminates. This is natural, because while each of the one-member families

$$\{1 - x\}, \ \{1 - 2x\} \tag{3.1}$$

is normally solvable over any algebra, the union is never solvable. As indicated in § 1, we do not know if this condition is enough to make even **Sol** (finite) hold, but we shall now show that **Sol** (normal, finite) is not generally true. Our example has the special merit of dealing with systems whose solution consists in constructing inverses, so that it destroys **Inv** (normal, finite) as well, as promised by (1.7).

3.2 THEOREM. *There exists a Banach algebra* A (2.11–2.14) *with elements* p, q *over which*

$$1 - qx \tag{3.21}$$

and

$$1 - py \tag{3.22}$$

are normally solvable, but

$$\{1 - qx, \ 1 - py\} \tag{3.23}$$

is not normally solvable.

Proof. The algebra A is isomorphic as a topological algebra, to the algebra of absolutely convergent power series on the unit disc. In order to reserve letters such as z for possible use as indeterminates, we use p for the "complex variable". Select a real number $\alpha, \alpha > 1$. For $a \in A$, say,

$$a = \lambda_0 + \lambda_1 p + \lambda_2 p^2 + \cdots \tag{3.24}$$

we define

$$||a|| = |\lambda_0| + \alpha(|\lambda_1| + |\lambda_2| + \cdots) \, . \tag{3.25}$$

The operations are the usual addition and multiplication of series. (2.11–2.14) are easily verified. (In fact, this algebra is a simple specimen of a ring of Shilov's type $K_{<\alpha n>}$ (§ 4) where $\{\alpha_n\} = \{1, \alpha, \alpha, \cdots\}$).

It is clear that

$$||pa|| \geqq ||a|| \qquad (a \in A) \, . \tag{3.26}$$

Moreover, for $0 \leqq \delta < 1$ we also have

$$||qa|| \geqq ||a|| \tag{3.27}$$

where

$$q = (1 - \delta)^{-1}(1 - \delta p)p \, . \tag{3.28}$$

To see (3.27), consider that

$$||(1 - \delta p)pa|| \geqq ||pa|| - \delta||p^2a|| .$$

Now

$$||p^2a|| = ||pa||, \text{ so } ||(1 - \delta p)pa|| \geqq ||pa||\,(1 - \delta) \geqq (1 - \delta)\,||a|| ,$$

by (3.26).

By [1, 3.5] each of the one-element systems

$$\{\gamma_1\} = \{1 - qx\} , \tag{3.29}$$

$$\{\gamma_2\} = \{1 - py\} , \tag{3.30}$$

is normally solvable (and, the combined system is solvable [1, 3.8]).

We submit that the following is an identity in x, y:

$$\delta(1 - \delta)(1 - \delta p)^{-1} = x - (1 - \delta)y + \gamma_1[\delta(1 - \delta)(1 - \delta p)^{-1} + (1 - \delta)y] + \gamma_2[- x(1 - \delta p)] . \tag{3.31}$$

This is readily verified by substituting (3.29), (3.30), and (3.28) into (3.31).

Let us now suppose $\{\gamma_1, \gamma_2\}$ is normally solvable. Let $a = \delta(1 - \delta)(1 - \delta p)^{-1}$. Then, from (3.31)

$$a - \gamma_1[\cdots] - \gamma_2[\cdots] = x - (1 - \delta)y .$$

Comparing this with (2.71), we see that

$$||a|| \leqq ||\, x - (1 - \delta)y|| = 1 + (1 - \delta) ,$$

where we have used (2.5) for the norm in $A(x, y)$. Now $a = \delta(1 - \delta)(1 + \delta p + \delta^2 p^2 + \cdots)$ and the norm of this is given by (3.25):

$$||a|| = \delta(1 - \delta)(1 + \delta\alpha + \delta^2\alpha + \cdots) = \delta - \delta^2 + \delta^2\alpha .$$

It thus appears that

$$\alpha \leqq 1 + 2(1 - \delta)\delta^{-2} . \tag{3.32}$$

Thus the desired counterexample is possible. In fact, if $\alpha > 1$ then some δ will make (3.32) false.

The next proposition shows how plentiful these counterexamples really are.

3.4 THEOREM. *Let A be any algebra satisfying* (2.11–2.14), *containing an element c which is not regular but is no topological zero-divisor. Then A contains p, q and can be given an equivalent norm such that* $\{1 - qx\}$, $\{1 - py\}$ *are each normally solvable, but* $\{1 - qx, 1 - py\}$ *is not.*

Proof. Select a complex-valued homomorphism ξ of A such that $\xi(c) = 0$. We may assume that $||ac|| \geqq ||a||$ for all $a \in A$. Select any δ such that $0 < \delta < 1$. *I* now present the p and q: $p = 3c$, $p = 3\beta c$ $(1 - \delta c)$ where $\beta = ||(1 - \delta c)^{-1}||$; and the new norm

$$|a| = |\xi(a)| + \alpha||a - \xi(a)|| \qquad (\alpha > 1) . \tag{3.41}$$

Here α is a parameter to be fixed later. It is not hard to see that (3.41) satisfies (2.11–2.14). Furthermore,

$$||a|| < |a| < 3\alpha\,||a|| . \tag{3.42}$$

Since $\xi(c) = 0$ we have $|3ca| = \alpha||3ca|| \geqq 3\alpha||a|| > |a|$. It follows that $1 - 3cy$ is normally solvable. It is similarly established that $1 - 3\beta c$ $(1 - \delta c)x$ is normally solvable.

Now suppose some extension B of A (A with the $|\cdot\cdot|$ norm, be it understood) had elements x, y of norm not exceeding 1 such that

$$3\beta c(1 - \delta c)x = 1 , \qquad 3cy = 1 .$$

Then

$$\beta(1 - \delta c)x = y , \qquad \beta x - y = \beta\delta c x .$$

Now $3\beta cx = (1 - \delta c)^{-1}$ so we have

$$\delta(1 - \delta c)^{-1} = 3\beta x - 3y$$

Whence

$$\delta|(1 - c)^{-1}| \leqq 3(\beta + 1) . \tag{3.43}$$

But

$$|(1 - \delta c)^{-1}| = 1 + \alpha||(1 - c)^{-1} - 1||$$

where the coefficient of α is not 0 because $\delta \neq 0$. Hence α can be chosen so that (3.43) is impossible.

The Banach algebra A used in (3.2) cannot yield a counterexample if the parameter α is taken as 1. This follows from the following.

3.5 THEOREM. *Let B satisfy* (2.11–214) *and let A be a subalgebra with unit. Let $\Delta(B)$ be the space of complex-valued homomorphisms of B, and $\Delta(A)$, the corresponding set for A. Suppose that every $\xi \in \Delta(B)$ when restricted to A falls into the Shilov boundary* [4, 5] $\partial_A\Delta(A)$. *Suppose moreover that there is a collection U of elements in B such that*

$$u \in U \text{ and } a \in A \text{ implies } ||ua|| = ||a|| \tag{3.51}$$

and

(3.52) $\{ua : u \in U,\ a \in A\}$ *is dense in* B .

Then each element c *of* A *which has an inverse of norm not exceeding* 1 *in some extension of* A, *has such an inverse in* B.

Proof. If $c \in A$ has an inverse in some superalgebra, then it cannot vanish on $\partial_A \Delta(A)$ as each of these homeomorphisms can be extended to any superalgebra of A. Thus c has an inverse b in B, and what remains to be proved is that $||b|| \leqq 1$. This will result from the fact that necessarily, $||ca|| \geqq ||a||$ for all $a \in A$ [1, 3.5].

By (3.52), there exist $\{u_n\}$, $\{a_n\}$ such that $u_n a_n \to b$. Therefore $||cu_n a_n|| \to ||cb|| = 1$. However (by 3.51)

$$||cu_n a_n|| = ||ca_n|| \geqq ||a_n|| = ||u_n a_n|| \to ||b|| .$$

This completes the proof of (3.5).

From this general proposition we now consider another which shows that for the A of (3.2) with $\alpha = 1$ there is a B to which A bears the relation described in (3.5). In fact, $B = L^1(\boldsymbol{Z})$ where $\boldsymbol{Z}$ is the discrete group of integers, and A can be identified with those elements of B which are supported by the semi-group $\boldsymbol{Z}_+$ of non-negative integers. This pair is discussed in [5. 23C and 24E].

3.6 Theorem. *Let* G *be a discrete abelian group and* S *a subsemigroup containing* $e \in G$. *Let* $B = L^1(G)$, *and* A *be the subalgebra of* B *consisting of those functions whose support lies in* S. *Let* U *be the group* G *as naturally imbedded in* $B : x \to \delta_x$, $\delta_x(y) = \delta(y - x)$, *where* $\delta(x) = 0$ *or* 1 *according to whether* $x \neq e$, *or* $x = e$. *Then* U, A, B *satisfy the conditions of* (3.5).

The specific properties of U are obvious, and the relation of $\Delta(B)$ and $\partial_A \Delta(A)$ is easily established, either by analogy with [5, 24E], or by [7, 4.6].

4. **The cortex.** Let A satisfy (2.11–2.14) as always. By $\Delta(A)$ we mean the space of complex linear homomorphisms of the algebra A onto the complex numbers $\boldsymbol{C}$, with the weak topology. By the *cortex* $\Gamma(A)$ of $\Delta(A)$ (or, more briefly, the *cortex* of A) we mean the set of those homomorphisms which can be extended to every extension B of A.

Now those $\xi \in \Delta(A)$ which can be extended to B form a compact set E_B which is the continuous image (under the restriction map) of $\Delta(B)$, and the cortex is evidently the intersection of these E_B. Moreover, each E_B contains the Shilov boundary $\partial_A \Delta(A)$ [4, 5] which is never void. Thus we have the following.

4.1 Theorem. *The cortex* $\Gamma(A)$ *is compact, and contains the Shilov* *boundary.*

When A has the *sup* norm, i.e, when $||a|| = \sup \{|\xi(a)| : \xi \in \Delta(A)\}$, then $\Gamma(A) = \partial_A \Delta(A)$ since the extension $B = \mathscr{C}(\partial_A(\Delta(A))$ admits only homomorphisms which are on the Shilov boundary. There are algebras in which the norm is not equivalent to the *sup-norm* in which $\Gamma(A)$ and $\partial_A(\Delta(A))$ coincide, for example the A of (3.6) above.

However, the work of Shilov [6] makes it possible to exhibit algebras with one generator in which $\Gamma(A) \neq \partial_A(\Delta(A))$. Because of the rarity of this paper in these parts it may be permissible to sketch proofs of some of Shilov's theorems.

Let $\{\alpha_n\} = \{\alpha_0, \alpha_1, \cdots\}$ be a sequence of real numbers where, for $m, n > 0$,

$$\alpha_0 = 1 \leqq \alpha_{m+n} \leqq \alpha_m \cdot \alpha_n \; . \tag{4.21}$$

Let $K(\boldsymbol{\alpha})$ be the space of these formal power series (which notation makes the algebraic operations more evident)

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \tag{4.22}$$

for which

$$||f|| = \Sigma \, |a_i| \alpha_i \; \infty \; . \tag{4.23}$$

$K(\boldsymbol{\alpha})$ satisfies (2.11–2.14). It follows from (4.21) that $r = \lim \; (\alpha_n)^{1/n}$ exists. Thus the spectrum of z (see 4.22) is the disc $\{|\lambda| \leqq r\}$, and this is a homeomorphic image of $\Delta(K(\boldsymbol{\alpha}))$ under the map $\xi \to \xi(z)$.

We now consider the possibility of enlarging the algebra by defining α_n also for $n > 0$, and norming formal Laurent series as in (4.23). It is easy to see that if such α_n $(n < 0)$ can be defined with (4.21) holding, then for each $r = 1, 2, \cdots$,

$$\beta_r = \sup \; \alpha_n(\alpha_{n+r})^{-1} : n = 1, 2, \cdots \tag{4.24}$$

must be finite. Moreover

4.25 PROPOSITION (Shilov). *Letting* $\alpha_{-n} = \beta_n$ for $n = 1, 2, \cdots$ *provides an extension of the* $K(\boldsymbol{\alpha})$ *when* β_1 *is finite.*

The proof lies in verifying (4.21), and then observing that $\beta_r < (\beta_1)^r$. Let us call this extension algebra $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The spectral radius of $1/z$ is evidently

$$s = \lim \; (\beta_n)^{1/n} \tag{4.26}$$

and $\Delta(K(\boldsymbol{\alpha}, \boldsymbol{\beta}))$ is homeomorphic to the spectrum of z, which is

$$\{\lambda : s^{-1} \leqq |\lambda| \leqq r\} \; . \tag{4.27}$$

When $\beta_1 = \infty$ we set $s^{-1} = 0$.

4.3 THEOREM (Shilov). *The element* $\lambda - z$ *is a generalized zero-divisor in* $K(\boldsymbol{\alpha})$ *if and only if* $s^{-1} < |\lambda| < r$.

Now $\lambda - z$ is *not* a generalized (or topological) zero-divisor if

$$\inf \{||(\lambda - z)f(z)|| : ||f(z)|| = 1\} \tag{4.31}$$

is positive. The evaluation of (4.31) is facilitated by

4.32 LEMMA. *Let* T *be a convex compact subset of a topological (real) linear space* L, *and let* $\varphi_0, \cdots, \varphi$ *be* $N_N + 1$ *real valued linear functionals on* L. Let $S(x) = |\varphi_0(x)| + \cdots + |\varphi_N(x)|$, $\mu = \min \{S(x) : x \in T\}$. *Then there exist* $i_1, i_2, \cdots, i_n$ *and a point* $x_1 \in T$ *such that*

$$S(x_1) = \mu \tag{4.33}$$

and

(4.34) x_1 *is an extreme point of* T *relative to* $Z(i_1, \cdots, i_n)$, *where the latter is the linear subspace defined by* $\varphi_{i_1} = \varphi_{i_2} = \cdots = \varphi_{i_n} = 0$.

Proof. Selection an x_0 such that $S(x_0) = \mu$ and such that the number of φ_i that vanish at x_0 is a maximum. Let

$$\varphi(x) = \textstyle\sum'(\operatorname{sgn} \varphi_i(x_0))\, \varphi_i(x) \tag{4.35}$$

where $'$ is to remind the reader that sgn $0 = 0$. Let $\{i_1, \cdots, i_n\}$ be those indices for which $\varphi_i(x_0) = 0$, and define Z as in (4.34). By the maximum-property of n, each sgn φ_i is constant on $T \cap Z$. Therefore $\varphi(x) = S(x)$ on $T \cap Z$. Now φ is linear and so there is an extreme point x_1 of the convex set $T \cap Z$ such that $\varphi(x_1) = \mu$.

Having established the Lemma we apply it as follows to the space L_N of polynomials of degree $< N$. Let T_N be the collection of those members of L_N whose $K(\boldsymbol{\alpha})$ norm is $\leqq 1$. For $f \in L_N$ let $\varphi_i(f)$ be α_i times the ith coefficient of $(\lambda - y)f(z)$. It is clear that the inf in (4.31) has the value

$$\lim_{N\to\infty} \inf_{f\in T_N} S_N(f)\,.$$

The set $Z(i_1, \cdots, i_n)$ of those $f \in L_N$ such that $\varphi_{i_1}(f) = \cdots = \varphi_{i_n}(f) = 0$ is clearly the set of those f such that $\lambda a_i = a_{i-1}$ for $i = i_1, i_2, \cdots, i_n$ (here a_{-1} and a_N are interpreted as 0.) Let the indices be arranged so that $i_1 < i_2 < \cdots < i_n$. This sequence decomposes into maximal blocks without gaps. If $\{m + 1, \cdots, m + p\} = \sigma$ is such a block then

$$f_\sigma = z^m(\lambda^p + \lambda^{p-1}z + \cdots + \lambda z^{p-1} z^p) \tag{4.36}$$

lies in $Z(i_1, \cdots, i_n)$. If $\sigma = \{0, \cdots\}$ let $f_\sigma = 0$. If $0 \neq i_1$ then $1 \in Z(i_1, \cdots, i_n)$. If i, $i + 1$ do not occur $(0 < i,\ i + 1 \leqq \mathrm{N})$ then

$z^i \in Z(i_1, \cdots, i_n)$. *Conversely*, every function in $Z(i_1, \cdots, i_n)$ is uniquely expressible as a linear combination of these functions just associated with $\{i_1, \cdots, i_n\}$. Considering how the norm is taken, the extreme points of $Z(i_1, \cdots, i_n) \cap T_N$ are obviously just functions of the type

$$cz^m(\lambda^p - z^p)(\lambda - z)^{-1} \tag{4.37}$$

where c is any number that makes the norm 1. (Here we allow $p = 1$ and also $m = 0$ to take care of those monomials mentioned above which are not due to gap-less blocks.)

4.4 LEMMA. *The inf* (4.31) *can be evaluated by letting f run through the system* (4.37).

Shilov does not seem to examine the inf (4.31) to the extent we do here, but the functions (4.37) occur in his considerations.

We now pass to a proof of (4.3). First of all, if $|\lambda| < s^{-1}$ then z and $z - \lambda$ have an inverse in $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$, whence $z - \lambda$ cannot be a topological zero-divisor. If $|\lambda| > r$ then $z - \lambda$ has an inverse in $K(\boldsymbol{\alpha})$.

Now suppose $z - \lambda$ is not a topological zero-divisor. We wish to show that $|\lambda| < s^{-1}$ if $|\lambda| \leqq r$. We confine ourselves to $\lambda \geqq 0$, and assume $\lambda \leqq r$. λ cannot be r or s^{-1} for these yield topological zero-divisors by Shilov's earlier theorem [6]. If $\lambda < r$ then for N, some we have $\lambda^p < \alpha_p$ for $p > N$. From (4.31) we obtain an $M < \infty$ such that $||(x - z)f(z)||\, M > ||f(z)||$ for all f. Inspired by (4.4) we select $f(z) = \lambda^p + \lambda^{p-1}z + \cdots + z^p$ and obtain

$$\lambda^p + \lambda^{p-1}\alpha_1 + \cdots + \alpha_p < M(\lambda^{p+1} + \alpha_{p+1}) < 2M\alpha_{p+1} .$$

For $q > N$ and $p = k + q - 1$ we obtain

$$\lambda^k \alpha_q < 2M\lambda\alpha_{k+q} ,$$

whence (by 4.24) $\lambda^k \beta_k < 1$, and (by 4.26) $\lambda s < 1$.

Using the notation of (4.3), we deduce the following [see 6. Th. 7].

4.4 COROLLARY. *For $K(\boldsymbol{\alpha})$ the Shilov boundary is $\{\lambda : |\lambda| = r\}$, and the cortex is $\{\lambda : s^{-1} \leqq |\lambda| \leqq r\}$. For $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ the Shilov boundary is $\{\lambda : s^{-1} = |\lambda|$ or $|\lambda| = r\}$ while the cortex is the same as for $K(\boldsymbol{\alpha})$. Thus if $s^{-1} < r$, then in each case the cortex is greater than the Shilov boundary.*

Shilov remarks that if $t < \rho$ then examples can be constructed such that $s^{-1} = t$, $r = \rho$. He gives no example, so we may just give one producing the interesting case $s^{-1} = 0$, $r = 1$. We have, of course, to define $\{\alpha_0, \alpha_1, \cdots\}$. Let $\alpha_m = \exp(\lambda(m))$ where $\lambda(m)$ is defined as follows. Set

$$\begin{aligned}\lambda_n(m) &= n^{-2}m \qquad \text{when} \quad m \leqq n^3 \\ &= 0 \qquad\quad\ \ \text{when} \quad m \geqq n^3 .\end{aligned}$$

Let $\lambda(m) = \lambda_1(m) + \lambda_2(m) + \cdots$. It is not hard to see that $\lambda(m) - \lambda(m+1) > n - \pi^2/6$. It follows that $\beta_1 = +\infty$ (see 4.24) and because $\beta_r \alpha_{r-1} \geqq \beta_1$, that $\beta_r = \infty$ whence $s^{-1} = 0$ (see 4.26).

We close this section by comparing the extension $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ which Shilov has provided for $K(\boldsymbol{\alpha})$ when $\beta_1 < \infty$ with that provided by [1, 3.1]

The least norm of z^{-1} in any extension of $K(\boldsymbol{\alpha})$ is easily seen to be β_1, for this least norm is by [1. 3.1] the reciprocal of the *inf* (4.31). Thus it is reasonable to compare Shilov's extension $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ with that provided by § 2, for normally solving $1 - \beta_1 zx$.

4.5 THEOREM. *Let $\beta_1 < \infty$. In $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ the norm of z^{-n} is β_n. In the "canonical" extension of $K(\boldsymbol{\alpha})$ normally solving $1 - \beta_1 zx$, the norm* of *z^{-n} is $(\beta_1)^n$.*

Proof. The statement about $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is obvious.

Denote β_1 by β. In the extension (2.6), the norm of x^n is inf $S[x^n - (1 - xy)g(x, y)]$ extended over all polynomials g, S meaning sum of all norms of coefficients of powers of x, where$\|y^n\| = \|\beta^n z^n\| = \beta^n \alpha_n$. Let $p + q$ be a maximum for the term $\gamma x^p y^q$ in some particular polynomial $g(x, y)$; and suppose $p + q > n - 1$. Then $x^n - (1 - xy)g(x, y)$ has $\gamma x^{p+1} y^{q+1}$ as its highest degree term. If we modify $g(x, y)$ by omitting the term $\gamma x^p y^q$, then the S-contribution from $x^p y^q$-terms in $x^n - (1 - xy)g(x, y)$ might become $|\gamma|\ \beta^p \alpha_q$ larger, but the contribution $|\gamma|\ \beta^{p+1} \alpha_{q+1}$ will disappear. Hence this modification changes the $S[\cdots]$ by at most a negative increment. Therefore, we may confine ourselves to g's whose terms have $p + q < n - 1$. Of these, $g = 0$ gives the minimum possible value for $S[\cdots]$, and it is β^n.

This theorem (4.5) shows that $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ gives the smallest possible inverse not only to z, but to all its powers (whereas the "canonical" one may not do justice, so to speak, to the inverses of $z^2, z^3, \cdots$). This being so, one wonders if $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ might not provide the best (i.e., least-in-norm) inverses to $z - \lambda$ for $|\lambda| < s^{-1}$. Our result (4.4), whose full force has not really been employed above, shows that this is not generally true.

4.6 THEOREM. *Consider $K(\boldsymbol{\alpha})$ with $\{\alpha_0, \alpha_1, \alpha_2, \cdots\} = \{1, 2, 1, 1, \cdots\}$. Then $s^{-1} = r = 1$. For $|\lambda_1| = \lambda < 1$ the norm of $(z - \lambda_1)^{-1}$ in $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is* $2(1 - \lambda)^{-1}$. *For each λ_1 such that $|\lambda_1| = \lambda < 1$ there is an extension of $K(\boldsymbol{\alpha})$ in which the norm of $(z - \lambda_1)^{-1}$ is*

(4.61) $\quad (2 + \lambda)(1 + \lambda^2)^{-1}$ *if* $\lambda \leqq 1/2$, *and* $(1 - \lambda)^{-1}$ *if* $\lambda \geqq 1/2$.

Proof. The details are tedious and should be accepted or verified by the reader. The system (4.24) comes out $\{\beta_1, \beta_2, \cdots\} = \{2, 2, 2, \cdots\}$ whence $\|(z - \lambda_1)^{-1}\| = 2(1 - \lambda)^{-1}$ in $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Taking however a fixed λ (one might as well suppose $\lambda = \lambda_1 \geqq 0$), the best that can be done by the canonical (and thus by any) method is a norm for $(z - \lambda)^{-1}$ equal to the reciprocal of the *inf* (4.31). A page of calculation, based on (4.4), yields the result stated. Curiously, the formula (4.61) gives a function which is not monotonely increasing, but has a minimum at $\lambda = 1/2$.

5. The tensor-product problem TPr. The proposition **TPr** (*norm, finite*) is false in general because it would conflict with (3.2). However, there is a simpler argument, which also destroys **TPr** (*top, finite*). It rests upon the following.

5.1 THEOREM. *Let B satisfy* (2.11–2.14), *and let $\{B_i : i \in I\}$ be a family of closed subalgebras; and let A be a closed subalgebra with the unit of B included in each of the B_i. Let Δ_i be the space of complex-valued homomorphisms of B_i, and let Γ_i be the cortex, $i \in I$. Let T_i^* be the restriction map*

$$T_i^* : \Delta_i \to \Delta \ ; \ T_i^*(\zeta) = \zeta|_{\Delta} \ . \tag{5.11}$$

Then for each $j \in I$ we must have

$$T_j^*(\Gamma_j) \subset \bigcap T_i^*(\Delta_i) \ . \tag{5.12}$$

Proof. Let ζ belong to the right hand side of (5.12). Then ζ extends to B, and thus ζ is for each i a restriction of some $\zeta_i \in \Delta_i$, to A.

For a pair of real numbers $r, \rho\,(0 \leqq \rho \leqq r)$ let $A(r, \rho)$ be the algebra of functions continuous for $\rho \leqq |\lambda| \leqq r$ and holomorphic for $\rho < |\lambda| < r$, with the *sup-norm*. The cortex is the set $\{|\lambda| = \rho\} \cup \{|\lambda| = r\}$ (deleting the former when $\rho = 0$). Let $A = A(0, 1)$, $B_1 = A(1/3, 1)$, $B_2 = A(2/3, 1)$. Then $A \subset B_1 \subset B_2$ and in some sense B_2 is the desired tensor product of B_1 and B_2 (over the ring A) but not in the sense **TPr** for the injection of $B_1 \to B_2$ is not *bi*-continuous. In fact, because $T^*(1/3) \notin T^*(\Delta_2)$, we have:

5.2 THEOREM. **TPr** *does not hold for* $A\{B_1\, B_2\}$.

Returing briefly to (5.12), we show that it is *sufficient* for **TPr** when all norms are *sup*-norms.

5.3 THEOREM. *If $\{B_i : i \in I\}$ is a family of Banach algebra extensions of A (all satisfying* 2.11–2.14) *and each B_i has the sup-morm then an algebra B as in* **TPr** (1.1) *exists such that the mappings f_i are isometries, provided condition* (5.12) *holds for each j.*

We sketch a demonstration with close reference to [3, Appendice I].

One forms $B_0 = \bigotimes_{(I)} B_i$. Let ζ be any element of $\times_{i \in I} \Delta_i$ such that $\zeta_i|_A$ is independent of i. Then $\bigotimes_{i \in I} \zeta_i = \zeta^*$ is a C–homomorphism of B_0. We define the norm of an element $b \in B_0$ as $||b|| = \sup |\zeta^*(b)|$, and complete B_0 in that norm. Now let $b_j \in B_j$ have norm 1. Then $|\zeta_j(b_j)| = 1$ 1 for some $\zeta_j \in \Gamma_j$. By (5.12), this ζ_j is part of a collection $\{\zeta_i\}$ of the type used in forming the homomorphisms ζ^*, and surely $|\zeta^*(f_j(b_j))| = 1$. Thus $||f_j(b_j)|| \geqq ||b_j||$. On the other hand, if $|\zeta^*(f_j(b_j)| > 1$ then $|\zeta_i(b_j)| > 1$ for some $\zeta_i \in \Delta_i$ which cannot be if $||b_i|| = 1$.

Thus (5.3) is proved.

6. The Sol problem for sup-normed algebras. The non-equivalence of **Sol** and **TPr** is brought out by the fact that **Sol** (norm, arb) is true when A has the *sup*-norm. For then, the algebra $M(\partial_A \Delta(A))$ of bounded functions on the Shilov boundary solves normally all normally solvable systems.

6.1 THEOREM. *Let B, satisfying* (2.11–2.14), *normally solve a system Σ over a subalgebra A having the* sup-*norm. Then $M(\partial_A \Delta(A))$ also normally solves Σ.*

Proof. Well-order the class $\Delta(B)$ of C-homomorphisms of B. For each $\zeta \in \partial_A(\Delta(A))$, let ζ' be the first element of $\Delta(B)$ which is an extension of ζ. Define $T(b)(\zeta) = \zeta'(b)$. This mapping is isometric on A, and of bound 1 on B. These two properties insure that the homomorphism T preserves "normal solution". Thus (6.1) is proved.

It is worth noting that $M(\partial_A \Delta(A))$ not only solves all systems solvable over A but also all systems solvable over itself.

7. A fragmentary result on joint removal of ideals. Let A satisfy (2.11–2.14), and let Δ be its space of C-homomorphisms. Let J be an ideal of A. The hull H of J is $\{\zeta : \zeta \in \Delta, \zeta = 0 \text{ on } J\}$. This is compact.

7.1 THEOREM. *Let $\{J_i : i \in I\}$ be a family of removable ideals (see* § 1) *and let J be a removable ideals. Let each J_i, and J be a principal ideal. Let the hulls H_i converge to the hull H of J in this sense: every neighborhood W of H contains all but finitely many of the H_i. Then the system $\{J\} \cup \{J_i : i \in I\}$ is removable.*

Proof. If $J = cA$ then c *is* subregular (see § 1); and we can select c so that $||ca|| \geqq ||a||$ for all $a \in A$. Elements c_i can be selected so that each $J_i = c_i A$ with $||c_i a|| \geqq ||a||$ for all a $\in A$. Let $W = \{\zeta : |\zeta(c)| < 1/2\}$. Let $H_{i_1}, \cdots, H_{i_n}$ include all those hulls which are not included in W. Let $d = c_{i_1} \cdots c_{i_n}$. One can find an integer p such that $|\zeta(dc^p)| < 1$ for all $H \cup \bigcup H_i$. In some extension algebra B, dc^p has an inverse b, $||b|| \leqq 1$. If the ideals J, J_i $(i \in I)$ are not all removed by B then B has a $\boldsymbol{C}$-homomorphism ζ_0 which is an extension of some ζ in $H \cup \bigcup H_i$. Now $\zeta_0(bdc^p) = 1$, $|\zeta_0(b)| \leqq 1$, but $|(dc^p)| < 1$. This is a contradiction.

A question. Suppose $c_1, \cdots, c_n$ are elements of A which generate a removable ideal. Then there are numbers $\mu_1, \cdots, \mu_n$ such that $||a|| \leqq ||c_1a||\,\mu_1 + \cdots + ||c_na||\mu_n$. (Indeed, if $1 = c_1x_1 + \cdots + c_nx_n$ in some superalgebra, then one can take $\mu_k = ||x_k||$.) Is the converse true? If so, we would have that every finite collection of removable ideals is a removable family of ideals, that is, **RId** (finite). The method would be, using the given systems

$$c_{1,i}, \cdots, c_{n,i} \qquad (i \in I)\,,$$

to construct a new system

$$c_k = \prod_{i \in I} c_{k_i, i}$$

$(k_1, \cdots, k_n$ a permutation of $1, \cdots, n)$ and apply that converse.

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