ON PARACOMPACTNESS

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It is well known that the product of a paracompact space with any compact space is paracompact and hence normal[1]. In this paper, we will establish the converse of this proposition by showing that *if* $X \times \beta X$ *is normal, then* X *is paracompact* (Theorem 2)[2].

The existence of a compactification is a characteristic property of a Tychonoff space, and the Stone-Čech compactification (the largest one) may reasonably be expected to play an important role in the theory of Tychonoff space. Indeed, some properties of a Tychonoff space X can be characterized by the properties of the Stone-Čech compactification βX (more precisely, by the properties of X as a dense subspace of βX)[3], and we shall give a new characterization of paracompactness in § 2 (Corollary of Theorem 1). In Theorem 1, we shall characterize paracompactness by the property of $\beta X \times \beta X$ in connection with the uniformity for X. This will yield an easy proof of the main theorem (Theorem 2).

1. Regularly open sets[4]. In the first place, we shall establish a lemma concerning regularly open sets, which will be used in the sequel. Let A be a subset of a topological space X. We shall denote by $\mathrm{Cl}_X(A)$ the closure of A and by $\mathrm{Int}_X(A)$ the interior of A.

A subset A of a topological space X is said to be regularly open if and only if $\mathrm{Int}_X(\mathrm{Cl}_X(A)) = A$. It is easy to verify that the intersection of two regularly open sets is again regularly open, but the union of them is, in general, not regularly open. The following lemma states that if X is a dense subspace of a topological space Y, then the family of all regularly open sets in X is in one to one correspondence with the family of all regularly open sets in Y.

LEMMA. *Let X be a dense subspace of a topological space Y.*

(a) *If A is regularly open in Y then the restriction $A \cap X$ of A on X is regularly open in X. Conversely, any regularly open set B in X is identical with the restriction of some regularly open set in Y.*

(b) *Let A be a regularly open set in Y and let A' be any open set in Y such that $A \cap X \supset A' \cap X$, then $A \supset A'$. Therefore two regularly open sets A, A' in Y are identical if and only if $A \cap X = A' \cap X$.*

* Received July 23, 1959.

[1] See, [**4**], Th. 5 and Th. 1.

[2] For [the] related results, the reader should refer to [**1**], [**3**], [**5**] and [**8**].

[3] C.f. [**6**], p. 96 and p. 97, [**7**], Th. 28, [**11**], p. 84 and [**9**].

[4] C.f. [**12**].

Proof of (a). An easy calculation shows that if X is a dense subspace of Y, then $\mathrm{Int}_X(\mathrm{Cl}_X(A)) = \mathrm{Int}_Y(\mathrm{Cl}_Y(A)) \cap X$. Therefore we have $A \cap X \subset \mathrm{Int}_X(\mathrm{Cl}_X(A \cap X)) = \mathrm{Int}_Y(\mathrm{Cl}_Y(A \cap X)) \cap X \subset \mathrm{Int}_Y(\mathrm{Cl}_Y(A)) \cap X = A \cap X$, and it follows that $\mathrm{Int}_X(\mathrm{Cl}_X(A \cap X)) = A \cap X$. Hence $A \cap X$ is regularly open. If B is regularly open in X, then $B = \mathrm{Int}_X(\mathrm{Cl}_X(B)) = \mathrm{Int}_Y(\mathrm{Cl}_Y(B)) \cap X$ and $\mathrm{Int}_Y(\mathrm{Cl}_Y(B))$ is obviously regularly open in Y.

Proof of (b). If $A' \not\subset A$, then we have $A' \not\subset \mathrm{Cl}_Y(A)$ because $A = \mathrm{Int}_Y(\mathrm{Cl}_Y(A))$. Therefore $A' \cap [\mathrm{Cl}_Y(A)]^c$ is a non-void open set in Y, where $[\mathrm{Cl}_Y(A)]^c$ denotes the complement of $\mathrm{Cl}_Y(A)$. Since X is dense in Y, there exists a point of X contained in $A' \cap [\mathrm{Cl}_Y(A)]^c$, and it follows that $A' \cap X \not\subset A \cap X$. The first part of (b) is therefore true. The last part of (b) follows immediately from the first.

2. Paracompactness. Throughout the sequel, we shall restrict ourselves to consideration of Tychonoff spaces (completely regular T_1-spaces). A compactification of X is a compact Hausdorff space containing X as a dense subspace. The Stom-Čech compactification βX is characterized among the compactifications of X by the fact that every bounded continuous function on X has a continuous extension over βX[5].

THEOREM 1. *X is paracompact if and only if for each compact set F in $\beta X - X$ there is a surrounding*[6] *V for X such that*

$$\tilde{V} \cap \Delta_F = \phi ,$$

where $\tilde{V}$ denotes the interior of the closure of V taken in $\beta X \times \beta X$; $\tilde{V} = \mathrm{Int}_{\beta X \times \beta X}(\mathrm{Cl}_{\beta X \times \beta X}(V))$, and $\Delta_F = \{(p, p) \in \beta X \times \beta X;\ p \in F\}$.

Proof. (*Necessity*) Assume that X is a paracompact space, and let F be a compacts set contained in $\beta X - X$. Then, there is for each point $x \in X$ an open neighborhood (in βX) U_x^* of x such that $\mathrm{Cl}_{\beta X}(U_x^*) \cap F = \phi$. Put $U_x = U_x^* \cap X$ and consider an open covering $\{U_x\}_{x \in X}$ of X. Take a locally finite open refinement $\{U_\lambda\}$ of $\{U_x\}_{x \in X}$, and let $\sum \varphi_\lambda = 1$ be a locally finite partition of unity subordinate to $\{U_\lambda\}$. Put $d(x, y) = \sum |\varphi_\lambda(x) - \varphi_\lambda(y)|$ and put $V_n = \{(x, y) \in X \times X;\ d(x, y) < 1/2^n\}$. We shall show that $\tilde{V}_1 \cap \Delta_F = \phi$, which will completes the proof.

Suppose, on the contrary, that there is a point $p \in F$ such that $(p, p) \in \tilde{V}_1$, then $U^*(p) \times U^*(p) \subset \tilde{V}_1$ for some open neighborhood (in βX) $U^*(p)$ of p, because $\tilde{V}_1$ is open in $\beta X \times \beta X$. Let x be a point of $U(p) = U^*(p) \cap X$ (there is surely such a point, since X is dense in βX), then there exists only a finite number of φ_λ's, say $\varphi_1, \cdots, \varphi_n$, which do not vanish at x. Put $H_k = \{y \in X;\ \varphi_k(y) > 0\}$, for $1 \leqq k \leqq n$. Clearly

[5] C.f. [**2**] p. 833.

[6] We call V a surrounding for X if V is a member of a uniformity for X compatible with the topology of X. (= "entourage")

$y \notin \bigcup_{k=1}^{n} H_k$ implies $d(x, y) > 1$, and it follows that $U^*(p) \cap X \subset \bigcup_{k=1}^{n} H_k$[7]. Hence p is contained in $\mathrm{Cl}_{\beta X}(\bigcup_{k=1}^{n} H_k)$. On the other hand, H_k is contained in some U_x because $\{U_\lambda\}$ is a refinement of $\{U_x\}_{x \in X}$, and $\mathrm{Cl}_{\beta X}(U_x) \cap F = \phi$. Therefore, no point of F is contained in $\mathrm{Cl}_{\beta X}(\bigcup_{k=1}^{n} H_k) = \bigcup_{k=1}^{n}(\mathrm{Cl}_{\beta X}(H_k))$. We have thus a contradiction. It follows that $\tilde{V}_1 \cap \Delta_F = \phi$.

(*Sufficiency*). Let $\{0_\nu\}$ be any open covering of X. For each 0_ν, we take (and fix) one open set 0_ν^*(in βX) such that $0_\nu^* \cap X = 0_\nu$. Put $F_\nu = [0_\nu^*]^c$, where $[0_\nu^*]^c$ denotes the complement of 0_ν^* in βX, and put $F = \bigcap_\nu F_\nu$, then F is a compact set contained in $\beta X - X$. By the hypothesis of our theorem, we can construct a sequence of surrounding $\{V_n\}$ such that $\tilde{V}_1 \cap \Delta_F = \phi$. Now, let us consider the uniform space $(X, \mathscr{U})$, where $\mathscr{U} = \{V_n\}$, and let τ be the uniform topology of $\mathscr{U}$. It is clear that topological the space (X, τ) is pseudo-metrizable hence is paracompact.[8] Let $d(x, y)$ be a pseudo-metric for X such that $\{(x, y) \in X \times X; d(x, y) < 1\} \subset V_1$, and put $W_n = \{(x, y) \in X \times X; d(x, y) < 1/2^n\}$. Since $V_1 \supset W_1$ implies $\tilde{V}_1 \supset \tilde{W}_1$ and since $\tilde{V}_1 \cap \Delta_F = \phi$, we have $\tilde{W}_1 \cap \Delta_F = \phi$. Consider an open covering $\{W_3(x)\}_{x \in X}$ of (X, τ) and let $\{U_\lambda\}$ be a locally finite open refinement of $\{W_3(x)\}_{x \in X}$. Since the original topology of X is stronger than τ, $\{U_\lambda\}$ is necessarily a locally finite open covering of X with respect to the original topology of X.

We shall show presently that $\mathrm{Cl}_{\beta X}(U_\lambda) \cap F = \phi$ for each U_λ. Notice first that the restriction $d_x(y)$ of $d(x, y)$ on $x \times X$ is a bounded continuous function on X with respect to the original topology and hence it has a continous extension d_x^* over βX. Suppose that $\mathrm{Cl}_{\beta X}(U_\lambda) \cap F \neq \phi$ for some U_λ, and let p be a point of $\mathrm{Cl}_{\beta X}(U_\lambda) \cap F$. Since $U_\lambda \subset W_3(X_0)$ for some $x_0 \in X$, p is an accumulation point of $\{y \in X; d(x_0, y) < 1/2^3\}$ for some $x_0 \in X$. Therefore $d_{x_0}^*(p) \leqq 1/2^3 < 1/2^2$, and there is a neighborhood (in βX) $0^*(p)$ of p such that $d_{x_0}(y) < 1/2^2$ for each $y \in 0(p) = 0^*(p) \cap X$. It follows that $0(p) \times 0(p) \subset W_1$, and consequently we have $(p, p) \in 0^*(p) \times 0^*(p) \subset \mathrm{Int}_{\beta X \times \beta X}(\mathrm{Cl}_{\beta X \times \beta X}(0(p) \times 0(p))$ $\subset \tilde{W}_1$. But this contradicts the above fact that $\tilde{W}_1 \cap \Delta_F = \phi$, Hence $\mathrm{Cl}_{\beta X}(U_\lambda) \cap F = \phi$.

Thus, we have a locally finite open covering $\{U_\lambda\}$ of X such that $\mathrm{Cl}_{\beta X}(U_\lambda) \cap F = \phi$ for each λ. Returning to the original covering $\{0_\nu\}$ of X, we find that $\{0_\nu^*\}$ covers $\mathrm{Cl}_{\beta X}(U_\lambda)$ for each U_λ, and, since $\mathrm{Cl}_{\beta X}(U_\lambda)$ is compact, there is a finite number of 0_ν^*'s, say $0_1^*, \cdots, 0_m^*$ such that $\mathrm{Cl}_{\beta X}(U_\lambda) \subset \bigcup_{k=1}^{m} 0_k^*$. It follows that $U_\lambda \subset \bigcup_{k=1}^{m} 0_k$, and we have $U_\lambda = \bigcup_{k=1}^{m} H_{\lambda,k}$, where $H_{\lambda,k} = U_\lambda \cap 0_k$. Thus, each U_λ can be represented as a finite union of open sets of the form $H_{\lambda k}$. Constructing $H_{\lambda k}$ for each U_λ in this way, we have a locally finite open refinement $\{H_{\lambda,k})$ of

[7] By virtue of our lemma, it follows that $U(p) \times U(p) \subset \mathrm{Int}_{X \times X}(\mathrm{Cl}_{X \times X}(V_1))$, and therefore $d(x, y) \leqq 1/2 < 1$ for each $y \in U(p)$.

[8] See [**10**], p. 160.

$\{0_\nu\}$. It follows that X is paracompact.

From the proof of the preceding theorem, we obtain the following characterization of paracompactness.

COROLLARY. *X is paracompact if and onl if for each compact set F in $\beta X - X$ there is a locally finite open covering $\{U_\lambda\}$ of X such that* $\mathrm{Cl}_{\beta X}(U_\lambda) \cap F = \phi$. *(Each U_λ is a subset of X.)*

The following theorem gives also a characterizotion of paracompactness.

THEOREM 2. *X is paracompact if and only if $X \times \beta X$ is normal.*

Proof. The necessity of the condition is clear [9]. To prove the sufficiency, we have only to show that for each compact set F in $\beta X - X$ there is a surrounding V for X such that $\tilde{V} \cap \Delta_F = \phi$, by virtue of Theorem 1.

Let F be a compact set contained in $\beta X - X$, then $X \times F$ and Δ_X are disjoint closed sets in $X \times \beta X$, and since $X \times \beta X$ is normal there are two open sets U_1, W_1 in $X \times \beta X$ such that $U_1 \supset \Delta_X$, $W_1 \supset X \times F$ and $U_1 \cap W_1 = \phi$. Put $U_0 = \mathrm{Int}_{X \times \beta X}(\mathrm{Cl}_{X \times \beta X}(U_1))$, then U_0 is a regularly open set in $X \times \beta X$ such that $U_0 \supset \Delta_X$ and $U_0 \cap (X \times F) = \phi$.

We now put $U = U_0 \cap (X \times X)$, and we will show that $\tilde{U} \cap \Delta_F = \phi$, where $\tilde{U}$ is the interior of the closure of U taken in $\beta X \times \beta X$ and $\Delta_F = \{(p, p) \in \beta X \times \beta X; p \in F\}$. Suppose, on the contrary, that $\tilde{U} \cap \Delta_F \neq \phi$, and let (p, p) be a point of $\tilde{U} \cap \Delta_F$. Then, there is a neighborhood (in βX) $0^*(p)$ of p such that $0^*(p) \times 0^*(p) \subset \tilde{U}$. Let x be a point of $0^*(p) \cap X$ (such a point exists, because X is dense in βX), then $x \times 0^*(p) \subset \tilde{U}$ and we have $(x, p) \in (0^*(p) \times 0^*(p)) \cap (X \times \beta X) \subset \tilde{U} \cap (X \times \beta X)$. On the other hand, it is true that $U_0 = \tilde{U} \cap (X \times \beta X)$. In fact, $U = \bar{U} \cap (X \times X)$ by (a) of our lemma, and we have $U_0 \cap (X \times X) = U = \tilde{U} \cap (X \times X) = [\tilde{U} \cap (X \times \beta X)] \cap (X \times X)$. That is, the restriction of U_0 on $(X \times X)$ is identical with that of $\tilde{U} \cap (X \times \beta X)$. Therefore we have $U_0 = \tilde{U} \cap (X \times \beta X)$ by (b) of the lemma, since both of U_0 and $\tilde{U} \cap (X \times \beta X)$ are regularly open in $X \times \beta X$. It follows that $(x, p) \in \tilde{U} \cap (X \times \beta X) = U_0$, but this contradicts the fact that $U_0 \cap (X \times F) = \phi$. Therefore $\tilde{U} \cap \Delta_F = \phi$.

We now consider the function $F(x, y) \in C(X \times \beta X)$ such that $F = 0$ on Δ_X and $F = 1$ outside of U_0 (such a function exists because $X \times \beta X$ is normal). Let F_x be the restriction of F on $x \times X$, and define a function $G(x, x')$ by letting

$$G(x, x') = ||F_x - F_{x'}|| = \sup_{z \in \beta X} |F(x, z) - F(x', z)| .$$

[9] See, [**4**], Th. 5 and Th. 1.

It is easy to verify that G is a continuous function on $X \times X$[10] and that

$$G(x, x') \geqq 0 \text{ for each } (x, x') \in X \times X, \quad G(x, x) = 0,$$
$$G(x, x') = G(x', x),$$
and
$$G(x_1, x_2) + G(x_2, x_3) \geqq G(x_1, x_3) .$$

Moreover, we have

$$F(x, x') = F(x, x') - F(x', x') \leqq \sup_{z \in \beta X} |F(x, z) - F(x', z)| = G(x, x') ,$$

and therefore $G(x, x') < 1$ implies that $F(x, x') < 1$. Put

$$V = \{(x, x') \in X \times X ;\ G(x, x') < 1\} ,$$

then V is evidently a surrounding for X and we have $V \subset U$. It follows that $\tilde{V} \subset \tilde{U}$, and, since $\tilde{U} \cap \Delta_F = \phi$, we have $\tilde{V} \cap \Delta_F = \phi$. The proof is completed.

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[10] Since $F(x, y)$ is a continuous function on $X \times \beta X$ and βX is compact, there is for each $\varepsilon > 0$ a neighborhood $U(x)$ of $x \in X$ such that $\|F_x - F_{x'}\| < \varepsilon$ for each $x' \in U(x)$. The continuity of G follows from this fact by an easy calculation.