SOME CHARACTERIZATIONS OF A CLASS OF UNAVOIDABLE COMPACT SETS IN THE GAME OF BANACH AND MAZUR

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1. Introduction. The game of Banach and Mazur is understood here[1] as follows:

Two players A and B choose alternately nonnegative numbers t_n, $(n = 0, 1, 2, \cdots)$ in the following manner: B chooses a number t_0 such that $0 \leqq t_0 < 1$. After t_i $(i = 0, 1, \cdots, 2n)$ have been chosen, A chooses t_{2n+1} such that

$$\text{(a)} \qquad 0 < t_{2n+1} < t_{2n} \qquad (\text{if } t_0 = 0,\ t_1 \text{ is arbitrary})$$

and subsequently B a number t_{2n+2} such that

$$\text{(b}'\text{)} \qquad 0 < t_{2n+2} < t_{2n+1}\,, \qquad (n = 0, 1, 2, \cdots)\,.$$

Given a set $S \subset [0, 1]$, A will be said to win on S if $s = \sum_{n=0}^{\infty} t_n \in S$; otherwise B wins.

We shall deal in this paper with a generalization of this game, consisting in replacing (b′) by

$$\text{(b)} \qquad 0 < t_{2n+2} < k \cdot t_{2n+1}\,, \qquad (n = 0, 1, 2, \cdots)$$

where $k > 0$ will be referred to as the game constant.[2]

We say that the set S is unavoidable, or that B cannot avoid it, if there exists a sequence of functions $t_1(t_0), t_3(t_0, t_1, t_2), \cdots, t_{2n+1}(t_0, t_1, \cdots, t_{2n}), \cdots$, satisfying (a) and such that $s = \sum_{n=0}^{\infty} t_n \in S$ whenever (b) holds. If, on the other hand, there exists a sequence of functions $t_0, t_2(t_0, t_1), \cdots, t_{2n}(t_0, t_1, \cdots t_{2n-1}), \cdots$ satisfying (b) and such that $s = \sum_{n=0}^{\infty} t_n \notin S$, whenever (a) holds, then S is said to be avoidable.

The sets. In this paper we shall consider closed subsets of $[0, 1]$ exclusively. Let S be an arbitrary closed set on the interval $f = [0, 1]$

Received July 27, 1960.

* Mathematics Research Center, U. S. Army, Madison, Wisconsin and Technion, Israel Institute of Technology, Haifa. Supported by U. S. Army under Contract No. DA 11-022-ORD-2059.

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[1] Various variants of the game are described in the so-called "Scottish Book", s. Coll. Math., **1** (1947), p. 57.

[2] The case of the constant k replaced by a variable k_n is considered in [**1**].

and suppose that 0 and 1 belong to S[3]. The complement $[0, 1] \sim S = \bigcup_{n=1}^{\infty} g_n$ is a union of open and disjoint intervals g_n. Denote by g the greatest of them. (If several such intervals of the same length exist, g will denote the one lying to the right of all others). Then $f \sim g = f_0 \cup f_1$ is a union of two closed intervals f_0 and f_1, where f_0 denotes the left and f_1 the right one. Suppose now the closed intervals $f_{\delta_1, \cdots \delta_n}$, $\delta_i = 0, 1$ are already defined and denote by $g_{\delta_1, \cdots, \delta_n}$ the greatest of the intervals g_n contained in $f_{\delta_1, \cdots, \delta_n}$ (if any). The set $f_{\delta_1, \cdots, \delta_n} \sim g_{\delta, \cdots, \delta_n} = f_{\delta_1, \cdots, \delta_n, 0} \cup f_{\delta_1, \cdots, \delta_n, 1}$ is a union of two closed intervals, where $f_{\delta_1 \cdots, \delta_n, 0}$ denotes the left and $f_{\delta_1, \cdots, \delta_n, 1}$ the right interval (Fig. 1)

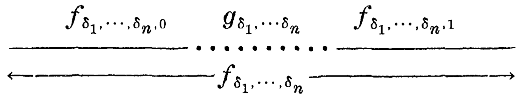

Fig. 1

It is clear that $S = \bigcap_{n=0}^{\infty} \bigcup_{\delta_i = 0, 1} f_{\delta_1, \cdots, \delta_n}$ $i = 1, 2, \cdots, n$ $((f_{\delta_1, \cdots, \delta_n})_{n=0}$ denotes the interval $f = [0, 1])$.

The class C of sets satisfying[4]

$$\text{(c)} \quad \frac{|g|}{|f_0|} = \frac{|g_{\delta_1, \cdots, \delta_n}|}{|f_{\delta_1, \cdots, \delta_n, 0}|} = c_1 > 0 \text{ and } \frac{|g|}{|f_1|} = \frac{|g_{\delta_1, \cdots, \delta_n}|}{|f_{\delta_1, \cdots, \delta_n, 1}|} = c_2 > 0$$

where c_1 and c_2 are constants (independent of $\delta_1, \cdots, \delta_n$) is called the Cantor class.

Evidently, each set belonging to C is perfect and its Lebesgue-measure is 0 (it is consequently also nowhere dense). We shall denote $x = |f_0|$, $y = |g|$ and $\alpha = 1 - x - y = |f_1|$. We can establish a one-to-one correspondence between the sets of C and the points of the triangle: $0 < x < 1$, $0 < y < 1 - x$ (see Fig. 2). A set of C corresponding to (x, y) is denoted by $S_{x,y}$. The sets $S_{x,y}$ of C for which $|f_0| = |f_1|$, i.e. the sets for which $y = 1 - 2x$, are called symmetric sets. In particular, the Cantor discontinuum $S_{1/3, 1/3}$ is a symmetric set.

Outline of results. S. Banach posed the problem of finding necessary and sufficient conditions which make a set S unavoidable.

In § 2 we find for every $k \geqq 1$ sufficient conditions for an arbitrary compact set S to be unavoidable for the constant k. These conditions are also necessary if the following additional condition (ā) is stipulated.

(ā) $t_1 \leqq \varepsilon$, where $\varepsilon > 0$ is a number chosen by B such that $(t_0, t_0 + \varepsilon] \cup S \neq 0$.

The condition (ā) implies a uniform structure (from the point of view of the game) of the set S; and under this restriction a solution of the problem of Banach in the case of compact sets is given.

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[3] This will be assumed throughout the paper.

[4] $|g|$ denotes the length of the interval g.

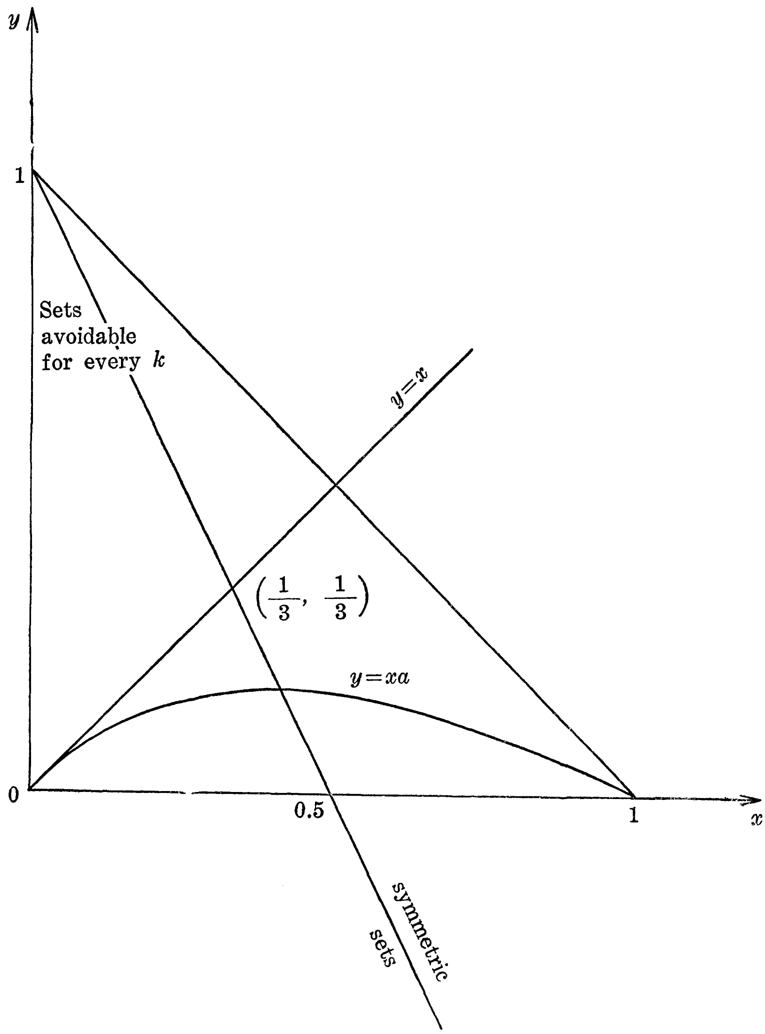

Fig. 2

In § 3 we give moreover a numerical solution of the problem of Banach for sets belonging to the Cantor class C. Namely, we define a function $\bar{k}(x, y)$:

$$\bar{k}(x, y) = \begin{cases} 0 & \text{for} \quad y \geqq x \\ \dfrac{\alpha(1 - x\alpha^p)}{y + x\alpha^{p+1}} & \text{for} \quad x\alpha^{p+1} \leqq y < x\alpha^p \,, \qquad (p = 0, 1, 2, \cdots) \end{cases}$$

($\alpha = 1 - x - y$, $0 < x < 1$, $0 < y < 1 - x$), such that the set $S_{x,y}$ is unavoidable if, and only if, the game-constant k satisfies $k \leqq \bar{k}(x, y)$. It can be easily seen that the lines $y = x\alpha^p$, ($p = 0, 1, \cdots$) are lines of discontinuity of this function and that a necessary and sufficient condition for a set $S_{x,y}$ of C to be avoidable for every $k > 0$ is that the point

(x, y) be on or above the diagonal $y = x$. In this sense the line $y = x$ separates the avoidable sets for every k from the others, and especially the Cantor discontinuum $S_{1/3,1/3}$ has this property with regard to the symmetric sets. The results of this section also include a generalization of a result obtained in [2], where, in answer to a question by H. Steinhaus, an unavoidable perfect set of measure 0 with the game-constant $k = 1$ was constructed. Since, as it turns out this is a set $S_{1/2,1/8}$ and $\bar{k}(\frac{1}{2}, \frac{1}{8}) = 39/25$, it is unavoidable if, and only if, $k \leqq 39/25$.

NOTATION. We denote by $\rho(h_1, h_2)$ the distance between the intervals h_1 and h_2; by $l(h)$ and $r(h)$ the left and right endpoints of the interval h; we also put $s_n = \sum_{j=0}^{n} t_j$.

Furthermore introduce the following definition:

(d) Let z be any point of the set S and $\{g^n\}_{n=0,1\cdots}$ a sequence of open intervals defined as follows $g^0 = (1, \infty)$ and g^{n+1} the greatest interval g_k lying between z and g^n (if several such intervals of the same length exist, g^{n+1} will denote the one lying to the right of all the others). The sequence $\{g^n\}$ and $\{f^n\}$ (where $f^n = [r(g^{n+1}), l(g^n)]$) may be finite e.g. if $z = l(g_m)$ for some m. The most interesting case is however when the sequence $\{g^n\}$ is infinite. It converges then to some point z' of S, $z' \geqq z$ and will be referred to as a descending sequence: $g^n \to z'$.

2. Arbitrary compact sets. In this section we consider arbitrary compact sets S in the interval $[0, 1]$. In addition to the assumptions (a) and (b) we also assume that $(\bar{a})$ holds. For every game-constant $k \geqq 1$, we shall give necessary and sufficient conditions for the set S to be unavoidable. We shall namely prove, that the three properties (p_1), (p_2) and (p_3), defined below, are equivalent. By means of a small modification of the proof it can be shown that (p_2) and (p_3) are equivalent for every $k > 0$ (not only $k \geqq 1$).

By g, $\tilde{g}$ (with or without subscripts (or superscripts)) we denote the open intervals g_n and the two intervals $(-\infty, 0)$ and $(1, \infty)$. We now choose a fixed $k \geqq 1$ and define for it the properties (p_1), (p_2) and (p_3).

(p_1) A compact set S is said to have the property (p_1) if the following conditions (p_1') and (p_1'') hold.

(p_1') If $\tilde{f}$ is an interval lying between two intervals g' and g'' at least one of which is other than $(-\infty, 0)$ and $(1, \infty)$ such that $r(g') = l(\tilde{f})$ and $r(\tilde{f}) = l(g'')$ then either $k \cdot |g'| \leqq |\tilde{f}|$ or $|g''| < |\tilde{f}|$ (Fig. 3)

Fig. 3

(p_1'') If $g^n \to z$, then there exist infinitely many integers n such that for every m, $m < n$ either $k \cdot \rho[z, r(g^n)] \leqq \rho(g^n, g^m)$ or $|g^m| < \rho(g^n, g^m)$.

Regarding sets having property (p_1) we note:
(1) If S satisfies (p_1) and $\tilde{f}$ is a segment lying between the intervals g^n and g^{n-1} which belong to some descending sequence $\{g^n\}_{n=0,1,\dots}$ then $\rho(g^n, \tilde{g}) > |\tilde{g}|$ holds for every interval $\tilde{g}$ contained in $\tilde{f}$.

Indeed, let f' be the interval defined by $f' = [r(g^n), l(\tilde{g})]$ (i.e. the interval lying between g^n and $\tilde{g}$). If $k\cdot|g^n| > |f'|$ then by (p_1') there is $|\tilde{g}| < |f'| = \rho(g^n, \tilde{g})$. If however $k\cdot|g^n| \leqq |f'|$ then by the definition (d) of a descending sequence of intervals $|\tilde{g}| < |g^n|$ and by the assumption $k \geqq 1$ we have $|\tilde{g}| < |f'| = \rho(g^n, \tilde{g})$.

We now introduce the following definition:
(h) A set S is said to have the property (h) in the interval $(z, z+\varepsilon)$ if for each interval $\hat{g}$ such that $\hat{g} \cap (z, z+\varepsilon) \neq 0$ there is $\rho(z, \hat{g}) > |\hat{g}|$.

We define the property
p(2) A set S is said to have the property (p_2) if the following two conditions (p_2') and (p_2'') are satisfied:
(p_2') The set S has the property (h) in each interval $(r(\tilde{g}), r(\tilde{g}) + k\cdot|\tilde{g}|)$.
(p_2'') For each $z \in S$ and $z \neq l(\tilde{g})$ there exists a point $z' > z$ arbitrarily close to z and such that S has the property (h) in the interval $(z', z' + k\cdot\rho(z, z'))$.

Finally
(p_3) A set S is said to have the property (p_3) if it is unavoidable (for the game constant k).

We shall now prove that for compact sets S the properties (p_1), (p_2) and (p_3) are equivalent. This will be done by proving the implications (p_1) $\to$ (p_2) $\to$ (p_3) $\to$ (p_1).

$$(2) \qquad\qquad (p_1) \longrightarrow (p_2)$$

Indeed, let $\hat{g}$ and $\tilde{g}$ be intervals such that $\hat{g} \cap (r(\tilde{g}), r(\tilde{g}) + k|\tilde{g}|) \neq 0$. Thus $\rho(\tilde{g}, \hat{g}) < k\cdot|\tilde{g}|$; ($p_2'$) holds by the condition (p_1') used for $g' = \tilde{g}$, $g'' = \hat{g}$ and $\tilde{f} = [r(\tilde{g}), l(\hat{g})]$. Thus ($p_1$) $\to$ (p_2'). It remains to prove (p_2''). Let $z \in S$ be a point such that $z \neq l(\tilde{g})$. If S contains an interval with the left endpoint[5] in z, then choosing z' sufficiently close to z, (p_2'') is satisfied in a trivial way. We therefore may assume that there exists an infinite sequence $g^n \to z$. By (p_1'') there are points $z' = r(g^n)$ arbitrarily close to z such that for each interval g^m lying to the right of z' there is either $k\rho(z, z') \leqq \rho(z', g^m)$ or $|g^m| < \rho(z', g^m)$. Let $m < n$ be the greatest integer such that $|g^m| \geqq \rho(z', g^m)$. Such a number exists, since for example there is always $|g^0| \geqq \rho(z', g^0)$. We have then by ($p_1''$): $k\rho(z, z') \leqq \rho(z', g^m)$ and for each t such that $m < t < n$, $|g^t| < \rho(z', g^t)$. By (1) we thus conclude, that S has the property (h) in the interval $(z', z' + \rho(z', g^m))$ which contains $(z', z' + k\rho(z, z'))$. Thus (p_1) $\to$ (p_2''), and (2) is proved.

We now prove that

[5] z may evidently be also an interior point of some interval contained in S.

$$(3) \qquad (p_2) \longrightarrow (p_3)$$

Indeed, let $0 \leqq t_0 < 1$ be an arbitrary number chosen by B. We then show that A can choose a number t_1 satisfying (a) and ($\bar{a}$) such that $s_1 \in S$ and that(h) holds in $(s_1, s_1 + kt_1)$: If $t_0 \in \tilde{g}$ or $t_0 = l(\tilde{g})$ A can choose $s_1 = r(\tilde{g})$ and our condition is satisfied by (p_2'). In the case $t_0 \in S$ and $t_0 \neq l(\tilde{g})$, A chooses $s_1 = z'$ and (p_2'') applies. Similarly A may after each step t_{2n} of B (satisfying (b)), choose t_{2n+1}, obtaining in particular $s_{2n+1} \in S$. By the compactness of S we then have $s = \lim_{n\to\infty} s_{2n+1} \in S$ and thus (p_3) holds.

REMARK 1. Note that the assumption $k \geqq 1$ is not used in the proof of (3). Hence, by (3) the property (p_2) (for $k > 0$ and not only for $k \geqq 1$) suffices for the unavoidability of the compact set S. It is easy to see, using ($\bar{a}$), that the condition (p_2) is also necessary for $k > 0$.

Before proving the implication $(p_3) \rightarrow (p_1)$ we note that

(4) If for some n there is $s_{2n-1} \notin S$ or $s_{2n-1} = l(\tilde{g})$ then B can avoid S, by choosing the numbers $t_{2n}, t_{2n+2}, \cdots$ sufficiently small.

We finally prove that

$$(5) \qquad (p_3) \longrightarrow (p_1) .$$

The proof is indirect. If (p_1') does not hold, then there exists an interval $\tilde{f} = [r(g'), l(g'')]$. (Fig. 3) such that $k \cdot |g'| > |\tilde{f}|$ and $|g''| \geqq |\tilde{f}|$. B can choose $t_0 = l(g')$ and $\varepsilon = |g'|$. Then by ($\bar{a}$) and (4) A has to choose $s_1 = r(g')$. Now B chooses $t_2 = |\tilde{f}| < k|g'| = kt_1$ and from $|g''| \geqq |\tilde{f}|$ and (a) follows $s_3 \in g''$. Hence by (4) B avoids S.

If, on the other hand, (p_1'') does not hold, then there exists a point z, a sequence $g^n \rightarrow z$ and an integer n_0, such that for every $n \geqq n_0$ there exists $m = m(n) < n$ with the property: $k\rho(z, r(g^n)) > \rho(g^n, g^m)$ and $|g^m| \geqq \rho(g^n, g^m)$. B chooses $t_0 = z$ and $\varepsilon < \rho(z, g^{n_0})$. By (4) it is sufficient to consider the case $r(g^{n+1}) \leqq s_1 < l(g^n)$ (Fig. 4) for some $n \geqq n_0$. In this case, however, B can, choosing $t_2 = \rho(s_1, g^m)$, satisfy (b) and by (a) there must be $s_3 \in g^m$. Thus by (4) the set S is avoidable.

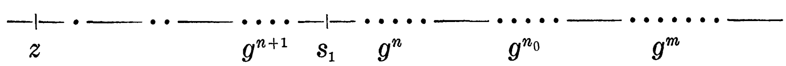

Fig. 4

From (2), (3) and (5) we obtain

THEOREM 1. *The properties* (p_1), (p_2) *and* (p_3) *are equivalent.*

This theorem solves the Banach problem in the case of compact sets on the additional assumption ($\bar{a}$).

3. Sets of the Cantor class. In this section we deal with sets $S_{x,y}$ of the Cantor-class C, only. We find for them a function $\bar{k}(x, y)$ defined within the triangle $0 < x < 1$; $0 < y < 1 - x$, such that the set $S_{x,y}$ is unavoidable if, and only if, the game-constant k satisfies: $k \leqq \bar{k}(x, y)$.

We begin with a few remarks. Denoting, as in the introduction, $x = |f_0|$, $y = |g|$ and $\alpha = 1 - x - y = |f_1|$ we obtain by (c) (s. Fig. 1)

(6) $|f_{\delta_1,\cdots,\delta_n}| = x^\nu \alpha^\mu$ and $|g_{\delta_1,\cdots,\delta_n}| = yx^\nu\alpha^\mu$ where $\mu = \sum_{i=1}^n \delta_i$ and $\nu = n - \mu$; it follows

$$(7) \qquad |g_{\delta_1,\cdots,\delta_n}| > |g_{\delta_1,\cdots,\delta_n,\delta_{n+1}}|, \qquad (n = 0, 1, \cdots).$$

Hence, if $g^n \rightarrow z$ and for some m, $g^m = g_{\delta_1,\cdots,\delta_{t_m}}$ then $g^{m+1} = g_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots1}_{q_m}}$ where $q_m \geqq 0$ (i.e. the interval g^{m+1} is obtained from g^m by adding one 0, or one 0 and several 1's, to the subscripts $\delta_1, \cdots, \delta_{t_m}$ of g^m).

By (c) we also have

(8) If $y < x$, then for every interval g_k contained in $f_{\delta_1,\cdots,\delta_n}$ there is $|g_k| < \rho[l(f_{\delta_1 \cdots \delta_n}), g_k]$.

We now introduce the following definition:

($\bar{\text{d}}$) Let $g^n \rightarrow z$ be a descending sequence such that there exist two infinite sequences $\{m'\}$ and $\{m''\}$—of integers with the property $|f^m| \leqq |g^m|$ for $m \in \{m'\}$ and $|f^m| > |g^m|$ for $m \in \{m''\}$, and such that for sufficiently large integers m, $m \in \{m'\}$ implies $m + 1 \in \{m''\}$ and $m - 1 \in \{m''\}$. Hence there exist an integer m_0 and an infinite sequence $\{r_j\}$ of integers such that $m_0 \in \{m'\}$, $(m_0 + i) \in \{m''\}$, $(1 \leqq i \leqq r_1)$, $(m_0 + r_1 + 1) \in \{m'\}$, $(m_0 + r_1 + 1 + i) \in \{m''\}$, $(1 \leqq i \leqq r_2)$, $(m_0 + r_1 + r_2 + 2) \in \{m'\}$, and so on. If $\overline{\lim}\, r_j = r$ is finite, then z is said to be a point of order r. If otherwise, $\overline{\lim}\, r_j = \infty$ then z is called a point of order ∞.

We prove now the following lemma.

LEMMA. *Let $g^n \rightarrow z$ and $y < x$. Denote by p the integer satisfying*

$$(9) \qquad x\cdot\alpha^{p+1} \leqq y < x\cdot\alpha^p$$

and put

$$\bar{k} = \bar{k}(x, y) = \frac{\alpha(1 - x\alpha^p)}{y + x\alpha^{p+1}}$$

then

(10) *at any arbitrarily small distance from the point z there exists a point $z' > z$ such that the inequality $\rho(z', g_k) > |g_k|$ holds for each interval g_k satisfying the condition*

$$g_k \cap (z', z' + \bar{k}\cdot\rho(z, z')) \neq 0\,, \quad (i.e.\ (p_2'')\ holds)\,.$$

Proof. By definition of the intervals g^m and f^m,

(11) if $g^m = g_{\delta_1,\cdots,\delta_{t_m}}$ then $f^m = f_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots1}_{q_m+1}}$, $q_m \geqq 0$

and $g^{m+1} = g_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots1}_{q_m}}$.

From (7) follows that $|g_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots1}_{q_m-1}}| > |g^{m+1}|$ for $q_m > 0$ and for $q_m = 0$ holds $|\tilde{g}| > |g^{m+1}|$ where $\tilde{g}$ is the interval satisfying $r(\tilde{g}) = l(f_{\delta_1,\cdots,\delta_{t_m}})$. In any case we have

$$z \in f_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots1}_{q_m}} . \tag{12}$$

The following cases will be considered:
(a) For infinitely may $m, q_m > p$.
(b) For every sufficiently large $m, q_m \leqq p$
(ba) For every sufficiently large $m, q_m = p$
(bb) For every sufficiently large $m, q_m < p$
(bc) There are two infinite sequences M' and M'' of integers such that for $m \in M'$, $q_m = p$, and for $m \in M''$, $q_m < p$.

By (11), (6) and (9) follows that

$$q_m = p \text{ is equivalent to } |f^m| \leqq |g^m| \tag{13}$$

$$q_m < p \text{ is equivalent to } |f^m| > |g^m| . \tag{14}$$

(bca) for infinitely many m holds

$$m \in M'' \quad \text{and} \quad q_m \geqq 1 \tag{15}$$

(bcb) for every sufficiently large $m \in M''$, $q_m = 0$

(bcba) For infinitely many m,

$m + 1 \in M'$ and $m + 2 \in M'$

(bcbb) For every sufficiently large m, from

$m + 1 \in M'$ follows $m + 2 \in M''$.

We shall now prove the lemma for each of the above cases separately:

(a) From (12) follows $\bar{k}\rho(z, f_{\delta_1,\cdots,\delta_{t_m},1}) \leqq \bar{k}(|f_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots1}_{q_m}}| + |g_m|)$

$$= \frac{\alpha(1 - x\alpha^p)}{y + x\cdot\alpha^{p+1}} |f_{\delta_1,\cdots,\delta_{t_m}}| \cdot (x\alpha^{q_m} + y) .$$

Thus for m satisfying $q_m > p$,

$$\bar{k}\cdot\rho(z, f_{\delta_1,\cdots,\delta_{t_m},1}) < \alpha |f_{\delta_1,\cdots,\delta_{t_m}}| = |f_{\delta_1,\cdots,\delta_{t_m},1}| .$$

If moreover m is sufficiently large then the distance $\rho(z, f_{\delta_1,\cdots,\delta_{t_m},1})$ is arbitrarily small and thus choosing $z' = l(f_{\delta_1,\cdots,\delta_{t_m},1})$ we conclude by (8) that (10) holds.

(ba) By (13) and (11) we have for m sufficiently large

$$f^m = f_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots 1}_{p+1}},\ g^{m+1} = g_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots 1}_{p}}$$

and $g^{m+\mu+1}(\mu \geqq 0)$ is obtained from $g^{m+\mu}$ by adding one 0 and p 1's to the subscripts of $g^{m+\mu}$. Hence

$$\begin{aligned}\rho(z, f_{\delta_1,\cdots,\delta_{t_m},1}) &= |g_{\delta_1,\cdots,\delta_{t_m}}| + |f_{\delta_1,\cdots\delta_{t_m},0,\underbrace{1\cdots 1}_{p+1}}| + |g_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots 1}_{p}}| + \\ &\quad + |f_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots 1}_{p},0,\underbrace{1\cdots 1}_{p+1}}| + \cdots = \\ &= |f_{\delta,\cdots,\delta_{t_m}}|(y + x\alpha^{p+1} + yx\alpha^p + x^2\alpha^{2p+1} + \cdots) = |f_{\delta_1,\cdots,\delta_{t_m}}|\cdot\frac{y + x\alpha^{p+1}}{1 - x\alpha^p}\,.\end{aligned}$$

Therefore $\bar{k}\cdot\rho(z, f_{\delta_1,\cdots,\delta_{t_m},1}) = \alpha|f_{\delta_1,\cdots,\delta_{t_m}}| = |f_{\delta_1,\cdots,\delta_{t_m},1}|$. Thus taking m sufficiently large (i.e. $f_{\delta_1,\cdots,\delta_{t_m},1}$ sufficiently near to z) and putting $z' = l(f_{\delta_1,\cdots,\delta_{t_m},1})$ we see, by (8), that (10) holds.

(bb) By (14) there exists a number μ_0, such that for $m \geqq \mu_0$, $|f^m| > |g^m|$. Now take $m \geqq \mu_0$ such that $\bar{k}\rho(z, l(f^m)) \leqq |f^{\mu_0}|$. Thus putting $z' = l(f^m)$ and taking m sufficiently large we obtain that (10) holds for every interval $g_k = g^n$ where $m \geqq n \geqq \mu_0$. Now for other intervals g_k (i.e. for $g_k \subset f^n$ $(m \geqq n \geqq \mu_0)$) (10) evidently holds by (8). Hence (10) holds in general.

(bca) Let m satisfy (15) and let r be the smallest integer such that $m + r \in M'$ (evidently $r \geqq 1$). Then, by (11) it follows that f^{m+i}, $(1 \leqq i \leqq r)$ are of the form

$$f^{m+i} = f_{\delta_1,\cdots\delta_{t_m},0,\underbrace{1\cdots 1}_{q_m},0,\underbrace{1\cdots 1}_{q_{m+1}},0,\underbrace{1\cdots 1}_{q_{m+2}},0,\cdots,0,\underbrace{1\cdots 1}_{q_{m+i}+1}}$$

where $0 \leqq q_{m+i} < p$ for $1 \leqq i < r$ and $q_{m+r} = p$, and the g^{m+j} are of the form $g^{m+j} = g_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots 1}_{q_m},0,\underbrace{1\cdots 1}_{q_{m+1}},0,\cdots,0,\underbrace{1\cdots 1}_{q_{m+j-1}}}$ for $1 \leqq j \leqq r$. By analogy with (12) we have

$$z \in f_{\delta_1,\cdots,\delta_{t_m},0,\underbrace{1\cdots 1}_{q_m},0,\underbrace{1\cdots 1}_{q_{m+1}},0,\cdots,0,\underbrace{1\cdots 1}_{q_{m+r}=p}}\,.$$

Therefore by (6)

$$\text{(16)}\qquad \begin{aligned}\rho \overset{\text{def.}}{=} \rho(z, f^{m+r-1}) &\leqq |f_{\delta_1,\cdots,\delta_{t_m}}|\cdot\left(x^{r+1}\alpha^{p+\sum_{i=0}^{r-1}q_{m+i}} + yx^r\cdot\alpha^{\sum_{i=0}^{r-1}q_{m+i}}\right) \\ &< |f_{\delta_1,\cdots,\delta_{t_m}}|(x^2\cdot\alpha^{p+q_m} + yx\alpha^{q_m})\,.\end{aligned}$$

Now evidently

$$\text{(17)}\qquad \begin{aligned}|f_{\delta_1,\cdots,\delta_{t_m},1}| + \sum_{i=0}^{r-1}(|g^{m+i}| + |f^{m+i}|) &\geqq |f_{\delta_1,\cdots,\delta_{t_m},1}| + |g^m| + |f^m| \\ &= |f_{\delta_1,\cdots\delta_{t_m}}|(\alpha + y + x\alpha^{q_m+1})\,.\end{aligned}$$

By (15)

$$\alpha(1 - x\alpha^p)(x^2 \cdot \alpha^{p+q_m} + yx\alpha^{q_m}) < (\alpha + y + x\alpha^{q_m+1})(x\alpha^{p+1} + y)$$

holds. Dividing both sides by $y + x\alpha^{p+1}$ we obtain

$$\bar{k}(x^2\alpha^{p+q_m} + yx\alpha^{q_m}) < \alpha + y + x\alpha^{q_m+1}$$

and therefore by (16) and (17)

$$\bar{k}\rho \leqq |f_{\delta_1,\cdots\delta_{t_m},1}| + \sum_{i=0}^{r-1}(|g^{m+i}| + |f^{m+i}|) .$$

Thus, putting $z' = l(f^{m+r-1})$ we see, by $|f^{m+i}| > |g^{m+i}|$ for $0 \leqq i < r$ and (8), that (10) holds.

In the case (bcb) we have for every sufficiently large $m \in M''$

$$|g^m| = |g_{\delta_1,\cdots,\delta_{t_m}}| < |f_{\delta_1,\cdots,\delta_{t_m},0,1}| = |f^m| .$$

Now turn to the case

(bcba) By (11) and (13) we have

$$g^{m+1} = g_{\delta_1,\cdots,\delta_{t_m},0},\ f^{m+1} = f_{\delta_1,\cdots,\delta_{t_m},0,0,\underbrace{1\cdots1}_{p+1}} ,$$

$$g^{m+2} = g_{\delta_1,\cdots,\delta_{t_m},0,0,\underbrace{1\cdots1}_{p}}$$

and

$$f^{m+2} = f_{\delta_1,\cdots,\delta_{t_m},0,0\underbrace{1\cdots1}_{p},0,\underbrace{1\cdots1}_{p+1}} .$$

Therefore, as in (12)

$$z \in f_{\delta_1,\cdots,\delta_{t_m},0,0,\underbrace{1\cdots1}_{p},0,\underbrace{1\cdots1}_{p}} .$$

Thus

(18) $$\rho(z, f^m) \leqq |f_{\delta_1,\cdots,\delta_{t_m}}| \cdot (x^3 \cdot \alpha^{2p} + yx^2\alpha^p + x^2\alpha^{p+1} + yx) .$$

Now, since for $p \geqq 1$, $x^3\alpha^{2p+1} < x^2\alpha^{p+2}$, we have

$$\alpha(x^3\alpha^{2p} + yx^2\alpha^p + x^2\alpha^{p+1} + yx) < (y + x\alpha^{p+1})(x\alpha + y + \alpha) .$$

Dividing both sides by $(y + x\alpha^{p+1})$ we obtain from (18) (since $1 - x\alpha^p < 1$) that

$$\bar{k} \cdot \rho(z, f^m) < |f^m| + |g^m| + |f_{\delta_1,\cdots,\delta_{t_m},1}| .$$

Taking now m sufficiently large and putting $z' = l(f^m)$ we see, by (8), that in this case again (10) holds.

We go over to the case

(bcbb) By $(\bar{\mathrm{d}})$ there are two possibilities

$$z \text{ is a point of order } r,$$
$$z \text{ is a point of order } \infty .$$

In the first case let $m_1, m_2, \cdots$ be the sequence $\{m'\} = M'$. By $q_{m_i} = p$ we have $f^{m_i} = f_{\delta_1,\cdots,\delta_{t_{m_i}},0,\underbrace{1\cdots 1}_{p+1}}$. If now for every sufficiently large i, $m_{i+1} - m_i = r + 1$ then for such i we have in view of (bcb)

$$\rho(z, f^{m_i+r}) = \sum_{j=i+1}^{\infty}\left[\sum_{h=0}^{r} |g^{m_j+h}| + \sum_{h=0}^{r} |f^{m_j+h}|\right] =$$
$$= x^{r+1}\alpha^p \frac{y\left(1 + \alpha^p \sum_1^r x^j\right) + x\alpha^{p+1} \sum_0^r x^j}{1 - x^{r+1}\alpha^p} |f_{\delta_1,\cdots,\delta_{t_{m_i}}}|$$

(see Fig. 5 where $\phi = |f_{\delta_1,\cdots,\delta_{t_{m_i}}}|$ and $r = 3$)

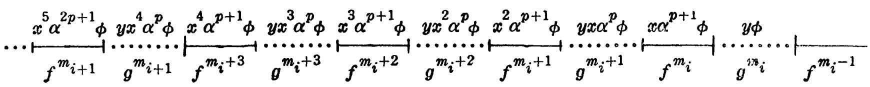

Fig. 5

Generally, there exist infinitely many integers i such that $m_{i+1} - m_i = r + 1$ and since $r = \overline{\lim}\, r_j$ we have for such integers i

$$\rho(z, f^{m_i+r}) \leqq x^{r+1}\alpha^p \frac{y\left(1 + \alpha^p \sum_1^r x^j\right) + x\alpha^{p+1} \sum_0^r x^j}{1 - x^{r+1}\alpha^p} |f_{\delta_1,\cdots,\delta_{t_{m_i}}}| .$$

On the other hand

$$\rho(l(f^{m_i+r}), r(f^{m_i})) = \alpha^p\left(y \sum_{j=1}^{r} x^j + x\alpha \sum_{j=0}^{r} x^j\right) \cdot |f_{\delta_1,\cdots,\delta_{t_{m_i}}}|$$

(see Fig. 5). Hence by $\{(1 - x\alpha^p)/(1 - x^{r+1}\alpha^p)\} < 1$, we have

$$\bar{k}\rho(z, f^{m_i+r}) < \rho(l(f^{m_i+r}), r(f^{m_i})) .$$

Putting $z' = l(f^{m_i+r})$ we see, considering $y < x\alpha^p$ and (8) that (10) holds.

Let finally z be a point of order ∞. We have $y = y(x + y + \alpha) = xy + y(y + \alpha)$ and hence by (9) $y < xy + x\alpha^p(y + \alpha)$, i.e. $y - xy = (1 - x)y < yx\alpha^p + x\alpha^{p+1}$. Thus for r sufficiently large also $(1 - x)y < yx\alpha^p + x\alpha^{p+1} - yx^{r+1}\alpha^p - x^{r+2}\alpha^{p+1}$ i.e.

$$y < yx\alpha^p \cdot \frac{1 - x^r}{1 - x} + x\alpha^{p+1}\frac{1 - x^{r+1}}{1 - x} = \alpha^p\left(y \sum_{j=1}^{r} x^j + x\alpha \sum_{j=0}^{r} x^j\right) . \tag{19}$$

Since z is a point of order ∞, there exist arbitrarily large integers r and m such that $m \in \{m'\}$, $m + r + 1 \in \{m'\}$ and $m + i \in \{m''\}$ for

$1 \leqq i \leqq r$. Now taking m and r sufficiently large and noting that

$$\rho(l(f^{m+r}), r(f^m)) = \alpha^p\Big(y \sum_{j=1}^{r} x^j + x\alpha \sum_{j=0}^{r} x^j\Big) |f_{\delta_1,\cdots,\delta_{t_m}}|$$

we obtain by (19) that there exist arbitrarily large integers m and r such that

$$|g^m| < \rho(l(f^{m+r}), g^m) . \tag{20}$$

We have also

$$\begin{aligned}\rho(l(f^{m+r}), r(f_{\delta_1,\cdots,\delta_{t_m},1})) &\geqq |f_{\delta_1,\cdots,\delta_{t_m},1}| + |g^m| + |f^m| = \\ &= (\alpha + y + x\alpha^{p+1}) |f_{\delta_1,\cdots,\delta_{t_m}}| .\end{aligned}$$

Further by (13) we have, by analogy with (16), (where r should be replaced by $r+1$) that

$$\rho(z, f^{m+r}) = \rho(z, l(f^{m+r})) \leqq |f_{\delta_1,\cdots,\delta_{t_m}}| (x^{r+2}\cdot\alpha^{2p} + yx^{r+1}\alpha^p)$$

and therefore

$$\bar{k}\rho(z, f^{m+r}) \leqq \rho(l(f^{m+r}), r(f_{\delta_1,\cdots,\delta_{t_m},1})) .$$

Thus putting $z' = l(f^{m+r})$ we see by (8) and (20) that (10) holds in this case again. The proof is completed.

We are now able to prove the following:

THEOREM 2. *Let $\bar{k}(x, y)$ be a function defined within the triangle $0 < x < 1$, $0 < y < 1 - x$ by the formula:*

$$\bar{k}(x, y) = \begin{cases} 0 & \text{for} \quad y \geqq x \\ \dfrac{\alpha(1 - x\alpha^p)}{y + x\alpha^{p+1}} & \text{for} \quad x\alpha^{p+1} \leqq y < x\alpha^p \end{cases}$$

where $\alpha = 1 - x - y$ and $p = 0, 1, 2, \cdots$

A set $S = S_{x,y} \in C$ is unavoidable if, and only if, the game-constant $k \leqq \bar{k}(x, y)$.

Proof. Proof of necessity: If $y \geqq x$, B can choose $t_0 = l(g)$ and wins for every game constant k.

In the case $y < x$, there exists an integer $p \geqq 0$ such that $x\alpha^{p+1} \leqq y > x\alpha^p$. We assume that $k > \bar{k}(x, y)$ and prove that B can avoid S. Let $\{g^n\}_{n=0,1,\cdots}$ be a descending sequence of intervals defined as follows:

$$g^0 = (1, \infty),\ g^1 = g,\ g^2 = g_{0,\underbrace{1\cdots1}_{p}},\ g^3 = g_{0,\underbrace{1\cdots1}_{p},0,\underbrace{1\cdots1}_{p}},\ \cdots$$

(i.e. g^{n+1} is obtained from g^n by adding one 0 and p 1's to the subscripts of g^n). Let now $g^n \to z$. We then have $\bar{k}\rho(z, f^n) = |f^n|$, for $n = 0, 1, \cdots$ and therefore, by $k > \bar{k}$

$$k\rho(z, f^n) > |f^n| \,. \tag{21}$$

By $x\alpha^{p+1} \leqq y$, we have

$$|g^n| \geqq |f^n| \,. \tag{22}$$

Now B chooses $t_0 = z$. If A makes $s_1 \in g_k$ (for some k) or $s_1 = l(g_k)$, then B avoids S by choosing $t_2, t_4, \cdots$ sufficiently small. Otherwise, $s_1 \in f^n$ for some n. B then moves to $s_2 = r(f^n)$ which by (21) satisfies (b). Evidently $t_2 < |f^n|$, and therefore from (22) and (a) follows $s_3 \in g^n$. Thus, choosing $t_4, t_6, \cdots$ sufficiently small, B wins.

Proof of sufficiency. By Remark 1 it suffices to show that the set $S_{x,y}$ satisfies (p_2). Now, since $y < x$ and $\bar{k}y < \alpha$, (p_2') is satisfied and by the lemma also (p_2'') is satisfied. Therefore (p_2) holds.

Theorem 2 solves the Banach problem for sets belonging to the Cantor class C. Putting $p = 0$ in the theorem we find, in particular, that the sets $S_{x,y}$ for $y \geqq x$ are avoidable for each $k > 0$. On the other hand the sets $S_{x,y}$ with $y < x$ are unavoidable for each $k \leqq \bar{k}(x, y)$. This can be formulated as follows:

REMARK 2. Sets $S_{x,y}$ for which $y = x$ separate, in the Cantor class C, all sets which are avoidable for every $k > 0$ from the others.

Since further, for $p = 0$ there is

$$\bar{k}(x, y) = \frac{(1 - x - y)(1 - x)}{y + x(1 - x - y)} = \frac{1 - x - y}{x + y}$$

we can obtain $\bar{k}(x, y)$ arbitrarily large (it is sufficient to choose x and $y < x$ sufficiently small). From Theorem 2 we thus obtain

REMARK 3. For every game-constant $k > 0$ there is a set $S_{x,y} \in C$ which is unavoidable.

Considering the symmetric sets, i.e. the sets $S_{x,y}$ for which $y = 1 - 2x$, then for x sufficiently close to $\frac{1}{2}$ (of course $x < \frac{1}{2}$) the condition $x\alpha^{p+1} \leqq y < x\alpha^p$, i.e. the condition $x^{p+2} \leqq 1 - 2x < x^{p+1}$ holds for sufficiently large p only (evidently $p = p(x)$). Hence $\bar{k} = \bar{k}(x, y) = \bar{k}(x, 1 - 2x) = [\{x(1 - x^{p+1})\}/(1 - 2x + x^{p+2})] \to \infty$ for $x \to \frac{1}{2}$. From Theorem 2 we thus obtain the following

REMARK 4. For each $k > 0$ there exists a symmetric unavoidable set.

Finally, since the only symmetric set for which $y = x$ is the Cantor discontinuum $S_{1/3,1/3}$, we obtain from Remark 2 the following

REMARK 5. The Cantor-discontinuum $S_{1/3,1/3}$ separates, in the class of symmetric sets, the sets which are avoidable for each $k > 0$ from the others.

The graph of the function $\bar{k}(x, 1-2x)$ is given in Fig. 6. The

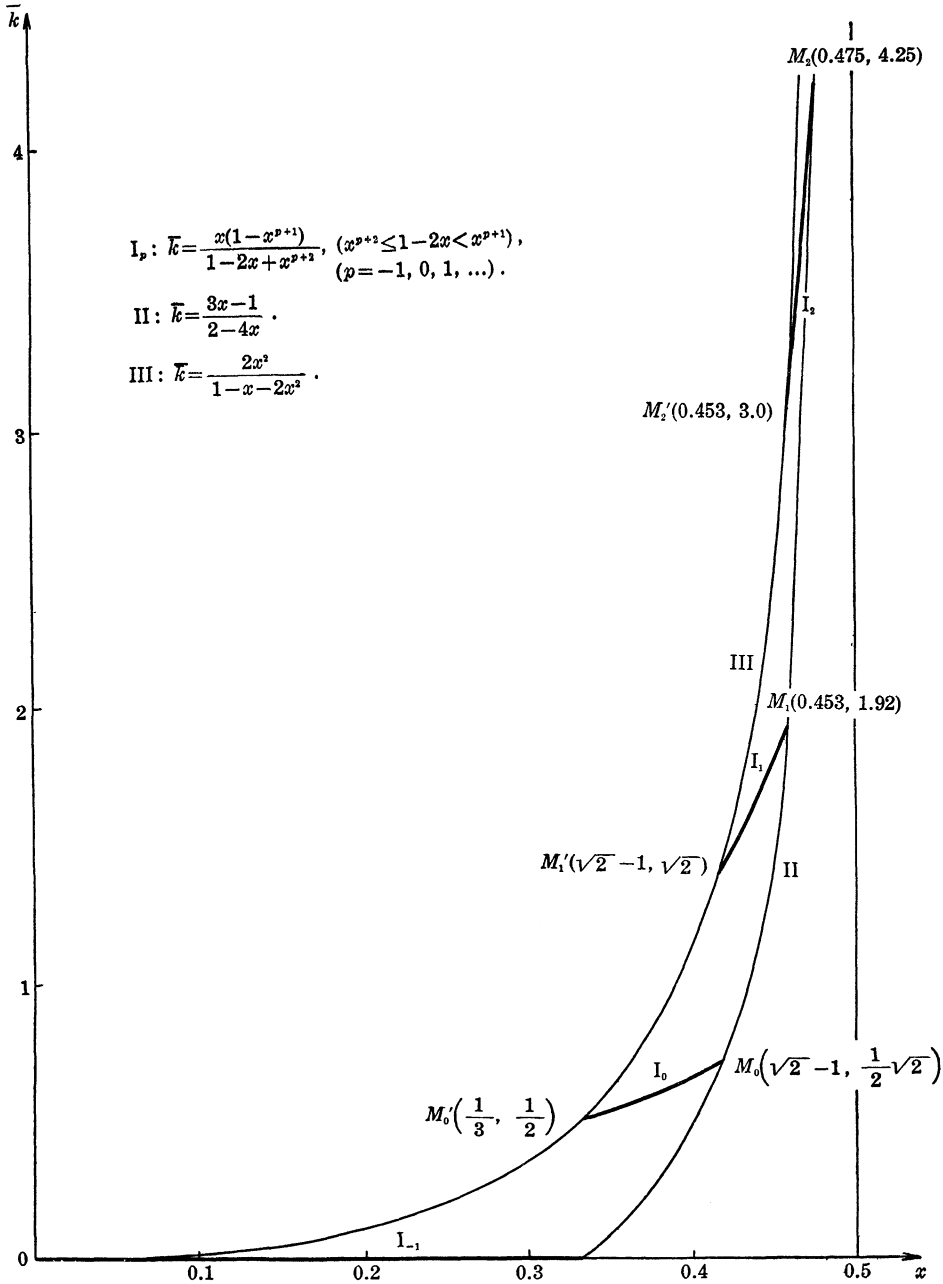

Fig. 6

points of discontinuity of this curve lie on the curves $\bar{k} = (3x - 1)/(2 - 4x)$ and $\bar{k} = 2x^2/(1 - x - 2x^2)$. The points M_p and M_p', $(p = 0, 1, \cdots)$ are the points of discontinuity of $\bar{k} = \{x(1 - x^{p+1})\}/(1 - 2x + x^{p+2})$ which lie on these curves respectively.

Note also that from the definition of $\bar{k}(x, y)$ it follows (see Fig. 2) that the lines $y = x\alpha^p$, $p = 0, 1, \cdots$ are lines of discontinuity of this function.

Finally, since for $x = 1/2$, $y = 1/8$ there is $x\alpha^2 \leqq y < x\alpha$ and thus $\bar{k}(1/2, 1/8) = 39/25$, we obtain

REMARK 6. The set $S_{1/2,1/8}$ constructed in [2] is unavoidable if and only if $k \leqq 39/25$.

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