WIRTINGER-TYPE INTEGRAL INEQUALITIES

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1. Introduction. The following inequalities (and other similar ones) are known:

(i) if $u'(x) \in L_2$ and $u(0) = 0$, then

$$\int_0^{\pi/2} u^2 dx \leqq \int_0^{\pi/2} u'^2 dx \qquad [4];$$

(ii) if $u''(x) \in L_2$ and $u(0) = u(\pi) = 0$, then

$$\int_0^{\pi} u^2 dx \leqq \int_0^{\pi} u''^2 dx \qquad [3];$$

in each case, equality occurs if and only if $u(x) \equiv A \sin x$. P. R. Beesack [1] has generalized these two types of inequalities by considering the underlying differential equations $y'' + py = 0$ and $y^{(iv)} - py = 0$ respectively, together with the equations satisfied by y'/y. In [2], a relation was obtained between the equation $y^{(2n)} - py = 0$ and the inequality

$$(-1)^n \int_a^b pu^2 dx \leqq \int_a^b u^{(n)^2} dx .$$

In this paper we let Ly be the general self-adjoint linear operator of even order

$$\sum_{i=0}^{n} (f_i y^{(i)})^{(i)}$$

and extend the methods of [2] to relate the equation

$$(1) \qquad Ly = 0$$

and the inequalities

$$(2) \qquad 0 \leqq \sum_{i=0}^{n} (-1)^{n+i} \int_a^b f_i u^{(i)^2} dx$$

and

$$(3) \qquad 0 \geqq \int_a^b \frac{1}{f_n} \cdot u^2 dx + (-1)^n \int_a^b \frac{1}{f_0} \cdot u^{(n)^2} dx .$$

2. Notation and lemmas. Let $y_i = f_i y^{(i)}$, $v_i = \sum_{k=0}^{i} y_{n-k}^{(i-k)}$,

$$u_{ij} = v_{n-i}/y^{(j)}, \text{ and } y_{ij} = y^{(i)}/y^{(j)} \qquad (i = 0, \cdots, n) .$$

Received August 10, 1960.

Then

$$v_i = v'_{i-1} + y_{n-i} \qquad (i = 1, \cdots, n) . \tag{4}$$

Let $(k_0 \cdots k_n)$ be an $(n+1)$-tuple consisting of 0's and 1's, such that $\sum_{i=1}^{n} k_i$ is even. Let

$$c_i = \begin{cases} a, & k_i = 0 \\ b, & k_i = 0 \end{cases} ; \qquad d_i = \begin{cases} a, & k_{i+1} = 1 \\ b, & k_{i+1} = 1 \end{cases} ; \tag{5}$$

$$\begin{matrix} c_i^* = a + b - c_i ; \\ d_i^* = a + b - d_i ; \end{matrix} \quad p_i = (-1)^{\sum_{j=0}^{i} k_j} ; \quad q_i = (-1)^i p_i ; \quad (i = 0, \cdots, n) .$$

We now and henceforth assume that (1) has a solution on $[a, b]$ such that

$$\begin{aligned} p_1 y^{(n-1)}(x) &> 0 \quad \text{on } (a, b) \text{ and at } c_1^* ; \\ p_i y^{(n-i)}(c_i) &\geqq 0 \qquad (i = 2, \cdots, n) ; \\ q_i v_i(d_i) &\geqq 0 \qquad (i = 0, \cdots, n-1) ; \end{aligned} \tag{6}$$

and that the $f_i(x) \in L[a, b]$, with $\int_a^b f_0(x)dx \neq 0$, and

$$\begin{aligned} (-1)^{n+i} f_i(x) &\leqq 0 \quad \text{on } [a, b] \qquad (i = 0, \cdots, n-1) ; \\ f_n(x) &\geqq 0 \quad \text{on } [a, b] . \end{aligned} \tag{7}$$

LEMMA 1. *We have*

$$p_i y^{(n-i)}(x) > 0 \quad \textit{on } (a, b) \textit{ and at } c_i^* \qquad (i = 1, \cdots, n) . \tag{8}$$

Proof. By hypothesis the lemma is true for $i = 1$. Suppose that, for some i such that $1 \leqq i \leqq n-1$, the statement holds. Integrating and multiplying by $(-1)^{k_{i+1}}$ we have

$$p_{i+1} y^{(n-i-1)}(x) = p_{i+1} y^{(n-i-1)}(c_{i+1}) + (-1)^{k_{i+1}} \int_{c_{i+1}}^{x} p_i y^{(n-i)}(t) dt > 0$$

on (a, b) and at c_{i+1}^*. This completes Lemma 1.

LEMMA 2. *We have*

$$q_i v_i(x) \geqq 0 \quad \textit{on } [a, b], > 0 \textit{ at } d_i^* \qquad (i = 0, \cdots, n-1) . \tag{9}$$

Proof. We proceed by induction on i $(i = n-1, \cdots, 1, 0)$. Now $v'_{n-1}(x) = v_n(x) - y_0 = -y_0$, so

$$q_{n-1} v_{n-1}(x) = q_{n-1} v_{n-1}(d_{n-1}) - (-1)^{1+k_n} \int_{d_{n-1}}^{x} (-1)^n f_0 p_n y dt \geqq 0 ;$$

since $|y| > 0$ and $\int_a^b f_0(x)dx \neq 0$, the inequality is strict at d_{n-1}^*.

Now suppose that, for some i ($n-1 \geqq i \geqq 1$), the statement holds. Then, integrating (4) and multiplying by q_{i-1},

$$q_{i-1}v_{i-1}(x) = q_{i-1}v_{i-1}(d_{i-1}) + (-1)^{1+k_i}\int_{d_{i-1}}^x q_i v_i dt - (-1)^{1+k_i}\int_{d_{i-1}}^x (-1)^i f_{n-i}p_i y^{(n-i)}dt ,$$

so $q_{i-1}v_{i-1}(x) \geqq 0$ on (a, b) and >0 at d_{i-1}^*. This completes Lemma 2.

3. The formal identity.

Since (at least formally)

$$u_{ii} = v'_{n-i-1}/y^{(i)} + f_i ,$$

we have

$$u_{ii} = u'_{i+1,i} + u_{i+1,i+1}y_{i+1,i}^2 + f_i . \tag{10}$$

Now we use (10) and induction to derive the formal identity

$$0 = \sum_{i=0}^{n-1}(-1)^{n+i}\left\{u_{i+1,i}u^{(i)^2}\Big|_a^b + \int_a^b u_{i+1,i+1}(u^{(i+1)} - y_{i+1,i}u^{(i)})^2 dx\right\} + \sum_{i=0}^{n}(-1)^{n+i}\int_a^b f_i u^{(i)2}dx ; \tag{11}$$

then we will justify the formal steps.

First,

$$\int_a^b u'_{i+1,i}u^{(i)^2}dx = u_{i+1,i}u^{(i)^2}\Big|_a^b - \int_a^b 2u_{i+1,i}u^{(i)}u^{(i+1)}dx = u_{i+1,i}u^{(i)^2}\Big|_a^b - \int_a^b 2u_{i+1,i+1}y_{i+1,i}u^{(i)}u^{(i+1)}dx ,$$

so

$$\int_a^b (u'_{i+1,i} + u_{i+1,i+1}y_{i+1,i}^2)u^{(i)2}dx = u_{i+1,i}u^{(i)^2}\Big|_a^b + \int_a^b u_{i+1,i+1}(u^{(i+1)} - y_{i+1,i}u^{(i)})^2 dx - \int_a^b u_{i+1,i+1}u^{(i+1)^2}dx . \tag{12}$$

Since $v_n(x) \equiv Ly \equiv 0$, $u_{00}(x) \equiv 0$; using (10) and (12) with $i = 0$,

$$0 = u_{10}u^2\Big|_a^b + \int_a^b u_{11}(u' - y_{10}u)^2 dx + \int_a^b f_0 u^2 dx - \int_a^b u_{11}u'^2 dx .$$

Suppose that, for some k such that $1 \leqq k \leqq n-1$,

$$(13) \qquad 0 = \sum_{i=0}^{k-1}(-1)^i\left\{u_{i+1,i}u^{(i)2}\Big|_a^b + \int_a^b u_{i+1,i+1}(u^{(i+1)} - y_{i+1,i}u^{(i)2})dx\right\} + \sum_{i=0}^{k-1}(-1)^i\int_a^b f_i u^{(i)2}dx + (-1)^k\int_a^b u_{kk}u^{(k)2}dx .$$

Using (10) and (12) with $i = k$, and substituting for the last term in (13), we obtain (13) with k replaced by $k+1$. Hence (13) holds for $k = 1, \cdots, n$; with $k = n$, using the fact that $u_{nn} \equiv f_n$, and multiplying by $(-1)^n$, we have (11).

LEMMA 3. *Let $u(x)$ be a function such that*

$$(14) \qquad u^{(n)} \in L_2[a, b];\ u^{(i)}(c_{n-i}) = 0 \qquad (i = 0, \cdots, n-1) .$$

(*Note that* (14) *implies that the zero of $u^{(i)}$ at c_{n-i} is of order* $\geqq 1$ $(i = 0, \cdots, n-2)$ *and* $> \frac{1}{2}$ $(i = n-1)$.) *Then* (11) *is valid.*

Proof. Our concern is with possible zeros of $y^{(i)}$ $(i = 0, \cdots, n-1)$ on $[a, b]$; by Lemma 1, the only possible zero of $y^{(i)}$ is at c_{n-i}. Let i be such that $0 \leqq i \leqq n-1$, and suppose that $y^{(i)}$ has a zero of order r at c_{n-i}. Then $r \leqq n-i$. For if $r > n-i$ then $y^{(i+k)}(c_{n-i}) = 0$ $(k = 1, \cdots, n-i)$, and so $c_{n-i} = c_{n-i-1} = \cdots c_1$; thus $y^{(n)}(c_1) = 0$. But, by Lemma 2, $v_0(c_1) \neq 0$ (since $c_1 = d_0^*$), and $v_0(x) = f_n(x)y^{(n)}(x)$. Thus $r \leqq n-i$. Now, since $c_{n-i} = \cdots = c_1$, $u^{(i)}$ has a zero of order $\geqq r$ at c_{n-i} $(i = 0, \cdots, n-2)$, and of order $> \frac{1}{2}$ $(i = n-1)$. The lemma now follows, as does the fact (to be used in the proof of Lemma 5) that $u_{i+1,i}(c_{n-i})u^{(i)2}(c_{n-i}) = 0$ $(i = 0, \cdots, n-1)$.

LEMMA 4. *On $[a, b]$, $(-1)^{n+i-1}u_{ii}(x) \leqq 0$ $(i = 1, \cdots, n)$.*

Proof. By Lemmas 1 and 2,

$$(-1)^{n+i-1}u_{ii} = (-1)^{n+i-1}\cdot(-1)^{n-i}\cdot q_{n-i}v_{n-i}/p_{n-i}y^{(i)} = -q_{n-i}v_{n-i}/p_{n-i}y^{(i)} \leqq 0 .$$

LEMMA 5. $$(-1)^{n+i}u_{i+1,i}u^{(i)2}\,|_a^b \leqq 0 \qquad (i = 0, \cdots, n-1) .$$

Proof. Since $c_j = d_{j-1}^*$,

$$(-1)^{n+i}u_{i+1,i}u^{(i)2}\,|_a^b = (-1)^{n+i+1+k_{n-i}}u_{i+1,i}u^{(i)2}\,|_{d_{n-i-1}}^{c_{n-i}} .$$

Evaluation at c_{n-i} gives zero, and

$$(-1)^{n+i+k_{n-i}}u_{i+1,i} = -q_{n-i-1}v_{n-i-1}/p_{n-i}y^{(i)} \leqq 0$$

on $[a, b]$ and so at d_{n-i-1}.

4. The inequality. We now state

THEOREM 1. *Let* $f_i(x) \in L[a, b]$ $(i = 0, \cdots, n)$, *with* $\int_a^b f_0(x)dx \neq 0$. *Let* $f_i(x)$ $(i = 0, \cdots, n)$ *satisfy* (7), *and let* $y(x)$ *be a solution of* (1) *which satisfies* (6). *Let* $u(x)$ *satisfy* (14). *Then*

$$0 \leqq \sum_{i=0}^{n} (-1)^{n+i} \int_a^b f_i(x)u^{(i)2}(x)dx . \tag{2}$$

Further, equality obtains if and only if $u(x) \equiv cy(x)$ *and* (6) *is modified to make* $q_i v_i(d_i) = 0$ $(i = 0, \cdots, n-1)$.

Proof. The Theorem follows immediately from the lemmas, except for the last statement, which follows from the fact that equality obtains if and only if $u^{(i+1)}(x) \equiv y_{i+1,i}(x)u^{(i)}(x)$ $(i = 0, \cdots, n-1)$ and $v_i(d_i) = 0$ $(i = 1, \cdots, n)$.

5. The reciprocal inequality. We now derive a set of inequalities which includes (3); we prove

THEOREM 2. *Let the* $f_i(x)$ $(i = 0, \cdots, n)$ *and* $y(x)$ *satisfy the hypothesis of Theorem* 1; *in addition, let* $f_i(x) \equiv 0$ *or* $f_i(x) \neq 0$ *on* $[a, b]$ $(i = 0, \cdots, n)$. *Let* $u(x)$ *satisfy*

$$u^{(n)} \in L_2[a, b];\ u^{(i)}(d_i) = 0 \qquad (i = 0, \cdots, n-1) . \tag{15}$$

Then, for each k $(1 \leqq k \leqq n)$ *such that* $f_{n-k}(x) \neq 0$,

$$0 \geqq \int_a^b \frac{1}{f_n(x)} u^2(x)dx + (-1)^k \int_a^b \frac{1}{f_{n-k}(x)} u^{(k)2}dx . \tag{16}$$

Proof. The proof is similar to that of Theorem 1, so we present it here in less detail. Let $r_{ij} = y^{(n-i)}/v_j$; then, formally,

$$r_{ii} = r'_{i+1,i} + r_{i+1,i}v_{i+1}/v_i - r^2_{i+1,i}f_{n-i-1} . \tag{17}$$

Thus

$$\int_a^b r_{ii}u^{(i)2}dx = r_{i+1,i}u^{(i)2}\Big|_a^b + \int_a^b r_{i+1,i+1}\Big(u^{(i+1)} - \frac{v_{i+1}}{v_i}u^{(i)}\Big)^2 dx \tag{18}$$

$$- \int_a^b f_{n-i-1}r^2_{i+1,i}u^{(i)2}dx - \int_a^b r_{i+1,i+1}u^{(i+1)2}dx \qquad (i=0, \cdots, n-2) ,$$

and

$$(19)\quad \int_a^b r_{ii}u^{(i)2}dx = r_{i+1,i}u^{(i)2}\Big|_a^b - \int_a^b \frac{1}{f_{n-i-1}}(u^{(i+1)} - r_{i+1,i}f_{n-i-1}u^{(i)})^2dx + \int_a^b r_{i+1,i}\frac{v_{i+1}}{v_i}u^{(i)2}dx + \int_a^b \frac{1}{f_{n-i-1}}u^{(i+1)2}dx \quad (i = 0, \cdots, n-1) .$$

Repeated application of (18) to $\int_a^b r_{00}u^2dx$ gives

$$\int_a^b \frac{1}{f_n}u^2dx = \sum_{i=0}^{k-2}(-1)^i\Big\{r_{i+1,i}u^{(i)2}\Big|_a^b + \int_a^b r_{i+1,i+1}\Big(u^{(i+1)} - \frac{v_{i+1}}{v_i}u^{(i)}\Big)^2dx - \int_a^b f_{n-i-1}r_{i+1,i}^2u^{(i)2}dx\Big\} + (-1)^{k-1}\int_a^b r_{k-1,k-1}u^{(k-1)2}dx ;$$

application of (19) to the last term gives

$$(20)\quad \int_a^b \frac{1}{f_n}u^2dx = \sum_{i=0}^{k-1}(-1)^i r_{i+1,i}u^{(i)2}\Big|_a^b + \sum_{i=0}^{k-2}(-1)^i\Big\{\int_a^b r_{i+1,i+1}\Big(u^{(i+1)} - \frac{v_{i+1}}{v_i}u^{(i)}\Big)^2dx - \int_a^b f_{n-i-1}r_{i+1,i}^2u^{(i)2}dx\Big\} + (-1)^{k-1}\Big\{\int_a^b r_{k,k-1}\frac{v_k}{v_{k-1}}u^{(k-1)2}dx - \int_a^b \frac{1}{f_{n-k}}(u^{(k)} - r_{k,k-1}f_{n-k}u^{(k-1)})^2dx + \int_a^b \frac{1}{f_{n-k}}u^{(k)2}dx\Big\} \quad (k = 1, \cdots, n) .$$

We now show that, if $f_{n-k}(x) \neq 0$, (20) is valid. Let a v_i have a zero of order r; such a zero must be at d_i. Now, $r \leqq n - i$. For we have

$$v_j' = q_{j+1}(q_{j+1}v_{j+1} + (-1)^j f_{n-j-1}p_{j+1}y^{(n-j-1)}) ;$$

since $y^{(n-j-1)}(d_j) \neq 0$, if $v_j'(d_j) = 0$ then $f_{n-j-1} \equiv 0$, and $v_j' \equiv v_{j+1}$. Thus, if $r > n - i$, $v_i^{(n-i-1)} = v_{n-1}$ and also $v_i^{(n-i)} = v_n \equiv 0$. The first of these implies that $v_i^{(n-i)} = v_{n-1}' = v_n - y_0 = -y_0 \neq 0$, a contradiction. Further, we have $d_i = \cdots = d_{i+r-1}$, so $u^{(i)}$ has a zero of order greater than $r - \frac{1}{2}$ at d_i. This suffices to justify (20). We note in addition that $r_{i+1,i}(d_i)u^{(i)2}(d_i) = 0$ $(i = 0, \cdots, n-1)$.

Now by hypothesis $(-1)^{i+1}f_{n-i-1} \leqq 0$ $(i = 0, \cdots, n-1)$. Lemma 4 implies that $(-1)^i r_{i+1,i+1} \leqq 0$ $(i = 0, \cdots, n-2)$. Finally,

$$(-1)^i r_{i+1,i} u^{(i)2} \Big|_a^b = -\frac{p_{i+1} y^{(n-i-1)} u^{(i)2}}{q_i v_i} \Big|_{d_i}^{d_i^*} ;$$

evaluation at d_i^* gives a non-positive quantity; evaluation at d_i gives zero. Hence the inequality (16) follows from (20).

6. Concluding remarks. If we want (16) for only one particular value of k ($k < n$), we need correspondingly less hypotheses on $y(x)$ and its derivatives, $u(x)$ and its derivatives, and $f_i(x)$ ($i = 0, \cdots, n$), since only $k + 1$ of the functions in each of these sets are actually involved in any of the proofs.

Since $(-1)^{n-i} f_i(x) \leqq 0$, from (2) we may delete any combination of terms excluding the last, and to the right-hand side of (16) we may add any terms of the form

$$(-1)^j \int_a^b \frac{1}{f_{n-j}} u^{(j)2} dx \qquad (i \leqq j \neq k) .$$

Thus, e.g., (2) implies

$$0 \leqq (-1)^k \int_a^b f_{n-k} u^{(k)2} dx + \int_a^b f_n u^{(n)2} dx ,$$

which perhaps corresponds more obviously to (16) than does (2).

Finally, the set of allowed values of $(k_0 \cdots k_n)$ can be split into halves such that one half, together with the inequality $Ly \geqq 0$, and also the other half, together with $Ly \leqq 0$, will produce the inequalities.

Bibliography

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