

OSCILLATION CRITERIA FOR THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. This paper is concerned with the oscillatory properties of the third-order linear differential equation

$$(1.1) \quad y''' + P(x)y'' + Q(x)y' + R(x)y = 0$$

where $P(x)$, $Q(x)$, and $R(x)$ are functions of C and the primes denote differentiation with respect to x . In all theorems dealing with the adjoint

$$y''' - (Py)'' + (Qy)' - Ry = 0$$

of (1.1) we make the additional assumption that $P(x)$ and $Q(x)$ are functions of C'' and C' , respectively. Unless otherwise noted, the interval under consideration is $(0, \infty)$.

The oscillatory properties of equation (1.1) were first investigated in a classical paper by G. D. Birkhoff [1], which appeared in 1911. Further results were obtained in papers by Mammana [5] and Sansone [7]; the latter, which appeared in 1948, contains a complete bibliography. More recent work on this equation can be found in [2], [3], [8], and [9].

A solution of (1.1) will be called *oscillatory* if it has an infinity of zeros in $(0, \infty)$ and *nonoscillatory* if it has but a finite number of zeros in this interval. An equation is termed *oscillatory* if there exists at least one oscillatory solution, and *nonoscillatory* if all its solutions are nonoscillatory. This latter definition is necessary since an equation (1.1) may have both oscillatory and nonoscillatory solutions. Also, we say that (1.1) is nonoscillatory in (a, ∞) if none of its solutions has more than *two* zeros in (a, ∞) . The number two is essential since there always exist solutions of (1.1) which have zeros at two arbitrary points.

In the study of the second- and fourth-order differential equations the self-adjoint forms are of special importance. The self-adjoint form of the third-order equation is

$$(1.2) \quad y''' + py' + \frac{1}{2}p'y = 0.$$

The general solutions of (1.2) is $y = c_1u^2 + c_2uv + c_3v^2$, where $u(x)$ and $v(x)$ are linearly independent solutions of the second-order equation

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$y'' + \frac{1}{4}py = 0$, and the study of this case is therefore without further interest.

2. Conjugate points. Our treatment of the third-order equation will be based on the concept of conjugate points, which we now define (cf. [4]). Let $y(x)$ be a solution of (1.1) which vanishes at $x=a$ and has at least $n+2$ ($n \geq 1$) zeros in $[a, \infty)$. If we designate these zeros by a_1, a_2, \dots, a_{n+2} ($a = a_1 \leq a_2 \leq \dots \leq a_{n+2}$) then the *n*th conjugate point of a is defined to be the smallest possible value of a_{n+2} as $y(x)$ ranges over all possible solutions of (1.1) for which $y(a) = 0$. The existence of these conjugate points can be established by the following compactness argument.

We first assume that there exists at least one solution $y(x)$ of (1.1) which vanishes at $x = a$ and has at least $n+2$ zeros in $[a, \infty)$. If $u(x)$ and $v(x)$ are two linearly independent solutions of (1.1) which vanish at $x = a$, then all other solutions vanishing at this point can be written in the form $y(x) = Au(x) + Bv(x)$. If there are only a finite number of these solutions $y(x)$ which have $n+2$ zeros in $[a, \infty)$ then the existence of the *n*th conjugate point needs no proof. If there are an infinity of these solutions $y(x)$ we consider the sequence of solutions $\{y_s(x)\}$, where $y_s(x) = A_s u(x) + B_s v(x)$. If we further normalize these functions by the condition $A_s^2 + B_s^2 = 1$ then it is easy to see that the resulting class of solutions is locally uniformly bounded and equicontinuous. Hence there exists a subsequence which converges to a solution $y_0(x)$ of (1.1). Let b equal the greatest lower bound of the a_{n+2} for this particular sequence. Since any limit point of zeros of the $y_s(x)$ belonging to the convergent subsequence is a zero of $y_0(x)$, we have $y_0(b) = 0$. If $b = a$, then $y_0(x) \equiv 0$ since this solution would have at least a triple zero at $x = a$. But this is impossible since $A_s^2 + B_s^2 = 1$, so that $b > a$ and this proves the existence of the *n*th conjugate point.

We call the solution $y_0(x)$ which produces the *n*th conjugate point the *extremal solution*. We now show that there can be at most two essentially different extremal solutions associated with any conjugate point $\eta_n(a)$ (two solutions will be called *essentially different* if they are not constant multiples of each other). This assertion is an immediate consequence of the following lemma.

LEMMA 2.1. *If $y_0(x)$ is the extremal solution of (1.1) and if $\eta_n(a)$ is the *n*th conjugate point of a , then the total number of zeros of $y_0(x)$ at $x = a$ and $x = \eta_n(a)$ (counting multiplicities) is at least three.*

Proof. Assume that $y_0(x)$ has simple zeros at both $x = a$ and $x = \eta_n(a)$ and, without loss of generality, let $y_0(x)$ be positive in $(b, \eta_n(a))$ where b is the first zero of $y_0(x)$ to the left of $\eta_n(a)$. Let $v(x)$ be a solution

of (1.1) such that $v(a) = 0$ and $v(\eta_n(a)) \neq 0$ and consider the function $w(x) = y_0(x) + \varepsilon v(x)$ where ε is sufficiently small. Then $w(a) = 0$ and the zeros of $w(x)$ are close to the zeros of $y_0(x)$. If we choose $\varepsilon v(\eta_n(a))$ to be negative then, since $w(\eta_n) = \varepsilon v(\eta_n)$ and $y(x) > 0$ in (b, η_n) , the $(n + 2)^{na}$ zero of $w(x)$ will occur before the $(n + 2)^{na}$ zero of $y_0(x)$. But this is absurd since $y_0(x)$ is the extremal solution. Hence $y_0(x)$ has a double zero either at a or at $\eta_n(a)$ (or possibly at both points).

By the uniqueness theorem of equation (1.1), all solutions which have a double zero at $x = a$ are constant multiples of each other. Hence, by Lemma 2.1, there are at most two essentially different extremal solutions associated with any conjugate point $\eta_n(a)$.

It may be conjectured that the extremal solutions are unique. This conjecture is easy to prove for the first conjugate point. For assume that this solution is not unique, that is, let $u(x)$ be an extremal solution such that $u(a) = u'(a) = u(\eta_1) = 0$ and let $v(x)$ be another extremal solution such that $v(a) = v(\eta_1) = v'(\eta_1) = 0$. Then it is not difficult to see that there exists a solution $y(x) = u(x) - \lambda v(x)$ which has three zeros in $[a, \eta_1]$, and is such that the zeros at a and $\eta_1(a)$ are simple. But this contradicts Lemma 2.1.

The distribution of the conjugate points is intimately related to the oscillatory properties of equation (1.1). As examples show, the oscillatory behavior of this equation may be rather complicated unless a distinction is made between a number of fundamentally different cases. The nature of these cases is reflected in the following properties of the conjugate points. (a) The conjugate points are distinct. This assumption introduces a certain amount of regularity in the separation pattern of the zeros of solutions of (1.1). (b) If the extremal solution associated with $\eta_n(a)$ has a double zero at $x = a$, the extremal solution belonging to $\eta_n(b)$ has a double zero at $x = b$. We define two classes of equations (1.1) for which these assumptions are satisfied, and which will be referred to as Class I and Class II, respectively.

Equations of Class I: An equation (1.1) is said to be of Class I (C_I), if any of its solutions $y(x)$ for which $y(a) = y'(a) = 0$, $y''(a) > 0$ ($0 < a < \infty$) satisfies $y(x) > 0$ in $(0, a)$.

Equations of Class II: An equation (1.1) is said to be of Class II (C_{II}), if any of its solutions $y(x)$ for which $y(a) = y'(a) = 0$, $y''(a) > 0$ ($0 < a < \infty$) satisfies $y(x) > 0$ in (a, ∞) .

We now derive a number of criteria which make it possible to decide whether a given equation (1.1) belongs to one of these classes. The following theorem first appeared in [5]. Generalizations of this theorem are given in [7] and [9].

THEOREM 2.2. *If, in the equation*

$$(2.1) \quad y''' + py' + qy = 0,$$

$2q - p' > 0$ ($2q - p' < 0$), except at isolated points at which $2q - p'$ may vanish, then (2.1) is of C_I ((2.1) is of C_{II}).

Proof. Let $y(x)$ be a solution of (2.1) such that $y(b) = y'(b) = 0$ and assume that (2.1) is not of Class I, that is let $x = a$ ($a < b$) be a zero of $y(x)$. Multiplying (2.1) by $y(x)$ and integrating from a to b , we obtain,

$$(2.2) \quad [yy'' - \frac{1}{2}y'^2 + \frac{1}{2}py^2]_a^b - \frac{1}{2} \int_a^b (p' - 2q)y^2 dx = 0,$$

$$- [y'(a)]^2 = \int_a^b (2q - p')y^2 dx.$$

This contradiction proves the theorem.

We note that if $2q - p' \equiv 0$ in any interval then (2.1) is selfadjoint and, by equation (2.2), if $y(x)$ has a double zero in this interval then all zeros in this interval are double zeros. Hence the equation is neither of Class I nor Class II.

THEOREM 2.3. *If the second-order differential equation*

$$(2.3) \quad y'' + p(x)y = 0$$

is nonoscillatory in $(0, \infty)$ and if $q \geq 0$ ($q \leq 0$) then (2.1) is of C_I ((2.1) is of C_{II}).

A stronger result than Theorem 2.3 can be proved. It is stated here as a separate lemma for future reference. Theorem 2.3 is an immediate consequence of this lemma.

LEMMA 2.4. *Let $y(x)$ be a solution of (2.1) such that $y'(a) = 0$, $y(a) \geq 0$, and $y''(a) > 0$. If the second-order equation (2.3) is nonoscillatory in $(0, \infty)$ and if $q \geq 0$ then $y'(x) < 0$ for $0 < x < a$ (if $q \leq 0$ then $y'(x) > 0$ for $x > a$).*

Proof. Assume that there exists a point $x = \alpha$ ($0 < \alpha < a$) such that $y'(\alpha) = 0$ and let α be the first such point to the left of $x = a$. Then $y'(x) < 0$ in (α, a) . Multiplying (2.1) by $y'(x)$ and integrating from α to a , we obtain,

$$(2.4) \quad \int_{\alpha}^a py'^2 dx + \int_{\alpha}^a qyy'dx = \int_{\alpha}^a y''^2 dx.$$

As shown in [6], if $u'' + pu = 0$ is nonoscillatory in $(0, \infty)$ and $v(x)$ is

any function of class D' such that $v(a) = 0$ then $\int_{\alpha}^a v'^2 dx > \int_{\alpha}^a p v^2 dx$. Since $y'(\alpha) = 0$, we have $\int_{\alpha}^a y''^2 dx > \int_{\alpha}^a p y'^2 dx$, and substituting this into (2.4), we obtain,

$$\int_{\alpha}^a p y'^2 dx + \int_{\alpha}^a q y y' dx > \int_{\alpha}^a p y'^2 dx$$

or

$$(2.5) \quad \int_{\alpha}^a q y y' dx > 0 .$$

Since $q(x)$ and $y(x)$ are positive and $y'(x)$ is negative in (α, a) , the inequality (2.5) is clearly impossible and this proves the lemma.

THEOREM 2.5. *If $q \leq 0 (q \leq 0)$ then the equations*

$$(2.6) \quad (r y')'' + q y = 0 , \quad r(x) > 0$$

and

$$(2.7) \quad (r y'')' + q y = 0 , \quad r(x) < 0$$

are of Class I (of Class II).

We prove the theorem for equation (2.6); the proof for (2.7) is analogous. Let $y(x)$ be a solution of (2.6) such that $y(b) = y'(b) = 0$ and, without loss of generality, let $y''(b)$ be positive. Assume (2.6) is not of Class I, that is, let $x = a$ be the first zero of $y(x)$ to the left of b . Then there exists at least one point α where $y'(x)$ vanishes. Let α be the first such point to the left of b . Integrating (2.6) from x to b , we obtain,

$$(2.8) \quad [r(x)y'(x)]' = (r y')' \Big|_b + \int_x^b q(t)y(t)dt .$$

Again integrating (2.8) from x to b , we have

$$(2.9) \quad -r(x)y'(x) = (b - x)r(b)y''(b) + \int_x^b (t - x)q(t)y(t)dt .$$

We now substitute $x = \alpha$ in (2.9) to obtain

$$(2.10) \quad 0 = (b - \alpha)r(b)y''(b) + \int_{\alpha}^b (t - \alpha)q(t)y(t)dt .$$

But the right-hand side of (2.10) is positive. The contradiction in (2.10) proves the theorem.

In § 5 we shall return to a more detailed discussion of the classes of equations (1.1) defined in Theorems 2.2 and 2.3. In this and the following two sections, we consider some of the properties common to

all such equations which belong to either Class I or Class II.

It follows from Lemma 2.1 that the n conjugate points of any point a of an equation of Class I are the zeros of a solution $u(x)$ which has a double zero at a . If this solution $u(x)$ is normalized by the condition $u''(a) = 1$, it will be called the *principal solution* $y(x, a)$ of (1.1). Thus, the n conjugate points of (1.1) (if they exist) are the zeros of $y(x, a)$.

Again, by Lemma 2.1, the n conjugate points of an equation of Class II are the points where the n essentially unique solutions $y_\nu(x)$, which vanish at $x = a$, have a double zero.

These results are summarized in the following two theorems.

THEOREM 2.6. *If equation (1.1) is of C_I and if $y(x)$ is a solution of (1.1) which vanishes at $x = a$ and has at least $n + 2$ zeros in $[a, \infty)$, then the principal solution $y(x, a)$ has $n + 2$ zeros at $a, \eta_1, \eta_2, \dots, \eta_n$ ($a < \eta_1 < \eta_2 < \dots < \eta_n$), (counting the double zero at $x = a$) and any other solution vanishing at $x = a$ has fewer than $\nu + 2$ zeros in $[a, \eta_\nu(a)]$.*

THEOREM 2.7. *If equation (1.1) is of C_{II} and $y(x)$ is a solution of (1.1) which vanishes at $x = a$ and has at least $n + 2$ zeros in $[a, \infty)$ then there exist n points $\eta_1, \eta_2, \dots, \eta_n$ ($\eta_1 < \eta_2 < \dots < \eta_n$) and n essentially unique solutions $y_\nu(x)$ such that*

- (a) $y_\nu(x)$ has a simple zero at $x = a$ and a double zero at $x = \eta_\nu(a)$.
- (b) $y_\nu(x)$ has exactly $\nu + 2$ zeros in $[a, \eta_\nu(a)]$ (properly counting multiplicities),
- (c) any other solution which vanishes at $x = a$ has fewer than $\nu + 2$ zeros in $[a, \eta_\nu(a)]$.

In the study of equation (1.1), its adjoint

$$(2.11) \quad -y''' + (Py)'' - (Qy)' + Ry = 0$$

plays an important role. The following theorem shows the relationship between their conjugate points.

THEOREM 2.8. *If a third-order differential equation is of Class I then its conjugate points are identical with the conjugate points of its adjoint.*

This theorem also applies to the case when $\eta_n(a) = \infty$, that is, when the n th conjugate point fails to exist. To prove the theorem, we need the following lemma.

LEMMA 2.9. *A third order differential equation is of Class I if, and only if, its adjoint is of Class II.*

Proof. Let $L(u) = 0$ and $M(v) = 0$ represent equations (1.1) and (2.11), respectively. Then

$$(2.12) \quad \int_a^b [vL(u) - uM(v)]dx = [vu'' - v'u' + v''u - (Pv)'u + Pvu' + Qvu]_a^b = 0 .$$

Let $u(x)$ be a solution of (1.1) such that $u(b) = u'(b) = 0$ and assume that (1.1) is not of Class I, that is, let $x = a$ ($a < b$) be a zero of $u(x)$. Construct the solution $v(x)$ of (2.11) such that $v(a) = v'(a) = 0$. Substituting these two solutions into (2.12) we find

$$(2.13) \quad v(b)u''(b) = 0 .$$

Since $u''(b) \neq 0$ (otherwise it would follow from the existence theorem that $u(x) \equiv 0$), (2.13) implies that $v(b) = 0$. This proves that if (1.1) is not of Class I, the adjoint (2.11) is not of Class II. Conversely, if we assume that (2.11) is not of Class II, we prove, by equation (2.12), that (1.1) is not of Class I. This completes the proof of the lemma.

We now prove Theorem 2.8. Let $u(x)$ and $v(x)$ be two linearly independent solutions of (2.11) which vanish at $x = a$ (e.g., let $u(a) = u'(a) = 0$, $u''(a) = 1$ and $v(a) = v''(a) = 0$, $v'(a) = 1$). Then all other solutions which vanish at $x = a$ can be written in the form $w(x) = Au(x) + Bv(x)$. By Lemma 2.9, if (1.1) is of C_I then (2.11) is of C_{II} . Hence, by Theorem 2.7, the conjugate points of (2.11) are the points where some $w(x)$ has a double zero. Using a result from § 4 (Theorem 4.4), we see that the converse is also true, that is, if $w(x)$ has a double zero at some point $x = b$, then b is a conjugate point of (2.11). Since $w(x)$ has a double zero if, and only if, the function

$$(2.14) \quad \sigma_1(x) = u(x)v'(x) - u'(x)v(x)$$

vanishes, the conjugate points of (2.11) are characterized by $\sigma_1(x) = 0$. It can be shown [1] that the function

$$(2.15) \quad \sigma(x) = \exp\left(\int^x P(x)dx\right)\sigma_1(x)$$

is a solution of the adjoint of (2.11), that is, of equation (1.1). It is easy to show that $\sigma(a) = \sigma'(a) = 0$, so that $\sigma(x)$ is a constant multiple of the principal solution $y(x, a)$ of (1.1). Since $\sigma(x)$ vanishes at the same points as $\sigma_1(x)$, the proof of the theorem is complete.

We complete the section by proving the following uniqueness theorem. (Sansone [7] proves this theorem for the equation (2.1) with the restriction that $2q - p'$ be of one sign.)

THEOREM 2.10. *If $u(x)$ and $v(x)$ are two nontrivial solutions of a third-order differential equation of Class I (or C_{II}) which are not*

constant multiples of each other, then $u(x)$ and $v(x)$ cannot have two zeros in common.

We first note that there always exists a solution of the third-order equation which vanishes at two arbitrary points. There may then, in general, be two essentially different solutions which have two zeros in common. In fact it is easy to construct examples in which two different solutions have an infinity of zeros in common. For example, the equation $y''' + y' = 0$ has the two particular solutions $u(x) = 1 - \cos x$ and $v(x) = \sin x$.

If the zeros in Theorem 2.10 are not distinct, the proof of the theorem follows from the uniqueness theorem. Assume then that $u(a) = u(b) = v(a) = v(b) = 0$ where $a < b$. By the definition of equations of Class I we know that $u'(b) \neq 0$ and $v'(b) \neq 0$. Therefore

$$w(x) = v'(b)u(x) - u'(b)v(x)$$

is a nontrivial solution of (1.1). Clearly $w(a) = w(b) = w'(b) = 0$, but this is absurd since equation (1.1) is of Class I. This proves the theorem. The proof of the theorem for equations of Class II is analogous.

3. Equations of Class I. The first part of this section deals with separation theorems and the second part is concerned with comparison theorems. We begin by proving a simple separation theorem of the Sturm type for equations of Class I.

THEOREM 3.1. *If (1.1) is of C_I and if $u(x)$ and $v(x)$ are two essentially different nontrivial solutions of (1.1) such that $u(a) = v(a) = 0$, then the zeros of $u(x)$ and $v(x)$ separate each in other (a, ∞) .*

In the proof of Theorem 3.1, we use the following elementary lemma [4].

LEMMA 3.2. *Let $u(x)$ and $v(x)$ be of Class C' in (a, b) , and let $v(x)$ be of constant sign in this interval. If $x = \alpha$ and $x = \beta$ ($a < \alpha < \beta < b$) are consecutive zeros of $u(x)$, then there exists a constant λ such that the function $u(x) - \lambda v(x)$ has a double zero in (α, β) .*

To prove Theorem 3.1, let α and β ($a < \alpha < \beta$) be two consecutive zeros of $u(x)$ and assume that $v(x)$ does not vanish in $[\alpha, \beta]$. Then, by Lemma 3.2, there exists a λ such that $w(x) = u(x) - \lambda v(x)$ has a double zero at some point in (α, β) . By hypothesis, $w(a) = 0$, and this leads to a contradiction since equation (1.1) is of Class I. Thus $v(x)$ must vanish at least once in (α, β) . By interchanging $u(x)$ and $v(x)$ we

easily prove that $v(x)$ vanishes only once between two consecutive zeros of $u(x)$. We might also remark that, by Theorem 2.10, $u(x)$ and $v(x)$ can have no common zeros other than the one at $x = a$.

The following corollary is an immediate consequence of Theorems 2.6 and 3.1.

COROLLARY 3.2. *If equation (1.1) is of C_I and if $u(x)$ is a solution of (1.1) which vanishes at $x = a$, then the conjugate points $\eta_n(a)$ separate the zeros of $u(x)$.*

Again, by Theorem 3.1, it is easy to show that if (1.1) is oscillatory then any solution of this equation which vanishes at least once, is oscillatory. To prove this assertion, let $v(x)$ be an oscillatory solution of (1.1) which vanishes at the points $x_1 \leq x_2 \leq x_3 \cdots$ and let $u(x)$ be another solution vanishing at $x = a$. We can construct the essentially different solution $w(x)$ such that $w(a) = w(x_1) = 0$. (If $a = x_1$, the proof of our assertion follows immediately by Theorem 3.1.) Applying Theorem 3.1, first to $w(x)$ and $v(x)$, and then to $w(x)$ and $u(x)$, we have proved our assertion. We state this as a separate theorem.

THEOREM 3.4. *If equation (1.1) is of C_I and if (1.1) is oscillatory, then any solution of this equation which vanishes at least once is oscillatory.*

Theorem 3.4 is proved in [8] for the equation $y''' + qy = 0$ ($q(x) \geq 0$). Theorem 2.6 implies the following corollary.

COROLLARY 3.5. *If equation (1.1) is of C_I , then (1.1) is oscillatory if, and only if, there exists an infinity of conjugate points $\eta_n(a)$ ($0 < a < \infty$).*

We can also show the following partial converse of Theorem 3.4.

THEOREM 3.6. *Let equation (1.1) be of Class I and let $u(x)$ be a solution of (1.1) such that $u(a) = 0$. If $u(x) \neq 0$ for $x > a$, then no solution of (1.1) can have more than two zeros in (a, ∞) .*

Proof. Assume that there exists a solution of (1.1) which has three zeros in (a, ∞) . By Theorem 3.1 we can consider the solution $v(x)$ where $v(x_1) = v'(x_1) = v(x_2) = 0$ ($a < x_1 < x_2$). Without loss of generality, let $v''(x_1)$ be positive so that $v(x) \geq 0$ for $x < x_2$. Again, without loss of generality, let $u(x)$ be positive for $x > a$. By Lemma 3.2, there exists a function $w(x) = u(x) - \lambda v(x)$ which has a double zero at some point between x_1 and x_2 . Since $u(x)$ and $v(x)$ are both positive in (x_1, x_2) we see that λ is positive. Hence

$$w(x_1) = u(x_1) > 0 \text{ and } w(a) = -\lambda v(a) < 0 ,$$

which implies a zero of $w(x)$ to the left of a double zero. But this is absurd, since equation (1.1) is of Class I, and the theorem is proved.

COROLLARY 3.7. *If equation (1.1) is of C_I , then between two consecutive zeros of any solution of (1.1) there are at most two zeros of any other solution.*

This corollary follows directly from Theorem 3.6.

We now show that the zeros of two principal solutions (that is, the conjugate points), say $u(x) = y(x, a)$ and $v(x) = y(x, b)$, must always separate. This separation, however, need not be of the simple Sturm type. The zeros may separate in pairs, that is between two consecutive zeros of $u(x)$ there may be two zeros of $v(x)$, and conversely.

THEOREM 3.8. *Let equation (1.1) be of C_I and let $u(x) = y(x, a)$ and $v(x) = y(x, b)$ be two principal solutions of (1.1) where $b > a$ and let $\eta_n(a)$ and $\eta_n(b)$ be the n th conjugate points of $u(x)$ and $v(x)$, respectively. If $v(x)$ vanishes exactly once in any interval $(\eta_k(a), \eta_{k+1}(a))$, or if $u(x)$ vanishes exactly once in any interval $(\eta_k(b), \eta_{k+1}(b))$, then the zeros of $u(x)$ and $v(x)$ interlace in $(\eta_k(a), \infty)$, or $(\eta_k(b), \infty)$. Otherwise the zeros separate in pairs.*

To prove the first part of the theorem, let $v(x)$ vanish exactly once in some interval $(\eta_k(a), \eta_{k+1}(a))$. Assume there exists a $j > k$ such that $v(x)$ does not vanish in the interval $(\eta_j(a), \eta_{j+1}(a))$. By Lemma 3.2, there exists a function $w(x) = u(x) - \lambda v(x)$ which has a double zero in this interval. Then $w(\eta_k(a)) = -\lambda v(\eta_k(a))$ and $w(\eta_{k+1}(a)) = -\lambda v(\eta_{k+1}(a))$. Since $v(x)$ vanished once in the interval $(\eta_k(a), \eta_{k+1}(a))$ we know that $v(\eta_k(a))$ and $v(\eta_{k+1}(a))$ must be of opposite sign, and this implies that $w(x)$ vanishes to the left of a double zero, which is incompatible with the hypothesis that (1.1) is of Class I. If we assume that $u(x)$ does not vanish in an interval $(\eta_j(b), \eta_{j+1}(b))$ where $j > k$, we arrive at the same conclusion. Therefore the zeros of $u(x)$ and $v(x)$ interlace in $(\eta_k(a), \infty)$.

If the zeros of $u(x)$ and $v(x)$ do not separate singly then, by Corollary 3.7, they must separate in pairs.

Theorem 3.8 enables us to bound the conjugate points of b by the conjugate points of a .

COROLLARY 3.9. *Let equation (1.1) be of Class I and let $\eta_n(a)$ and $\eta_n(b)$ be the n th conjugate points of a and b , respectively. If $\eta_k(a) < b < \eta_{k+1}(a)$ (where $\eta_0(a) = a$), then*

$$\eta_{k+n}(a) \leq \eta_n(b) \leq \eta_{k+n+2}(a) .$$

We can now prove a comparison theorem using Corollary 3.9. We compare equation (1.1) with

$$(3.1) \quad y''' + P(x)y'' + Q(x)y' + R_1(x)y = 0$$

where

$$(3.2) \quad R_1(x) \geq R(x) .$$

THEOREM 3.10. *Let equation (1.1) be of Class I. If $\eta_n(a)$ and $\eta'_n(a)$ are the n th conjugate points of (1.1) and (3.1) respectively, and if the coefficients of these equations are related by (3.2), then equation (3.1) belongs to Class I and*

$$(3.3) \quad \eta'_n(a) \leq \eta_{2n-1}(a) .$$

To prove the first part of this theorem, assume the contrary, that is, assume that equation (3.1) does not belong to C_I . Let $v(x)$ be a solution of (3.1) such that $v(b) = v'(b) = 0$, and $v''(b) > 0$ and let $x = a$ ($0 < a < b$) be the first zero of $v(x)$ to the left of $x = b$. Construct the solution $u(x)$ of the adjoint of (1.1) (that is, equation (2.11)) such that $u(a) = u'(a) = 0$ and $u''(a) > 0$. By Lemma 2.9, equation (2.11) belongs to C_{II} , so that $u(x) > 0$ for $x > a$. Multiplying (2.11) by $v(x)$ and (3.1) by $u(x)$, subtracting and integrating between a and b , we obtain

$$[vu'' - v'u' + v''u + Puv' - (Pu)'v + Qvu]_a^b = - \int_a^b (R_1 - R)uv dx ,$$

$$v''(b)u(b) = - \int_a^b (R_1 - R)uv dx .$$

Clearly, the left-hand side of this last equation is positive while the right-hand side is negative. This contradiction proves that equation (3.1) belongs to C_I . Using this fact, we now prove the second part of the theorem.

Let $u(x) = y(x, a)$ be a principle solution of (1.1) and let $b = \eta_1(a)$ be the first zero of $u(x)$ in (a, ∞) . Then, by Theorem 2.8, we can construct the extremal solution $w(x)$ of the adjoint of (1.1) such that $w(a) = w(b) = w'(b) = 0$ and $w(x) \neq 0$ in (a, b) . Let $v(x) = y(x, a)$ be a principal solution of (3.1). We first show that $v(x)$ vanishes in (a, b) . To this end, we assume that $v(x) > 0$ in (a, b) (by construction $v(x)$ cannot be strictly negative in this interval). Multiplying (2.11) by $v(x)$ and (3.1) by $w(x)$, subtracting, and integrating between a and b , we obtain.

$$(3.4) \quad v(b)w''(b) = - \int_a^b (R_1 - R)vw dx .$$

From the construction of $w(x)$, it is clear that $w''(b) > 0$ so that the left-hand side of (3.4) is positive (or zero if $v(b) = 0$) while the right-

hand side of (3.4) is negative. This contradiction proves that $v(x)$ must vanish in (a, b) , that is,

$$(3.5) \quad \eta'_1(a) \leq \eta_1(a) .$$

We now prove (3.3) by induction. Let $b_1 = \eta'_1(a)$ and let $u_1(x) = y(x, b_1)$ and $v_1(x) = y(x, b_1)$ be principal solutions of (1.1) and (3.1), respectively. By Theorem 3.1, the zeros of $v(x)$ and $v_1(x)$ separate in (b_1, ∞) . Since $v_1(x)$ has a double zero and $v(x)$ has a simple zero at $x = b_1$, it can be shown, by a slight modification in the proof of Lemma 3.2 (cf. [4]), that the first zero of $v(x)$ to the right of b_1 occurs before the first zero of $v_1(x)$. This implies that

$$(3.6) \quad \eta'_{n+1}(a) < \eta'_n(b_1) < \eta'_{n+2}(a) .$$

Applying Corollary 3.9, we have,

$$(3.7) \quad \eta_n(a) < \eta_n(b_1) < \eta_{n+2}(a) .$$

Assuming now that (3.3) is true for $n = k$, we prove it to be true for $n = k + 1$. The first inequality in (3.6) gives

$$\eta'_{k+1}(a) < \eta'_k(b_1)$$

and, by the assumption,

$$\eta'_k(b_1) < \eta_{2k-1}(b_1) .$$

The second inequality of (3.7) gives

$$\eta_{2k-1}(b_1) < \eta_{2k+1}(a) .$$

Combining these last three inequalities, we obtain

$$\eta'_{k+1}(a) < \eta_{2k+1}(a) ,$$

which proves (3.3) for $n = k + 1$. Since we have already proved it to hold for $n = 1$, the inequality (3.3) is true for all n .

Theorem 3.10 shows that if (1.1) has an infinity of conjugate points the same must be true for (3.1). In view of Corollary 3.5, this implies the following result.

THEOREM 3.11. *Let equation (1.1) be of Class I and let the coefficients of (1.1) and (3.1) be related by (3.2). If equation (1.1) is oscillatory, then equation (3.1) is likewise oscillatory.*

4. Equations of Class II. Since an equation of Class I is essentially (that is, if due allowance is made for the length of the corresponding intervals) equivalent to an equation of Class II in which the variable x has been replaced by $c - x$ ($c = \text{const.}$), results for equations of Class II can be obtained, by appropriate transformations, from

the corresponding results of equations of Class I. We therefore merely state here the theorems in question.

THEOREM 4.1. *Let equation (1.1) be of Class II and let $u(x)$ and $v(x)$ be two essentially different nontrivial solutions of (1.1) If $u(a) = v(a) = 0$, then the zeros of $u(x)$ and $v(x)$ separate in $(0, a)$.*

THEOREM 4.2. *If equation (1.1) is of Class II, then between two consecutive zeros of any solution of (1.1) there are at most two zeros of any other solution.*

THEOREM 4.3. *If equation (1.1) is of Class II and if $u(x)$ and $v(x)$ are two essentially different nontrivial solutions of (1.1) such that $u(a) = v(b) = 0$ ($a < b$), then the number of zeros of $u(x)$ and $v(x)$ differ by at most two in $(0, a)$*

Theorem 2.7 shows that the extremal solution $y_\nu(x)$ of (1.1) has a double zero at the ν th conjugate point. The following theorem shows the converse of this.

THEOREM 4.4. *Let equation (1.1) be of Class II and let $u(x)$ be a solution of (1.1) such that $u(a) = 0$ and let $u(x)$ have $n + 2$ zeros in $[a, b]$. If $u(b) = u'(b) = 0$, then b is the n th conjugate point of a .*

Proof. Assume that the theorem is not true, that is, assume that $\eta_n(a) < b$. Then, by Theorem 2.7, there exists a solution $v(x)$ such that $v(a) = v(\eta_n) = v'(\eta_n) = 0$. Since $u'(a)$ and $v'(a)$ are not zero, there exists a constant λ such that the solution $w(x) = u(x) - \lambda v(x)$ has a double zero at $x = a$. Without loss of generality, let $u'(a)$ and $v'(a)$ be positive, so that λ is also positive. This implies that $u(x)$ and $\lambda v(x)$ do not intersect for $x > a$, for if they did, then $w(x)$ would have a zero to the right of a double zero. By Theorem 4.2, there are at most two zeros of $\lambda v(x)$ between two consecutive zeros of $u(x)$. Since $u(x)$ has a double zero at $x = b$, a simple count shows that $v(x)$ has at most $n + 1$ zeros in $[a, b]$ and $v(x)$ is therefore not the extremal solution for $\eta_n(a)$. Hence $b = \eta_n(a)$.

In order to obtain more information about the distribution of the conjugate points for equations of Class II, we now construct the extremal solution leading to the n th conjugate point. Assume that there exists a solution $u(x)$ of (1.1) which vanishes at $x = a$ and has at least $n + 2$ zeros in $[a, \infty)$, say at a_1, a_2, \dots, a_{n+2} ($a = a_1 < a_2 < \dots < a_{n+2}$). Let $v(x)$ have a double zero at $x = a$ and let $v(x)$ be positive for $x > a$. If we assume that the last zero of $u(x)$ is not a double zero, then, by Lemma 3.2, the solution $w(x) = u(x) - \lambda v(x)$ has a double zero in (a_{n+1}, a_{n+2}) , say at $x = \alpha$. Without loss of generality, we can also assume

that $u(x)$ is positive in this interval. Then λ is positive and it is easy to see that $z = \lambda v(x)$ intersects every positive arch of $v(x)$ twice. For if it did not, the function $w_1(x) = u(x) - \lambda_1 v(x)$ would, by Lemma 3.2, have a double zero at some point in this interval. Clearly $\lambda_1 < \lambda$, so that $z_1 = \lambda_1 v(x)$ intersects the arch of $u(x)$ in the interval (a_{n+1}, a_{n+2}) , which implies that $w_1(x)$ has a zero to the right of a double zero. But this is impossible since equation (1.1) is of Class II. A simple count shows that $w(x)$ has $n + 2$ zeros in $[a, \alpha]$. By Theorem 4.4, $w(x)$ is the extremal solution. We have proved the following theorem.

THEOREM 4.5. *If equation (1.1) is of C_{II} and if $y(x)$ is a solution of (1.1) which vanishes at $x = a$ and has at least $n + 2$ zeros in $[a, \infty)$ then the n conjugate points of a separate the zeros of $y(x)$.*

The connection between the conjugate points and the oscillatory and nonoscillatory behavior of equations of Class II is illustrated by the following theorem which is analogous to Corollary 3.5 for equations of Class I (but does not follow from it).

THEOREM 4.6. *If equation (1.1) is of C_{II} , then (1.1) is oscillatory if, and only if, for every positive a there exists an infinity of conjugate points $\eta_n(a)$.*

By definition, equation (1.1) is oscillatory if there exists at least one solution with an infinity of zeros. Hence if $y(x)$ is an oscillatory solution, for every point a , there exists a point b and an arbitrary number n , such that $y(x)$ has n zeros in (a, b) . If we let $u(x)$ be a solution of (1.1) such that $u(a) = u(b) = 0$ then, by Theorem 4.3, $u(x)$ has at least n zeros in $[a, b]$. Since n is arbitrary, Theorem 4.5 implies an infinity of conjugate points.

Conversely, assume that there exists an infinity of conjugate points. By Theorem 2.7, there exist solutions of (1.1) with arbitrary many zeros in (a, ∞) . This, however, is not sufficient to prove the existence of a solution with an infinity of zeros. To prove this, we note, as in § 2, that all solutions of (1.1) which vanish at $x = a$ can be written in the form $y(x) = c_1 u(x) + c_2 v(x)$ where $u(x)$ and $v(x)$ are linearly independent solutions which vanish at $x = a$. Again, as in § 2, if we normalize $y(x)$ by the condition $c_1^2 + c_2^2 = 1$, we can find a subsequence of $\{y_\nu(x)\}$ which converges uniformly to a solution $y_0(x)$ of (1.1). By Theorem 4.5, all solutions $y_\nu(x)$ ($\nu > n$) vanish in the interval (η_{n-1}, η_n) . Since any limit point of zeros of the $y_\nu(x)$ belonging to the convergent subsequence is a zero of $y_0(x)$, we see that $y_0(x)$ must have a zero between two consecutive conjugate points. This completes the proof of Theorem 4.6.

We have now shown that when equation (1.1) is either of Class I or Class II then it is oscillatory if, and only if, it has an infinity of

conjugate points. This fact, together with Theorem 2.8 and Lemma 2.9, proves the following theorem.

THEOREM 4.7. *A third-order differential equation of Class I (or C_{II}) is oscillatory if, and only if, its adjoint is oscillatory.*

Theorem 4.7 enables us to prove comparison theorems for equations of Class II by using comparison theorems for equations of Class I.

THEOREM 4.8. *Let equations (1.1) and (3.1) be of Class II, and let*

$$R_1(x) \leq R(x).$$

If (1.1) is oscillatory, then (3.1) is oscillatory.

The proof is immediate. Since (1.1) and (3.1) are of Class II, by Lemma 2.9, their adjoints are of Class I. Therefore, Theorem 3.11 applies to the adjoints and, by Theorem 4.7, the oscillatory behavior of (1.1) and (3.1) is characterized by the oscillatory behavior of their adjoints.

5. Examples. The examples of equations of Class I and Class II mentioned in § 2 are now discussed in greater detail. We first note that equation (1.1) can be transformed into

$$(5.1) \quad y''' + p(x)y' + q(x)y = 0$$

by a simple transformation of the form $y(x) = v(x) \exp\left(-\frac{1}{3} \int^x P(x) dx.\right)$

The adjoint to (5.1) is

$$(5.2) \quad y''' + py' + (p' - q)y = 0 .$$

1. We first consider the case in which the coefficients of equation (5.1) are such that $2q - p'$ is of one sign in $(0, \infty)$ (cf. Theorem 2.2). If, in addition, we assume that $p(x) \geq 0$, then the comparison theorems of §§ 3 and 4 can be strengthened. We compare equation (5.1) with the equation

$$(5.3) \quad y''' + p_1y' + q_1y = 0$$

where

$$(5.4) \quad p_1 \geq p \geq 0 \text{ and } q_1 \geq q$$

and

$$(5.5) \quad 2q - p' > 0 \text{ and } 2q_1 - p'_1 > 0 .$$

The inequalities (5.5) guarantee that equations (5.1) and (5.3) are of Class I (Theorem 2.2); hence the theorems of § 3 apply.

THEOREM 5.1. *If $\eta_n(a)$ and $\eta'_n(a)$ are the n th conjugate points of (5.1) and (5.3), respectively, and the coefficients of these equations satisfy (5.4) and (5.5), then*

$$(5.6) \quad \eta'_n(a) \leq \eta_{(k+1)n-1}(a)$$

where k is a positive integer.

To prove the theorem we need the following lemma.

LEMMA 5.2. *Let $u(x)$ be a solution of (5.1) such that $u(a) = u'(b) = 0$ ($b > a$). If $2q - p' > 0$, except at isolated points at which $2q - p'$ may vanish, and if $p \geq 0$, then $u(x)$ vanishes in (b, ∞) .*

Proof. Multiplying (5.1) by $u(x)$ and integrating from a to x , we obtain

$$(5.7) \quad u'^2(x) = u'^2(a) + 2u(x)u''(x) + p(x)u^2(x) + \int_a^x (2q - p')u^2 dx .$$

If we assume that $u(x) > 0$ for $x > a$, then $u'(x)$ can vanish but once in (a, ∞) for, by equation (5.7), $u(x)u''(x) < 0$ whenever $u'(x) = 0$. Hence $u'(x) < 0$ for $x > b$. There are now three possibilities;

- (i) $u''(x) < 0$,
- (ii) $u''(x) > 0$, or
- (iii) $u''(x)$ has an infinity of zeros for $x > b$.

We shall show that in all three cases $u(x)$ vanishes for $x > b$. This is trivially true in case (i); if two consecutive derivatives of $u(x)$ are negative then $u(x)$ must ultimately be negative. In case (ii), the right-hand side of (5.7) is positive so that $u'^2(x)$ approaches some constant $k^2 \neq 0$ (since the last term of (5.7) is an increasing function). Hence $u'(x)$ approaches $-k$, but this implies that $u(x)$ is ultimately negative. In case (iii) we let x approach infinity along the points in which $u''(x)$ vanishes and we arrive at the same conclusion as in case (ii). This completes the proof of the lemma.

To prove Theorem 5.1, let $u(x) = y(x, a)$ and $v(x) = y_1(x, a)$ be principal solutions of (5.1) and (5.3), respectively, and let $b = \eta_1(a)$ be the first zero of $u(x)$. By Theorem 2.8, the extremal solution $w(x)$ of the adjoint of (5.1) vanishes at a , has a double zero at b , and vanishes nowhere in the interval (a, b) . Without loss of generality, we may assume that $w(x)$ is positive in (a, b) . Multiplying (5.2) by $v(x)$ and (5.3) by $w(x)$, and integrating from a to b , we obtain,

$$\begin{aligned} & [wv'' - w'v' + w''v]_a^b + \int_a^b [p_1wv' + pw'v + p'wv]dx \\ & + \int_a^b (q_1 - q)wvdx = 0 \end{aligned}$$

or

$$(5.8) \quad w''(b)v(b) + \int_a^b (p_1 - p)wv'dx + \int_a^b (q_1 - q)wvdx = 0 .$$

By construction, $w''(b) > 0$, and $v(x)$ and $v'(x)$ are positive for all x immediately to the right of the point $x = a$. There are now three possibilities which we must consider;

- (i) $v(x)$ and $v'(x)$ are positive in $[a, b]$,
- (ii) $v(x) > 0$ but $v'(x)$ vanishes in $[a, b]$, and
- (iii) $v(x)$ vanishes in $[a, b]$.

It is clear that the conditions of (i) are incompatible since this implies that all three terms in (5.8) are positive. If the conditions (ii) hold, then Lemma 5.2 guarantees the vanishing of $v(x)$ at some point, say $x = b_1 = \eta'_k(a)$. If this point is in the interval $[\eta_{k-1}(a), \eta_k(a)]$, that is, if

$$(5.9) \quad \eta_{k-1}(a) \leq b_1 \leq \eta_k(a) ,$$

then the remainder of the proof of the theorem is analagous to the proof of Theorem 3.10. The integer k appearing in (5.9) is, of course, the same k appearing in (5.6).

It is clear that if conditions (iii) hold, then the conclusion of Theorem 5.1 is the same as that of Theorem 3.10, that is, the inequality (5.6) can be replaced by the stronger inequality (3.3).

Although the conclusion of Theorem 5.1 is not so strong as that of Theorem 3.10, Theorem 5.1 still implies that if there are an infinity of conjugate points of equation (5.1) then the same is true of equation (5.3). In view of Corollary 3.5, this implies the following result.

THEOREM 5.3. *If equation (5.1) is oscillatory, and if inequalities (5.4) and (5.5) are satisfied, then equation (5.3) is also oscillatory.*

If $2q - p' < 0$ then, by Theorem 2.2, equation (5.1) is of Class II and, by Lemma 2.9, the adjoint (5.2) is of Class I. Theorem 5.3 can be applied to the adjoints of equations (5.1) and (5.3) if the coefficients are related by

$$(5.10) \quad p_1 \geq p \geq 0 \text{ and } p'_1 - q_1 \geq p' - q .$$

In view of Theorem 4.7, we have the following result.

THEOREM 5.4. *Let the coefficicints of equations (5.1) and (5.3) satisfy the inequalities*

$$2q - p' < 0 \text{ and } 2q_1 - p'_1 < 0 ,$$

respectively. In addition, let the coefficients of these equations be

related by (5.10). If equation (5.1) is oscillatory, then equation (5.3) is likewise oscillatory.

These comparison theorems will lead to oscillation criteria whenever the oscillatory behavior of a given equation is known. As a first example, consider the Euler Equation

$$(5.11) \quad y''' + \frac{a}{x^2}y' + \frac{b}{x^3}y = 0.$$

It is not difficult to show that if $a \geq 1$ then equation (5.11) is oscillatory for all values of b . If $a < 1$, we must determine the sign of the term $2q - p'$ in order to apply Theorem 5.3. For equation (5.11), we have $2q - p' = \frac{2}{x^3}(a + b)$, and it can be shown that if $a + b > 0$, (5.11) is oscillatory if, and only if, $a + b - 2\left(\frac{1-a}{3}\right)^{3/2} > 0$. If $a + b < 0$, (5.11) is oscillatory if, and only if, $a + b + 2\left(\frac{1-a}{3}\right)^{3/2} < 0$. These remarks and Theorem 5.3 are sufficient to establish the following results.

THEOREM 5.5. *If $2q - p' > 0$ and if there exists a number α such that*

$$\liminf_{x \rightarrow \infty} x^2 p(x) > \alpha > 1, \quad \liminf_{x \rightarrow \infty} x^3 q(x) > -\alpha$$

then equation (5.1) is oscillatory.

THEOREM 5.6. *If $2q - p' > 0$ and if there exists a positive number α such that*

$$\liminf_{x \rightarrow \infty} x^2 p(x) > \alpha, \quad \liminf_{x \leftarrow \infty} x^3 q(x) > 2\left(\frac{1-\alpha}{3}\right)^{3/2} - \alpha$$

then equation (5.1) is oscillatory. If

$$\limsup_{x \rightarrow \infty} x^2 p(x) < \alpha, \quad \limsup_{x \rightarrow \infty} x^3 q(x) < 2\left(\frac{1-\alpha}{3}\right)^{3/2} - \alpha$$

then equation (5.1) is nonoscillatory.

Consider the equation

$$(5.12) \quad y''' + q(x)y = 0.$$

By Theorem 2.2, equation (5.12) is of Class I when $q \geq 0$ and is of Class II when $q \leq 0$. The adjoint of (5.12) is

$$(5.13) \quad y''' - q(x)y = 0.$$

Let $q(x)$ be of one sign in $(0, \infty)$, then, by Theorem 4.7, equation (5.12) is oscillatory if, and only if, equation (5.13) is oscillatory.

THEOREM 5.7. *If $q(x)$ is of one sign, then equation (5.12) is oscillatory if*

$$\liminf_{x \rightarrow \infty} x^3 |q(x)| > \frac{2}{3\sqrt{3}}$$

and nonoscillatory if

$$\limsup_{x \rightarrow \infty} x^3 |q(x)| < \frac{2}{3\sqrt{3}} .$$

Here, the Euler Equation $y''' + \frac{b}{x^3}y = 0$ was used as the equation of comparison.

2. We now consider the case in which the coefficient $p(x)$ of (5.1) is such that the second-order differential equation

$$(5.14) \quad y'' + py = 0$$

is nonoscillatory and $q(x)$ does not change sign in $(0, \infty)$ (cf. Theorem 2.3). In this case we obtain a separation theorem of a different type. The problem now considered refers to the relation between the number of zeros of $y(x)$ and those of $y'(x)$ when $y(x)$ is a solution of (5.1). In the case of a second-order equation, say, $y'' + py' + qy = 0$, two consecutive zeros of the derivative $y'(x)$ of a solution $y(x)$ are—for trivial reasons—separated by a zero of $y(x)$. In the case of higher-order equations this is, in general, not true. It is shown in [7] that if $p(x) \geq 0$ and $2q - p' > 0$ and $y(x)$ is a solution of (5.1) such that $F[y(a)] \geq 0$ where

$$F[y(x)] = y'^2 - 2yy'' - py^2 ,$$

then the zeros of $y(x)$ and $y'(x)$ separate in (b, ∞) , where b is the first zero of $y'(x)$ to the right of $x = a$. (This can be proved by using a slightly modified form of equation (5.7).) These conditions imply the regular alternation of the zeros of $y(x)$ and $y'(x)$ only at the right of a point $x = a$ at which $F[y(a)] \geq 0$. The following result shows that the regularity in the distribution of the zeros of $y(x)$ and $y'(x)$ is essentially guaranteed by the assumption that the second-order differential equation (5.14) be nonoscillatory.

THEOREM 5.8. *Let (5.14) be nonoscillatory and let $q(x)$ be of one sign in $(0, \infty)$. If $y(x)$ is any solution of (5.1) such that $y'(x)$ is oscillatory, then $y(x)$ is likewise oscillatory.*

We prove the theorem for $q(x) \geq 0$. The inequality (2.5) shows that if $y(x)$ does not vanish between two consecutive zeros of $y'(x)$ then $y(x)$ and $y'(x)$ must be of the same sign in this interval. We may assume that $y(x) > 0$ (and, therefore, $y'(x) \geq 0$). This, however, is not sufficient to prove the theorem, since $y'(x)$ may have an infinity of zeros and still be nonnegative in $(0, \infty)$. We now show that this is impossible. If $x = \alpha$ is a point such that $y'(\alpha) = y''(\alpha) = 0$, then $y'(x)$ must be negative in the neighborhood of $x = \alpha$ since, by equation (5.1), $y'''(\alpha) < 0$. This completes the proof.

We have shown that between two consecutive zeros of $y'(x)$ there is in general one zero of $y(x)$. This regularity can fail *only once* in the interval $(0, \infty)$, that is, there can exist one interval $[\alpha, \beta]$ such that $y'(x)$ vanishes at the end points and $y(x) \neq 0$ in (α, β) . If this occurs, then, by Lemma 2.4, $y'(x) \neq 0$ for $0 < x < \alpha$. Hence the separation of the zeros of $y(x)$ and $y'(x)$ is essentially guaranteed.

With the restrictions imposed on the coefficients of (5.1) in this section, additional information about the separation of the conjugate points of two principal solutions of (5.1) can be obtained. This is illustrated by the following theorem.

THEOREM 5.9. *Let (5.14) be nonoscillatory and let $q(x)$ be positive in $(0, \infty)$. Let $u(x) = y(x, a)$ and $v(x) = y(x, b)$ be two principal solutions of (5.1) such that $b > a$. If $\text{sgn } u(b) = \text{sgn } u'(b)$, then the zeros of $u(x)$ and $v(x)$ separate each other in (b, ∞) .*

Proof. We may assume that both $u(b)$ and $u'(b)$ are positive. As previously mentioned (§ 2), if $u(x)$ and $v(x)$ are two linearly independent solutions of (5.1), then $\sigma(x) = uv' - u'v$ is a solution of the adjoint (5.3). If, in particular, $u(x)$ and $v(x)$ are taken as the two principal solutions defined above, then

$$(5.15) \quad \begin{cases} \sigma(b) = 0 \\ \sigma'(b) = u(b)v''(b) > 0 \\ \sigma''(b) = u'(b)v''(b) > 0. \end{cases}$$

We now show that if $\sigma(x)$ satisfies the conditions (5.15), then $\sigma(x) > 0$ follow $x > b$. Assume this is not true, that is, let $\sigma(c) = 0$ where $c > b$. Let $w(x)$ be a solution of (5.1) such that $w(c) = w'(c) = 0$ and $w''(c) > 0$. Multiply (5.1) by $\sigma(x)$ and (5.3) by $w(x)$, add, and integrate from b to c , to obtain,

$$(5.16) \quad \sigma'(b)w'(b) - \sigma''(b)w(b) = 0.$$

Since $w(x)$ has a double zero at $x = c$ and $w''(c) > 0$, it follows, by Lemma 2.4, that $w(x) > 0$ and $w'(x) < 0$ for $x < c$. By the existence

theorem for equation (5.1), $\sigma'(b)$ and $\sigma''(b)$ are not both zero, so that the left-hand side of (5.16) is strictly negative. This contradiction proves that $\sigma(x) > 0$ for $x > b$. Using this fact it is now possible to show that the zeros of $u(x)$ and $v(x)$ (that is, the conjugate points) separate in (b, ∞) . If $v(x)$ does not vanish in some interval $[\eta_\nu(a), \eta_{\nu+1}(a)]$, where $\eta_\nu(a) > b$, then the function $\frac{u}{v}$ is well-defined and vanishes at the end points of this interval. Thus $\left(\frac{u}{v}\right)'$ must vanish at some point inside this interval. But

$$\left(\frac{u}{v}\right)' = \frac{vu' - v'u}{v^2} = -\frac{\sigma}{v^2},$$

and this is impossible since we have just shown that $\sigma(x) > 0$ for $x > b$. Hence $v(x)$ must vanish at least once between two consecutive zeros of $u(x)$ for $x > b$. Interchanging the roles of $u(x)$ and $v(x)$ we have proved that $v(x)$ vanishes exactly once between two consecutive zeros of $u(x)$.

One might conjecture—in view of Theorem 3.8—that if $\text{sgn } u(b) \neq \text{sgn } u'(b)$ then the zeros of $u(x)$ and $v(x)$ must separate in pairs. This, however, can be shown to be false by considering the example $y''' + qy = 0$ where $q = \text{const.}$ (This example clearly satisfies the hypothesis of Theorem 5.9.) We can construct two principal solutions $u(x) = y(x, a)$ and $v(x) = y(x, b)$ such that $\text{sgn } u(b) \neq \text{sgn } u'(b)$ for which the zeros of $u(x)$ and $v(x)$ separate singly in (b, ∞) .

The following theorem is an immediate application of Theorem 5.9.

THEOREM 5.10. *Let (5.14) be nonoscillatory and let $q(x)$ be positive in $(0, \infty)$. If equation (5.1) is oscillatory and if $u(x)$ is a nonoscillatory solution then $u(x)$ is monotonically decreasing.*

By Theorem 3.4, $u(x)$ cannot vanish in $(0, \infty)$, hence we may assume that $u(x)$ is positive in this interval. Assume $u(x)$ is not monotonically decreasing, that is, assume that $u'(x) > 0$ in some interval (a, b) . Let $v(x) = y(x, a)$ be a principal solution of (5.1). By Theorem 3.4, $v(x)$ is oscillatory in (a, ∞) . Let $x = b_1$ be the first zero of $v(x)$ to the right of $x = a$, then $v(x) > 0$ in (a, b_1) . Hence, by Lemma 3.2, there exists a function $w(x) = v(x) - \lambda u(x)$ which has a double zero at some point α in (a, b) . Since $u(x)$ and $v(x)$ are both positive in (a, b_1) , we see that λ is positive and since $u'(\alpha) > 0$, we find that $v'(\alpha) > 0$. By Theorem 5.9, the zeros of the two principal solutions $v(x)$ and $w(x)$ must separate in (α, ∞) . This, however, is impossible since $\lambda u(x)$ intersects each positive arch of $v(x)$ twice. This proves that $u'(x)$ cannot be positive in any interval (a, b) and hence is monotonically decreasing.

We now derive some oscillation criteria for these equations. The

first such criterion is given by the following theorem.

THEOREM 5.11. *Let (5.14) be nonoscillatory. If either of the inequalities*

$$(a) \quad p'(x) \geq q(x) \geq 0$$

or

$$(b) \quad p'(x) \leq q(x) \leq 0$$

is satisfied, then equation (5.1) is nonoscillatory.

We prove the theorem for (a); the proof for (b) is analagous. Since $p' - q \geq 0$, it follows from Theorem 2.3 that equation (5.2) is of Class I. Hence, by Lemma 2.9, the adjoint of (5.2), that is equation (5.1), belongs to Class II. Accordingly, for any point $x = a$ there exists a solution of (5.1) which has a double zero at a and does not vanish for $x > a$. But $q(x) \geq 0$, which implies that (5.1) belongs to Class I (Theorem 2.3). It follows, by Theorem 3.6, that (5.1) is nonoscillatory.

We have proved, in effect, that if a third-order differential equation is both of Class I and Class II, then the equation is nonoscillatory.

The following two theorems give oscillation criteria for equations (5.1) which depend on the integrability of the functions $x(q - p')$ and $x^2(q - p')$. We assume that $q \geq 0$ and $p' - q \leq 0$. The preceding theorem shows that if $q \geq 0$ and $p' - q \geq 0$ then (5.1) is nonoscillatory.

THEOREM 5.12. *Let (5.14) be nonoscillatory and let the functions $p(x)$, $q(x)$, and $q(x) - p'(x)$ be positive in (a, ∞) . If*

$$\int_a^\infty x[q(x) - p'(x)]dx = \infty,$$

then equation (5.1) is oscillatory.

Proof. Assume that equation (5.1) is nonoscillatory. Since $q(x) > 0$, equation (5.1) is of Class I and hence, by Theorem 3.6, there exists a point $x = a$ such that no solution of (5.1) has more than two zeros in (a, ∞) . Therefore, the principal solution $u(x) = y(x, a)$ is positive for $x > a$. We now show that $u'(x)$ is also positive for $x > a$. To this end, assume that $u'(x) < 0$. Since $u'(x) > 0$ for x immediately to the right of $x = a$, there exists a point b such that $u'(b) = 0$. By Lemma 2.4, $u'(x)$ has no other zeros in (b, ∞) . It is clear that $u''(b) < 0$. If two consecutive derivatives of $u(x)$ are negative for $x > b$, then $u(x)$ must ultimately be negative. Hence $u''(x)$ must ultimately be positive, that is, there exists a point $x = c$ such that $u''(c) = 0$. It is not difficult to see that the points b and c can be used in the inequality (2.5).

Since $u(x) > 0$ and $u'(x) < 0$ in (b, c) , the left-hand side of (2.5) is negative. This contradiction proves that $u'(x) > 0$ for $x > a$. Using this fact, we now see, by equation (5.1), that $u'''(x) < 0$ for $x > a$, so that $u''(x)$ is decreasing. Again, if two consecutive derivatives of $u(x)$ are negative, $u(x)$ must become negative. Therefore $u''(x) > 0$, and $u'(x)$ is thus increasing for $x > a$. Since $u'(x) > 0$ for some $x = b$ immediately to the right of a , we have,

$$u'(x) > A > 0$$

and, since $u(b) > 0$,

$$u(x) = u(b) + (x - b)u'(c)$$

or,

$$u(x) > (x - b)u'(c) > (x - b)A$$

where $a < c < b$.

Integrating (5.1) from b to x , we obtain,

$$\begin{aligned} u''(x) - u''(b) &= p(b)u(b) - p(x)u(x) + \int_b^x (p' - q)u(t)dt, \\ u''(b) + p(b)u(b) &= u''(x) + p(x)u(x) + \int_b^x (q - p')u(t)dt \\ &\geq \int_b^x (q - p')u(t)dt \\ &\geq A \int_b^x [q(t) - p'(t)](t - b)dt. \end{aligned}$$

Since the left-hand side of the inequality is independent of x , we have,

$$\int_b^\infty (t - b)[q(t) - p'(t)]dt < \infty$$

which implies that

$$\int_b^\infty t[q(t) - p'(t)]dt < \infty$$

and this completes the proof of Theorem 5.12.

THEOREM 5.13. *Let (5.14) be nonoscillatory and let the functions $p(x)$, $q(x)$, and $q(x) - p'(x)$ be positive in (a, ∞) . If*

$$(5.17) \quad \int_a^\infty x^2[q(x) - p'(x)]dx < \infty,$$

then equation (5.1) is nonoscillatory.

Proof. Integrating (5.1) from a to x yields

$$y(x) = y(a) + (x - a)y'(a) + \frac{1}{2}(x - a)^2[y''(a) + p(a)y(a)] - \int_a^x (x - t)p(t)y(t)dt - \frac{1}{2}\int_a^x (x - t)^2[q(t) - p'(t)]y(t)dt .$$

If we let $u(x) = y(x, a)$ be the principal solution of (5.1) then this equation becomes

$$(5.18) \quad u(x) = \frac{1}{2}(x - a)^2 - \int_a^x (x - t)p(t)u(t)dt - \frac{1}{2}\int_a^x (x - t)^2[q(t) - p'(t)]u(t)dt .$$

Assuming that (5.1) is oscillatory, the principal solution $u(x)$ is an oscillatory solution (Theorem 3.4). Let $x = b$ be the first zero of $u(x)$ in (a, ∞) . Since $u(x) > 0$ in (a, b) , equation (5.18) yields the inequality

$$(5.19) \quad u(x) \leq \frac{1}{2}(x - a)^2 \text{ for } x \in (a, b) .$$

Substituting $x = b$ into (5.18), we have

$$\begin{aligned} \frac{1}{2}(b - a)^2 &= \int_a^b (b - t)p(t)u(t)dt + \frac{1}{2}\int_a^b (b - t)^2[q(t) - p'(t)]u(t)dt \\ &\leq (b - a)\int_a^b p(t)u(t)dt + \frac{1}{2}(b - a)^2\int_a^b [q(t) - p'(t)]u(t)dt \end{aligned}$$

or,

$$1 \leq \int_a^b [q - p']u(t)dt + \frac{2}{b - a}\int_a^b p(t)u(t)dt .$$

Using (5.19) in this last inequality, we find that

$$(5.20) \quad 1 \leq \frac{1}{2}\int_a^b (t - a)^2[q - p']dt + \frac{1}{b - a}\int_a^b (t - a)^2p(t)dt .$$

We can find a bound on the second integral in (5.20) by using a result in [6] which states that if $y'' + py = 0$ is nonoscillatory and $p(x) \geq 0$, then

$$(5.21) \quad \frac{1}{b - a}\int_a^b (t - a)^2p(t)dt < 1 - (b - a)\int_b^\infty p(t)dt .$$

There are two cases of interest;

- (i) $\limsup_{x \rightarrow \infty} x \int_x^\infty p(t)dt = \alpha \neq 0 \quad (\alpha < 1)$
- (ii) $\limsup_{x \rightarrow \infty} x \int_x^\infty p(t)dt = 0 .$

We now show that in both cases (5.20) implies a contradiction to (5.17).

Let us first consider case (i). By (5.17), we can choose a large enough so that

$$(5.22) \quad \int_a^\infty x^2[q - p']dx \leq \alpha .$$

Substituting (5.21) into (5.20), we have

$$\begin{aligned} (b - a) \int_b^\infty p(t)dt &< \frac{1}{2} \int_a^b (t - a)^2 [q - p']dt \\ &< \frac{1}{2} \int_a^b t^2 [q - p']dt . \end{aligned}$$

Letting b approach infinity along a sequence of points for which $b \int_b^\infty p(t) dt$ approaches its upper limit, the last inequality yields

$$2\alpha < \int_a^b t^2 [q - p']dt ,$$

which contradicts (5.22). Therefore, in this case, equation (5.1) is nonoscillatory.

We can treat case (ii) by letting

$$r(t) = (t - a) \int_t^\infty p(x)dx$$

and integrating the second term on the right-hand side of (5.20) by parts, that is,

$$\begin{aligned} \frac{1}{b - a} \int_a^b (t - a)^2 p(t)dt &= (b - a) \int_b^\infty p(x)dx - \frac{2}{b - a} \int_a^b (t - a) \left(\int_t^\infty p(x)dx \right) dt \\ &= (b - a) \int_b^\infty p(x)dx - \frac{2}{b - a} \int_a^b r(t)dt . \end{aligned}$$

Letting b approach infinity through a sequence of points for which $b \int_b^\infty p(x)dx$ approaches its upper limit, this last equation yields

$$\lim_{b \rightarrow \infty} \left[\frac{1}{b - a} \int_a^b (t - a)^2 p(t)dt \right] = 0 .$$

Using this fact, we let b approach infinity in equation (5.20), to obtain

$$2 \leq \int_a^\infty (t - a)^2 [q - p']dt < \int_a^b t^2 [q - p']dt ,$$

again contradicting (5.22). This completes the proof.

If $q \leq 0$ and if $q - p' \leq 0$, then equation (5.1) is of C_{II} and, by Lemma 2.9, the adjoint (5.2) belongs to Class I. Hence, the last two theorems can be applied to equation (5.2). Since (5.1) is oscillatory if,

and only if, (5.2) is oscillatory (Theorem 4.7), the following result is apparent.

THEOREM 5.14. *Let (5.14) be nonoscillatory and let $p(x)$ be positive and $q(x)$ negative in (a, ∞) . If*

$$\int_a^{\infty} x[-q(x)]dx = \infty ,$$

then (5.1) is oscillatory. If

$$\int_a^{\infty} x^2[-q(x)]dx < \infty ,$$

then (5.1) is nonoscillatory.

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