

GROUP MEMBERSHIP IN RINGS AND SEMIGROUPS

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1. Introduction. Let R be a semigroup or associative ring. A group G in R is a subset of R which is a group under the multiplication on R . That is, G contains an idempotent e which acts as a multiplicative identity on G and if $\alpha \in G$ then there exists an element $\alpha' \in G$ such that $\alpha\alpha' = \alpha'\alpha = e$. An element α of R is said to be a group element in R if α belongs to some group in R .

The problem of deciding whether a given element of R is a group element has been investigated in various types of rings in [3], [4], [5], [6], [10], [11]. The purpose of the present paper is the generalization and extension of some results of Barnes and Schneider [3], Drazin [4] and Farahat and Mirsky [6].

Section 2 of this paper extends some results of [6] on the imbedding of the groups contained in a ring with identity in the group of units of the ring.

In § 3 use is made of the concept of left π -regularity. McCoy [9] introduced the concept of π -regularity, the consequences of which have developed in [1], [2], and [8]. It imposes a finiteness condition satisfied, for example, by rings with minimum condition, by nil rings, by the "divided" rings of [6] and by direct sums of such rings. This condition is found to be sufficient in many of the cases where [6] uses the condition that the ring be a direct sum of divided rings. Moreover, the condition of left π -regularity is applicable to the case of semigroups. Under this condition, it is shown that if S is an extension of a semigroup or ring R , $\alpha \in R$ and α is a group element in S , then α is a group element of R .

Section 4 deals with conditions under which some power of a given element of R is a group element.

Section 5 gives a necessary and sufficient condition for the same property in terms of annihilators.

In order to point up the comparative weakness of the condition of left π -regularity of a ring necessary and sufficient conditions are given in § 6 that a left π -regular ring be a direct sum of divided rings.

2. Groups in rings with identity. Throughout this section R will denote a ring with an identity element 1 and U will denote the group of units of R .

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LEMMA 2.1. (Farahat and Mirsky [6]) *Let G be a group with idempotent e in R . Then the mapping $\eta: G \rightarrow U$ defined by $\eta(g) = g + (1 - e)$ is an isomorphism of G onto a subgroup G_1 of U . The idempotent e commutes with every element of G_1 and $G = eG_1$. Moreover, if $x \in G_1$ then $x = ex + (1 - e)$.*

Proof. Let $g \in G$ and let g' be its inverse in G . Then

$$\{g + (1 - e)\}\{g' + (1 - e)\} = gg' + (1 - e)^2 = e + (1 - e) = 1,$$

whence $g + (1 - e) \in U$. The verification that η is an isomorphism is routine.

If $x \in G_1$ then $x = g + (1 - e)$ for some $g \in G$. Hence $ex = xe = g$. It follows that $x = ex + (1 - e)$ and that $G = eG_1$. Thus the lemma is proved.

Now let $C(e)$ denote the set of all elements of U which commute with e . Then, clearly, $eC(e)$ is a group with idempotent e in R . It follows from Lemma 2.1 that every group with idempotent e in R is contained $eC(e)$, whence $eC(e)$ is the unique maximal group with idempotent e in R . We set $M(e) = eC(e)$. If we now apply the isomorphism η of Lemma 2.1 to $M(e)$ we obtain a subgroup $M_1(e)$ of U . It also follows from Lemma 2.1 that $M_1(e) \subseteq C(e)$. We shall show that $M_1(e)$ is not only a subgroup of $C(e)$, but is, in fact, a direct factor. This will follow from the more general Theorem 2.2.

If e_1, e_2, \dots, e_n are idempotents of R , let $C(e_1, e_2, \dots, e_n)$ denote the set of all elements of U which commute with each of e_1, e_2, \dots, e_n .

THEOREM 2.2. *If e_1, e_2, \dots, e_n are mutually orthogonal idempotents in R and $e_1 + e_2 + \dots + e_n = 1$ then*

$$\begin{aligned} C(e_1, e_2, \dots, e_n) &= M_1(e_1) \otimes M_1(e_2) \otimes \dots \otimes M_1(e_n) \\ &\cong M(e_1) \otimes M(e_2) \otimes \dots \otimes M(e_n). \end{aligned}$$

Proof. Let $C = C(e_1, e_2, \dots, e_n)$.

(1) We shall first show that if $i \neq j$ then $M_1(e_i)$ and $M_1(e_j)$ commute elementwise. Let $x \in M_1(e_i)$ and $y \in M_1(e_j)$. Then $x = e_i x + (1 - e_i)$ and $y = e_j y + (1 - e_j)$. Therefore

$$xy = e_i x (1 - e_j) + (1 - e_i) e_j y = e_i x + e_j y$$

which, by symmetry, is also equal to yx .

(2) Next, $C = M_1(e_1) \times M_1(e_2) \times \dots \times M_1(e_n)$. For suppose $x \in C$. Then $e_i x + (1 - e_i) \in M_1(e_i)$, $i = 1, \dots, n$. Now

$$\begin{aligned} & \{e_1x + (1 - e_1)\} \cdots \{e_nx + (1 - e_n)\} \\ & = e_1x + \cdots + e_nx = (e_1 + \cdots + e_n)x = x. \end{aligned}$$

(3) We now prove that $M_1(e_i) \cap \prod_{j \neq i} M_1(e_j) = 1$. For let x belong to this intersection. Then

$$x = e_ix + (1 - e_i) = \prod_{j \neq i} \{e_jx_j + (1 - e_j)\},$$

where $x_j \in U$. Hence, from the commutativity of the $e_jx_j + (1 - e_j)$ and the fact that $e_i\{e_jx_j + (1 - e_j)\} = e_i$, it follows that

$$e_ix = xe_i = e_i \times e_i \times \cdots \times e_i = e_i,$$

and so

$$x = e_ix + (1 - e_i) = e_i + (1 - e_i) = 1.$$

From (1), (2) and (3) it follows that C is the direct product of the $M_1(e_i)$.

COROLLARY 2.3. $C(e) = M_1(e) \otimes M_1(1 - e) \cong M(e) \otimes M(1 - e)$.

Proof. It is merely necessary to notice that $C(e) = C(e, 1 - e)$.

3. Group elements in extensions of π -regular semigroups and rings.

In this section R will generally denote a semigroup; results in which R must be assumed to be a ring will be so indicated.

Let R be a semigroup and $\alpha \in R$. We say that α is left π -regular ([8], [2]) if there exists an element x in R and a positive integer n such that $x\alpha^{n+1} = \alpha^n$. The semigroup R is said to be left π -regular if every element of R is left π -regular. Similar definitions are made for right π -regularity. Evidently, if α is both left and right π -regular then there exist x and y in R and a positive integer n for which $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$.

Left π -regularity is a finiteness condition in the following sense: The element α is left π -regular if and only if the descending sequence of left ideals $R\alpha \supseteq R\alpha^2 \supseteq R\alpha^3 \supseteq \cdots$ terminates in a finite number of steps. More precisely, $x\alpha^{n+1} = \alpha^n$ implies that $R\alpha^{n+1} = R\alpha^n$, and conversely, $R\alpha^{n+1} = R\alpha^n$ implies that $x\alpha^{n+2} = \alpha^{n+1}$ for some $x \in R$, and, if R has an identity, implies $x\alpha^{n+1} = \alpha^n$.

Left π -regularity does not imply right π -regularity and, of course, conversely. In the case of semigroups this is shown by the following example. Let σ and τ be two infinite cardinals with $\tau \leq \sigma$, and let E be a set of cardinal σ . Let B be the semigroup of all one-to-one mappings of E into itself for which the complement of αE in E is of cardinal τ . It is easy to see that for each $\alpha \in B$ there is an $x \in B$ such that $x\alpha$ is the identity map on αE and so $x\alpha^2 = \alpha$, whence B is left π -regular. But for all integers n and all $y \in B$, $\alpha^{n+1}yE$ is properly

contained in $\alpha^n E$ and, hence, no element of B is right π -regular. (In the case that $\sigma = \tau = \aleph_0$, the semigroup B is called the semigroup of Baer and Levi.)

THEOREM 3.1. *Suppose R is a semigroup and S is an extension of R . If $\alpha \in R$ is left π -regular and α is a group element in S , then α is also a group element in R .*

Proof. Suppose α is a group element in S . Then there exists α' , $e \in S$ such that

$$\alpha e = e\alpha = \alpha, \quad \alpha'e = e\alpha' = \alpha', \quad \alpha\alpha' = \alpha'\alpha = e.$$

Since $\alpha \in R$ is left π -regular there exists an element $x \in R$ and a positive integer n for which $\alpha^n = x\alpha^{n+1}$. Hence $e = \alpha^n\alpha'^n = x\alpha^{n+1}\alpha'^n = x\alpha e = x\alpha \in R$. Moreover, $\alpha' = e\alpha' = x\alpha\alpha' = xe \in R$. Consequently, α is a group element in R and the theorem is proved.

We note that it follows from this theorem that if S is any extension of the semigroup B of our example and $\alpha \in B$, then α is not a group element in S . For if α were a group element in S it would also be a group element in B and hence right π -regular in B .

An element α of a semigroup R is called *cancellable* (often called regular) if α is both right and left cancellable, viz: $\alpha x = \alpha y$ implies $x = y$ and $x\alpha = y\alpha$ implies $x = y$. In a ring an element is cancellable if and only if it is not a proper divisor of zero.

COROLLARY 3.2. *Let R be a semigroup and let T be an extension of R . Suppose*

- (i) *Every element of R is cancellable in T ,*
- (ii) *For each $\alpha \in R$, $x \in T$ there exist $\alpha' \in R$, $x' \in T$ such that $\alpha x' = x\alpha'$,*
- (iii) *Every element of R is left π -regular in R .*

Then T contains an identity and R is a group.

Note that if R is a ring and R^* is the set of non-zero elements of R , then if R^* satisfies (i), (ii) and (iii), the conclusion of the corollary tells us that R is a division ring.

Proof. By a slight modification of an argument of Jacobson [7], p. 118, we may form a semigroup of fractions x/α , $x \in T$, $\alpha \in R$. If we denote this semigroup of fractions by S , we may imbed T , and consequently R , in S by the mapping $x \rightarrow x\alpha/\alpha$. The element α/α is an identity for S and, of course, also for T . Every element of R is invertible in S with respect to 1, namely its inverse is α/α^2 . In view of (iii) it follows that $1 \in R$, and thus to T , and that each element of R has an inverse in R , whence R is a group.

We note that if R is assumed to be in the center of T then condition (ii) is automatically satisfied with $\alpha = \alpha'$ and $x = x'$.

COROLLARY 3.3. *Let R be a ring, finitely generated as a module over its center C . Suppose*

- (i) *Every non-zero element of R is cancellable, that is, R has no proper zero divisors,*
 - (ii) *Every element of C is left π -regular in C ,*
- then R is a division ring.*

Proof. By Corollary 3.2, C is a field. Thus R is a finite dimensional algebra over C . But, by (i), this algebra has no zero divisors and a finite dimension algebra without zero divisors is a division ring.

4. Powering elements into group elements. Under certain conditions an element α of a semigroup R may not be a group element itself although some power of it may be. An example of this is the case when R contains a zero element and α is a nilpotent element of R .

THEOREM 4.1. *Let R be a semigroup and S an extension of R such that for each $x \in S$ there is a positive integer $m = m(x)$ for which $x^m \in R$. Suppose $\alpha \in R$ and α is both left and right π -regular in S .¹ Then α^n is a group element in R for some n and conversely.*

Proof. Since α is both left and right π -regular, find $x, y \in S$ and a positive integer n such that $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$. If $p \geq n$ then it follows that $x\alpha^{p+1} = \alpha^p = \alpha^{p+1}y$. Find $m(x)$ and $m(y)$ such that $x^{m(x)}, y^{m(y)} \in R$ and set $p = m(x)m(y)n$. Then $x^p, y^p \in R$ and $x\alpha^{p+1} = \alpha^p = \alpha^{p+1}y$. Note that

$$x\alpha^p = x^2\alpha^{p+1} = \alpha^{p+1}y^2 = \alpha^py$$

and so we may set $\beta = x^{2p}\alpha^p = \alpha^py^{2p}$ and $e = x^p\alpha^p = \alpha^py^p$. Then $\beta, e \in R$ and

$$\beta\alpha^p = x^{2p}\alpha^{2p} = x^{2p-1}x\alpha^{p+1}\alpha^{p-1} = x^{2p-1}\alpha^{2p-1} = \dots = x^p\alpha^p = e.$$

Similarly $\alpha^p\beta = e$. Also

$$e\alpha^p = x^p\alpha^{2p} = x^{p-1}x\alpha^{p+1}\alpha^{p-1} = x^{p-1}\alpha^{2p-1} = \dots = \alpha^p.$$

By another similar argument $\alpha^pe = \alpha^p$. Further,

$$e\beta = x^p\alpha^{2p}y^{2p} = x^p\alpha^py^p = x^{2p}\alpha^p = \beta$$

¹ Drazin [4] calls an element which is both left and right π -regular a pseudo-invertible element.

and $\beta e = \beta$. Thus, in order to show that α^p is a group element, it is now sufficient to show that e is idempotent. But $e^2 = x^a \alpha^{2p} y^p = \alpha^p y^p = e$.

Conversely, if α^n is a group element in R and β is the group inverse of α^n , then $\beta \alpha^{2n} = \alpha^n = \alpha^{2n} \beta$. Hence, α is both left and right π -regular in R , and therefore in S .

COROLLARY 4.2. *If R is a semigroup, $\alpha \in R$ and α is both left and right π -regular, then α^n is a group element in R for some n and conversely. More precisely, α^n is a group element if and only if*

$$x \alpha^{n+1} = \alpha^n = \alpha^{n+1} y$$

for some $x, y \in R$.

Proof. Take $S = R$ in the preceding theorem.

This result appears in a somewhat different guise in a paper by M. P. Drazin, [4].

COROLLARY 4.3. *Let R be a semigroup and α^n a group element with identity e in R . If $e\alpha = \alpha$ then α is a group element in R , and conversely.*

Proof. Let α^n be a group element with identity e and inverse β . Then α^{n+1} is also a group element with the same identity and so $\alpha = e\alpha = \beta \alpha^n \cdot \alpha = \beta \alpha^{n+1}$ a product of group elements with idempotent e and thus a group element itself.

Conversely, if α is a group element with identity f and α^n is a group element with identity e , then $e = f$ since α^n is also a group element with identity f .

5. Annihilator conditions that a given element be a group element.

In a ring R we define the left and right annihilators of an element α in the usual manner:

$$A_l(0, \alpha) = \{z \in R: z\alpha = 0\} \quad \text{and} \quad A_r(0, \alpha) = \{z \in R: \alpha z = 0\}.$$

So that we may state our next results for semigroups as well as rings we shall generalize the concept of an annihilator. In a semigroup R we shall set

$$\begin{aligned} A_l(x, \alpha) &= \{z \in R: z\alpha = x\alpha\}, \\ A_r(x, \alpha) &= \{z \in R: \alpha z = \alpha x\}. \end{aligned}$$

Several consequences of these definitions are easily proved (though we shall make no use of these properties). The sets $A(x, \alpha)$ are equivalence

classes modulo the equivalence relation $z \equiv_i x$ defined by $z\alpha = x\alpha$, and similarly for the sets $A_r(x, \alpha)$. An element $\alpha \in R$ is cancellable if and only if $A_l(x, \alpha) = A_r(x, \alpha) = \{x\}$ for all $x \in R$. If R is a ring, then $A_l(x, \alpha) = x + A_l(0, \alpha)$ and $A_r(x, \alpha) = x + A_r(0, \alpha)$. Finally, (and this we shall need) it is evident that

$$A_l(x, \alpha) \subseteq A_l(x, \alpha^2) \subseteq A_l(x, \alpha^3) \subseteq \dots$$

and that $A_l(x, \alpha^n) = A_l(x, \alpha^{n+1})$ implies that $A_l(x, \alpha^n) = A_l(x, \alpha^m)$ for all $m \geq n$, and similarly for the sets $A_r(x, \alpha)$, $A_r(x, \alpha^2)$, etc.

In analogy with the phrase left π -regular we introduce the following terminology: An element α of a semigroup R is called *left A-regular* in R if there exists a positive integer n for which $x\alpha^{n+1} = z\alpha^{n+1}$ implies that $x\alpha^n = z\alpha^n$ for all x and z in R . Thus α in R is left A-regular if and only if the ascending chains $A_l(x, \alpha) \subseteq A_l(x, \alpha^2) \subseteq \dots$ terminate in finitely many steps for all $x \in R$.

It is easy to see that a left π -regular element of a semigroup R is right A-regular, and a right π -regular element is left A-regular. In this connection, a slight generalization of a theorem Azumaya [2] proved for rings is of interest.

THEOREM 5.1. *Let R be a semigroup and α a left π -regular element of R . If α is left A-regular then α is right π -regular, and conversely.*

Proof. Suppose α is left A-regular in R . Then we may choose a positive integer n such that $z\alpha^{n+1} = z'\alpha^{n+1}$ implies $z\alpha^n = z'\alpha^n$ for all $z, z' \in R$ and $x\alpha^{n+1} = \alpha^n$ for some $x \in R$. We wish to prove that $\alpha^{n+1}y = \alpha^n$ for some $y \in R$. It is clearly sufficient to prove that $\alpha^m x^m \alpha^n = \alpha^n$ for $m = 0, 1, 2, \dots$. Since the case $m = 0$ is trivial, we proceed by induction. Thus, assume that $\alpha^m x^m \alpha^n = \alpha^n$. Then

$$\alpha^{m+1} x^{m+1} \alpha^{n+1} = \alpha^{m+1} x^m \alpha^n = \alpha^{n+1} = x\alpha \cdot \alpha^{n+1},$$

whence $\alpha^{m+1} x^{m+1} \alpha^n = x\alpha \cdot \alpha^n = \alpha^n$.

That the converse is true has already been remarked before the statement of the theorem.

The second theorem of this section relates the integers n which occur in the definitions of π -regularity and A-regularity. Let n be a positive integer and let z and α be elements of the semigroup R . We shall say that condition $A(z, \alpha, n)$ holds if both $A_l(z, \alpha^{n+1}) = A_l(z, \alpha^n)$ and $A_r(z, \alpha^{n+1}) = A_r(z, \alpha^n)$. Thus an element α is both right and left A-regular if there exists a positive integer n such that $A(z, \alpha, n)$ holds for all z , and conversely. If R is a ring and $A(z, \alpha, n)$ is satisfied for some $z \in R$ then $A(z, \alpha, n)$ is satisfied for all $z \in R$.

THEOREM 5.2. *Let R be a semigroup and let α be an element of R*

which is both left and right π -regular in R . The following results hold:

- (i) If α^n is a group element in R then $A(z, \alpha, n)$ holds for all $z \in R$.
- (ii) If $A(\alpha, \alpha, n)$ holds then α^{n+1} is a group element in R .
- (iii) If R has identity 1 and $A(1, \alpha, n)$ holds then α^n is a group element.

We remark that under the hypotheses of the theorem, α^n is a group element for some n and α is both left and right A -regular.

Proof. (i) If α^n is a group element, then it follows from Corollary 4.2 that $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$ for some $x, y \in R$. Suppose that u is any element of $A_l(z, \alpha^{n+1})$ where $z \in R$. Then $z\alpha^n = z\alpha^{n+1}y = u\alpha^{n+1}y = u\alpha^n$, whence $u \in A_l(z, \alpha^n)$ which implies $A_l(z, \alpha^{n+1}) = A_l(z, \alpha^n)$. The proof of $A_r(z, \alpha^{n+1}) = A_r(z, \alpha^n)$ is similar.

(ii) Suppose that $A(\alpha, \alpha, n)$ holds. Since α is left π -regular there exists a positive integer m and an $x \in R$ such that $x\alpha^{m+1} = \alpha^m$. We shall show that $x\alpha^{n+2} = \alpha^{n+1}$. If $m < n + 1$, we obtain this equality by multiplying the previous equality by α^{n-m+1} . If $m = n + 1$ there is nothing to prove. If $m > n + 1$ then $x\alpha^2\alpha^{m-1} = \alpha\alpha^{m-1}$ and $A(\alpha, \alpha, n)$ implies that

$$x\alpha^{n+2} = x\alpha^2\alpha^n = \alpha\alpha^n = \alpha^{n+1}.$$

The existence of an element $y \in R$ satisfying $\alpha^{n+2}y = \alpha^{n+1}$ is proved similarly. It now follows from Corollary 4.2 that α^{n+1} is a group element.

(iii) The proof is similar to the proof of (ii). This time it follows from $x\alpha^{m+1} = 1\alpha^m$ when $m = n$ that $x\alpha^{n+1} = x\alpha\alpha^n = 1\alpha^n = \alpha^n$ by virtue of $A(1, \alpha, n)$. Hence α^n is a group element.

6. A criterion that a ring be semi-divided. A ring is said to be *divided* if it has an identity and every element is invertible or nilpotent. A ring is *semi-divided* if it is the direct sum of (possibly infinitely many) divided rings. The terminology is that of [6]. In this section we shall give necessary and sufficient conditions that a left π -regular ring be semi-divided.

LEMMA 6.1. *Let R be a semigroup both left and right π -regular. Then every non-nil (left) ideal of R contains a non-zero idempotent.*

Proof. Let I be a non-nil left ideal of R and α a non-nilpotent

element of I . α^n is a group element with respect to the non-zero idempotent e for some n . Let β be the inverse of α^n . Then $e = \beta\alpha^n \in I$.

If e and f are idempotents then we say that e *dominates* f if $ef = fe = f$. An idempotent $e \neq 0$ is *primitive* if it dominates only 0 and itself. For rings this is equivalent to saying that e is primitive if it is not the sum of two non-zero orthogonal idempotents.

THEOREM 6.2. *Let R be a ring satisfying the following conditions:*

- (i) R is left and right π -regular;
- (ii) Every primitive idempotent of R is in the center of R ;
- (iii) Every non-zero idempotent of R dominates a primitive idempotent;
- (iv) If $x \in R$ then $x e = 0$ for all but finitely many primitive idempotents e .

Then R is the direct sum of a semi-divided ring and a nil ring, and conversely. If, in addition, R satisfies the condition:

- (v) Every element of R has a left or right identity,
- then R is semi-divided, and conversely.*

Proof. Let e be a primitive idempotent. Then $Re = eR$, since e is in the center of R , and e is the identity of Re . Since e is primitive e is unique non-zero idempotent of Re . If $\alpha \in Re$ is not nilpotent then α^n is invertible in Re . But $e\alpha = \alpha$ and so, by 4.3, α is invertible in Re . Hence Re is a divided ring.

Let $\{e_i\}$ be the set of all primitive idempotents of R . $e_i e_j = 0$ if $e_i \neq e_j$. The sum ΣRe_i is direct; for if $x \in Re_j \cap \sum_{i \neq j} Re_i$ then $x = x e_j = \sum_{i \neq j} x_i e_i e_j = 0$. Thus $R_1 = \Sigma Re_i$ is semi-divided.

Let R_2 be the set of all $x \in R$ for which $x e_i = 0$ for all primitive idempotents e_i . R_2 is an ideal of R . If R_2 contains a non-zero idempotent then, by condition (iii) R_2 contains a primitive idempotent e . But then we would have $e = e^2 = 0$. Hence, by 6.1, R_2 must be nil.

The sum $R_1 + R_2$ is direct; for if $x \neq 0$ is an element of R_1 then $x e_i \neq 0$ for at least one e_i . Hence, $R_1 \cap R_2 = 0$. We now wish to show that $R = R_1 + R_2$. If $x \in R$ then $x e_i \neq 0$ for only finitely many primitive e_i . Hence, $x' = x - \Sigma x e_i$ is well defined. Moreover, $x' e_i = x e_i - x e_i^2 = 0$ and so $x' \in R_2$. Therefore, $x = \Sigma x e_i + x' \in R_1 + R_2$. Hence, R is the direct sum of a semi-divided ring and a nil ring.

The converse is directly verified.

Now suppose in addition to (i)-(iv) R also satisfies (v). Let $x \in R_2$. Then x has a (say left) identity $e = e_1 + e_2$, $e_1 \in R_1$, $e_2 \in R_2$, and

$$x = ex = e_1 x + e_2 x = e_2 x,$$

since $e_1 x = 0$. But then $x = e_2^m x$ for all $m \geq 0$. Since R_2 is nil, $e_2^m = 0$ for some m and so $x = 0$. Thus $R_2 = 0$ and $R = R_1$, a semi-divided ring.

Again, the converse is easily verified.

That condition (iv) is actually necessary may be seen from the following example. Let S be the strong direct sum of countably many copies of Z_4 , the ring of integers mod 4. Let R be the subring of S generated by the weak direct sum and the element $(2, 2, \dots, 2, \dots)$. Then R satisfies (i)–(iii) and is not the direct sum of a semi-divided ring and nil ring.

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