ON THE THEORY OF SPATIAL INVARIANTS

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1. Introduction. By an operator we mean in this paper a bounded linear transformation of a Hilbert space into itself. The adjoint of an operator T we will denote T^*. An operator is said to be normal if it commutes with its adjoint.

Given operators A_1, A_2 on the respective Hilbert spaces h_1, h_2 we say A_1 and A_2 are spatially equivalent if there exists a norm preserving linear transformation (called a spatial isometry) W of h_1 onto h_2 such that $WA_1W^{-1} = A_2$ on h_2, or equivalently, $W^{-1}A_2W = A_1$ on h_1. A major problem of operator theory is to determine a complete set of spatial invariants for an operator; two operators should be spatially equivalent if and only if they are assigned the same invariants.

This problem has been completely solved for the class of normal operators (see [5]). The weighted spectrum theorem for normal operators which generate maximal Abelian self adjoint algebras (henceforth denoted "masa") on separable Hilbert space can be stated as follows; any such operator T is determined within spatial equivalence by a family of equivalent finite Borel measures on the complex plane, concentrated on the spectrum of T, and this family is a spatial invariant (see bibliography of [5]). By the decomposition of Abelian W^*-algebras (see [5]) a certain sequence of such families constitutes a complete set of spatial invariants for an arbitrary normal operator on separable Hilbert space (whether generating a masa or not).

Let $\{A_n\}$ and $\{B_n\}$ be sequences of operators on the respective Hilbert spaces h and k. We say that $\{A_n\}$ is spatially equivalent $\{B_n\}$ if there exists an isometry W of h onto k such that $WA_nW^{-1} = B_n$ on k simultaneously for all n. Among other things we will find a complete set of spatial invariants for certain kinds of sequences of normal operators. And this we will apply to the theory of spatial invariants of some operators on separable Hilbert space which are not normal. Throughout this paper we will assume that all Hilbert spaces employed are separable in order that families of equivalent measures are at our disposal.

Each of the sections in this paper carries its own introduction and technical preliminaries. However an acquaintance with multiplication algebras of finite measure spaces, the weighted spectrum theorem on separable Hilbert space (cited above), and the decomposition of abelian W^*-algebras, is presupposed throughout.

2. Normal sequences. In this paper C will usually denote the com-

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plex plane. We define the topological space $\boldsymbol{C}$ to be the cartesian product of a countable infinity of copies of C. For each integer $n > 0$ let C_n be a copy of C; then we may express $\boldsymbol{C}$, for example, as $\prod_1^\infty C_n$. The nth coordinate function f_n, defined $f_n(x_1, x_2, \cdots, x_n, \cdots) = x_n$, is continuous on $\boldsymbol{C}$. The family of coordinate functions separates points in $\boldsymbol{C}$.

DEFINITION. *The proper sets of* $\boldsymbol{C}$ *are the sets in the* σ*-ring generated by all the sets of the form* $f_n^{-1}(E)$ *where* n *is a positive integer and* E *is a Baire subset of* C. *A complex valued function* f *on* $\boldsymbol{C}$ *is said to be proper if* $f^{-1}(E)$ *is a proper subset of* $\boldsymbol{C}$ *for each Baire subset* E *of* C. *A measure on the proper sets of* $\boldsymbol{C}$ *is called a proper measure.*

Clearly this definition would be unchanged if we employed open sets E in lieu of Baire sets, or closed sets in lieu of Baire sets, or compact sets in lieu of Baire sets.

Sets of the form $f_n^{-1}(E)$, E open in C, are open. Hence all the proper sets lie in the σ-ring generated by the open sets in $\boldsymbol{C}$. On the other hand each open set in $\boldsymbol{C}$ is proper because the open sets in the topological basis of $\boldsymbol{C}$ usually employed are proper and $\boldsymbol{C}$ is second countable. Hence the proper sets in $\boldsymbol{C}$ constitute the σ-ring generated by the open sets in $\boldsymbol{C}$. Likewise the proper sets constitute the σ-ring generated by the closed sets in $\boldsymbol{C}$.

We will be concerned with families of equivalent proper measures on $\boldsymbol{C}$, i.e., a proper measure on $\boldsymbol{C}$ and all proper measures equivalent to it. If a proper set is null with respect to one measure in such a family, it is null with respect to all members of the family. This defines the null sets of a family of equivalent proper measures on $\boldsymbol{C}$. A family of equivalent proper measures is said to be concentrated on a set E in $\boldsymbol{C}$ if the complement of E is a null set with respect to the family.

DEFINITION. *A normal sequence* $\{A_n\}$ *is a sequence of mutually commuting normal operators on a separable Hilbert space. A normal sequence* $\{A_n\}$ *is said to have simple spectrum if the* W^**-algebra generated by all the* A_n *together is a masa.*

In the normal sequence $\{A_n\}$, A_i commutes with A_j, all i, j. Hence A_i commutes with A_j^*, all i, j, by the Fuglede Theorem (see bibliography of [1]). Likewise A_i^* commutes with A_j^*, all i, j. Hence the *-algebra generated by the A_i together is Abelian.

The terms "normal sequence" and "sequence with simple spectrum" were so selected as to be analogous to the terms "normal operator" and "operator with simple spectrum". In this paper the phrase "with simple spectrum" will be abbreviated "*wss*".

In this section we will show that any *wss* sequence on a separable Hilbert space is spatially equivalent to the sequence $\{L_{fn}\}$ where m is some appropriate proper measure on $\boldsymbol{C}$ and where L_{f_n} is multiplication on the complex Hilbert space $L_2(C, m)$ by the nth coordinate function f_n.

In order to prove this let $\{A_n\}$ be a *wss* sequence. Let X be the spectrum of the abelian C^*-algebra M generated by all the A_n together with the identity. The weak closure of M is a masa on a separable Hilbert space h; consequently there is a vector $v \in h$ such that Mv is dense in h and $\| v \| = 1$ (see [5]). Let $T \leftrightarrow T(.)$ be the canonical correspondence between M and $C(X)$. For each vector Tv, $T \in M$, define $\phi(Tv) = T(.)$ in $C(X)$. ϕ is well defined, for if $T_1 v = T_2 v$, $(0) = M(T_1 - T_2)v = (T_1 - T_2)Mv$ and $T_1 - T_2 = 0$, $T_1(.) = T_2(.)$. We define the following integral on $C(X)$; $\lambda(T(.)) = (Tv, v)$. $\| \phi(Tv) \|_2^2 = \lambda(T(.)^- T(.)) = (T^* Tv, v) = \| Tv \|^2$. λ is faithful for v is a separating vector for M. This integral is implemented by a unique measure on X as in [4], pp. 29–37. To save notation we call this measure λ also. ϕ is norm preserving where the norm on $C(X)$ is the $L_2(X, \lambda)$ norm. The closure of the manifold $[Mv]$ is h and the L_2 closure of $C(X)$ is $L_2(X, \lambda)$. ϕ extends in a unique manner to an isometry of h onto $L_2(X, \lambda)$. If $T \in M$, $\phi(A_n Tv) = A_n T(.) = A_n(.)T(.) = L_{A_n(.)}\phi(Tv)$ on $L_2(X, \lambda)$ where $L_{A_n(.)}$ is multiplication on $L_2(X, \lambda)$ by $A_n(.)$. Since $\phi A_n = A_n \phi$ on a dense manifold in h, $\phi A_n \phi^{-1} = L_{A_n(.)}$ on $L_2(X, \lambda)$. This holds simultaneously for all n. The functions $A_n(.)$ separate points in X because M is the C^*-algebra generated by the operators A_n and the identity. We can consequently identify X with a compact subset E of $\boldsymbol{C}$ as in [3], p. 152; for $x \in X$, $x \leftrightarrow (A_1(x), A_2(x), A_3(x), \cdots, A_n(x), \cdots)$. Observe that the function $A_n(.)$ is carried into f_n on $\boldsymbol{C}$. λ implements a finite measure m on E in the obvious manner. We extend m to all of $\boldsymbol{C}$ by making all subsets of the complement of E m null. The multiplication algebra of $(\boldsymbol{C}, m)$ is essentially the same as the multiplication algebra of (X, λ). Hence we are able to state

LEMMA 1. *For any wss sequence $\{A_n\}$ on a separable Hilbert space h there exists a finite measure m on $\boldsymbol{C}$ concentrated on a compact subset such that $\{A_n\}$ is spatially equivalent $\{L_{f_n}\}$ on $L_2(\boldsymbol{C}, m)$ where f_n is the nth coordinate function.*

It remains only to show that there exists a *proper* measure with the same property. By means of topological arguments we could adjust m so that it is proper, but the following result is more elegant and will be particularly significant in our future papers on unbounded operators. For a measure m on $\boldsymbol{C}$ as in Lemma 1 there exists a finite proper measure m_1 on $\boldsymbol{C}$ with the same property.

To prove this assertion let v be the vector in $L_2(\boldsymbol{C}, m)$ which is identically 1. $L_{f_n}v$ is a vector in $L_2(\boldsymbol{C}, m)$, hence the coordinate functions f_n are m measurable. All the proper sets are m measurable and so are all the proper functions. We need only prove that every bounded m measurable function is a.e. equal to a proper function, then make m_1

the reduction of m to the proper sets. The operators of the form L_f, f bounded, proper, constitute a *-algebra the weak closure of which contains all the L_{f_n}. Since $\{A_n\}$, $\{L_{f_n}\}$ are *wss*, the strong closure of this algebra is a masa, more precisely the multiplication algebra of $(\boldsymbol{C}, m)$. Let S be an m measurable set. Let g_n be a proper bounded function such that $\| g_n - \chi_S \| < 2^{-n}$, all n. $\| Re\, g_n - \chi S \| \leqq \| g_n - \chi_S \|$ so we can suppose g_n is real valued. But $| \chi_{\{x; g_n(x) \geqq \frac{1}{2}\}} - \chi_S | \leqq 2 | g_n - \chi_S |$. $\| \chi_{\{x; g_n(x) \geqq \frac{1}{2}\}} - \chi_S \| < 2^{1-n}$. Define the proper set $E_n = \bigcup_{i \geqq n} \{x; g_i(x) \geqq \frac{1}{2}\}$. $\| \chi_{E_n} - \chi_S \| < 2^{2-n}$ and $\chi_{E_n} \chi_S = \chi_S$ a. e., all n. Define the proper set $E = \bigcap_n E_n$. $\| \chi_E - \chi_S \| = 0$. Each m measurable set S is equal to a proper set E modulo null sets. Now each bounded nonnegative m measurable function f is the uniform limit of a nondecreasing sequence of m measurable step functions s_n. Each s_n is a. e. equal to a proper step function s'_n such that s'_n is nondecreasing as n increases. An argument employing bounded monotone sequences of numbers shows that s'_n converges pointwise to a proper function which must equal f a.e. This proves

LEMMA 2. *For any wss sequence $\{A_n\}$ on separable Hilbert space there exists a finite proper measure m on $\boldsymbol{C}$ such that $\{A_n\}$ is spatially equivalent $\{L_{f_n}\}$ on $L_2(\boldsymbol{C}, m)$ where f_n is the* nth *coordinate function.*

3. Spatial invariants for wss sequences. Suppose m is a finite proper measure on $\boldsymbol{C}$ concentrated on a compact set E. $L_2(\boldsymbol{C}, m)$ is a separable Hilbert space because $\boldsymbol{C}$ is second countable. $\{L_{f_n}\}$ is a normal sequence on $L_2(\boldsymbol{C}, m)$ where f_n is the nth coordinate function. Let "$Re\, f_n$" and "$Im\, f_n$" denote respectively the real and imaginary parts of f_n. Let M be the abelian W^*-algebra generated by all the L_{f_n}. $L_{f_n^*} = L_{\bar{f_n}} \in M$. $L_{Re\, f_n}, L_{Im\, f_n} \in M$. For any real r, the function $r \wedge (Re\, f_n)$ is the uniform limit on E of polynomials in $Re\, f_n$. Consequently $L_{r \wedge (Re\, f_n)} \in M$. Likewise $L_{r \wedge (Im\, f_n)} \in M$. It follows that for any open set U in C $L_{\chi_{f_n^{-1}(U)}} \in M$, all n. M is the multiplication algebra of $(\boldsymbol{C}, m)$ and $\{L_{f_n}\}$ is wss. Thus we have

LEMMA 3. *Any finite proper measure on $\boldsymbol{C}$ concentrated on a compact set defines a wss sequence as described above.*

Now suppose m and m_1 are two finite proper measures concentrated on the respective compact sets E_1 and E_2. Let L_{f_n} be multiplication on $L_2(\boldsymbol{C}, m)$ by the nth coordinate function f_n, and let T_{f_n} be multiplication on $L_2(\boldsymbol{C}, m_1)$ by f_n. Suppose furthermore there exists an isometry W of $L_2(\boldsymbol{C}, m)$ onto $L_2(\boldsymbol{C}, m_1)$ such that $W L_{f_n} W^{-1} = T_{f_n}$ on $L_2(\boldsymbol{C}, m_1)$, all n. $L_{f_n^*} = L_{\bar{f_n}}$, $T_{f_n^*} = T_{\bar{f_n}}$. $W L_{Re\, f_n} W^{-1} = T_{Re\, f_n}$ and $W L_{Im\, f_n} W^{-1} = T_{Im\, f_n}$, all n. $E_1 \cup E_2$ is compact. For any real number r, $r \wedge (Re\, f_n)$ is the uniform limit on $E_1 \cup E_2$ of appropriate polynomials in $Re\, f_n$. Hence $WL_{r\wedge(Re\, f_n)}W^{-1} = T_{r\wedge(Re\, f_n)}$ and likewise $WL_{r\wedge(Im\, f_n)}W^{-1} = T_{r\wedge(Im\, f_n)}$. Consequently for any open set U in C, $WL_{\chi_{f_n^{-1}(U)}}W^{-1} = T_{\chi_{f_n^{-1}(U)}}$. And for any proper set E in $\boldsymbol{C}$, $WL_{\chi_E}W^{-1} = T_{\chi_E}$. Thus a proper set E in $\boldsymbol{C}$ is not see copy the 0 operator on $L_2(\boldsymbol{C}, m_1)$ if and only if T_{χ_E} is not the 0 operator on $L_2(\boldsymbol{C}, m_1)$ if and only if E is not m_1 null. m and m_1 are equivalent measures. We state

LEMMA 4. *If two finite proper measures concentrated on compact subsets of* $\boldsymbol{C}$ *define spatially equivalent wss sequences, they are equivalent measures.*

Next we suppose m and m_1 are equivalent finite proper measures concentrated on compact sets. Let dm_1/dm denote the Radon-Nikodym derivative function. We define the mapping W of $L_2(\boldsymbol{C}, m)$ onto $L_2(\boldsymbol{C}, m_1)$ as follows; for $f \in L_2(\boldsymbol{C}, m)$, $Wf = (dm_1/dm)^{\frac{1}{2}}f$. The proofs that W is an isometry of $L_2(\boldsymbol{C}, m)$ onto $L_2(\boldsymbol{C}, m_1)$ and that $W(.)W^{-1}$ demonstrates the spatial equivalence of the *wss* sequences defined by the respective measures m and m_1, we leave to the reader.

LEMMA 5. *If two finite proper measures concentrated on compact subsets of* $\boldsymbol{C}$ *are equivalent, the wss sequences respectively defined by them are spatially equivalent.*

With each *wss* sequence on separable Hilbert space we associate a finite proper measure concentrated on a compact subset of $\boldsymbol{C}$ as was done in Lemma 2, and all such finite proper measures equivalent to it. Thus every *wss* sequence is assigned a family of equivalent finite proper measures concentrated on a compact subset of $\boldsymbol{C}$, and two *wss* sequences are assigned the same family if and only if they are spatially equivalent. These families are the spatial invariants we seek. Furthermore any such family is assigned to a *wss* sequence unique within spatial equivalence by Lemmas 3 and 5. Consequently such families are models of *wss* sequences. To recapitulate

THEOREM 1. *To each wss sequence on separable Hilbert space we have assigned a family of equivalent finite proper measures concentrated on a compact subset of* $\boldsymbol{C}$, *and two wss sequences are assigned the same family if and only if they are spatially equivalent. Such families are models of wss sequences in the sense that each is assigned to a wss sequence unique within spatial equivalence.*

Suppose finally that A is a *wss* operator on separable Hilbert space. Then $(A, 0, 0, \cdots, 0, \cdots)$ is a *wss* sequence. Let $\mathscr{F}$ be the family assigned to this sequence in Theorem 1. We claim that the proper sets $\{x; f_n(x) \neq 0\}$, $n > 1$ are null with respect to $\mathscr{F}$. Otherwise there would be a positive number r such that $E = \{x; |f_n(x)| \geqq r\}$ is not null with respect to $\mathscr{F}$, some $n > 1$, and $L_{f_n}\chi_E$ (in any of the measures in $\mathscr{F}$) is not the 0 vector contrary to the fact that all operators in the *wss* sequence after the first are zero. Consequently the set $\bigcup_{n>1}\{x; f_n(x) \neq 0\}$ is null with respect to $\mathscr{F}$. $\mathscr{F}$ is concentrated on the set of elements in $\boldsymbol{C}$ constructed by restricting all coordinates after the first to 0. Each measure in $\mathscr{F}$ induces a Baire measure on C (consider only the first coordinate in $\boldsymbol{C}$) and these Baire measures are equivalent and are concentrated on a compact subset of C. It is evident that these Baire measures on C constitute the spatial invariant constructed for A in the original weighted spectrum theorem (see the bibliography of [5]). Thus Theorem 1 is a bona fide generalization of the weighted spectrum theorem for *wss* operators on separable Hilbert space.

4. Spatial invariants for normal sequences. Any normal sequence on separable Hilbert space is the direct sum of unique i-fold copies of *wss* sequences, $i = \infty, 1, 2, 3, \cdots$ (see [5]). We could easily make our spatial invariants for normal sequences mappings from the cardinals $\infty, 1, 2, 3, \cdots$ to families of the form described in § 3. However the invariants we will employ are much simpler.

Let $\mathscr{F}$ be a family of equivalent finite proper measures concentrated on a compact subset of $\boldsymbol{C}$ and let $\{E_i\}$ be a sequence of disjoint proper sets modulo null sets, that is $E_i \cap E_j$, $i \neq j$ are null with respect to $\mathscr{F}$. $\{L_{f_n}\}$ restricted to $L_2(E_i, \mathscr{F})$ is a *wss* sequence where $L_2(E_i, \mathscr{F})$ is the Hilbert space $L_2(E_i, m)$ for some $m \in \mathscr{F}$ and L_{f_n} is multiplication on $L_2(E_i, m)$ by the nth coordinate function (see Lemma 5). Let $\{B_{in}\}$ denote this *wss* sequence. Let $\{A_n\}$ be the direct sum of the i-fold copies of $\{B_{in}\}$, $0 < i < \infty$, together with the ∞-fold copy of $\{L_{f_n}\}$ restricted to $L_2(\boldsymbol{C} - \bigcup_{i<\infty} E_i, \mathscr{F})$. $\{A_n\}$ is a normal sequence on separable Hilbert space, and by Lemma 3 the W^*-algebra generated by the A_n is the direct sum of the i-fold copies of the multiplication algebras of $(E_i, \mathscr{F})$ together with the ∞-fold copy of the multiplication algebra of $(\boldsymbol{C} - \bigcup_{i<\infty} E_i, \mathscr{F})$.

LEMMA 6. *Any family of equivalent finite proper measures concentrated on a compact subset of $\boldsymbol{C}$ together with a sequence of mutually disjoint proper sets ($E_i \cap E_j$, $i \neq j$, null) defines a normal sequence on separable Hilbert space unique within spatial equivalence as described above.*

Now let $\{A_n\}$ be a normal sequence on separable Hilbert space h, and let M be the abelian W^*-algebra generated by the operators A_n.

For each cardinal $i = \infty, 1, 2, \cdots$, there is a unique subspace h_i of h which together generate h, such that M_{h_i} (the reduction of M to h_i) is an i-fold copy of a masa. Let k_i be subspaces of the respective h_i such that M_{k_i} is a masa algebraically *-isomorphic to M_{h_i}. Let k be the subspace of h generated by all the k_i. M_k is the direct sum of the masas M_{k_i}; hence M_k is a masa and the reduction of $\{A_n\}$ to k is a *wss* sequence, call it $\{B_n\}$. Let $\mathscr{F}$ be the family assigned to $\{B_n\}$ in Theorem 1. Let P_{k_i} be the projection with range k_i. $P_{k_i} \in M_k$. P_{k_1} defines a proper set E_1 modulo null sets (with respect to $\mathscr{F}$); more precisely P_{k_1} is multiplication by χ_{E_1} on $L_2(\boldsymbol{C}, \mathscr{F})$. In a similar manner the projection P_{k_2} defines the proper set E_2 modulo null sets, P_{k_3} defines E_3, etc. Thus we have associated with the normal sequence $\{A_n\}$ a family of equivalent finite proper measures concentrated on a compact subset of $\boldsymbol{C}$ and a sequence of mutually disjoint proper sets modulo null sets. A review of our construction discloses that these are spatial invariants. If $\{A_n\}$ and $\{A'_n\}$ are spatially equivalent normal sequences on separable Hilbert space we have assigned the same entities to them. By the procedure described in the second paragraph of § 4 the sequence $\{A_n\}$ may be constructed within spatial equivalence from the invariants just assigned to it. And finally any such family and sequence of proper sets described in that paragraph are the invariants assigned to some appropriate normal sequence on separable Hilbert space. To recapitulate

THEOREM 2. *We have assigned to each normal sequence on separable Hilbert space a family of equivalent finite proper measures concentrated on a compact subset of* $\boldsymbol{C}$ *together with a sequence of mutually disjoint proper subsets of* $\boldsymbol{C}$ *modulo null sets, such that two normal sequences are spatially equivalent if and only if they are assigned the same entities. Furthermore such a family and sequence of subsets are assigned to some normal sequence unique within spatial equivalence. This defines models of normal sequences within spatial equivalence.*

The following conclusions concerning the invariants assigned to a normal sequence are evident. The sequence is *wss* if and only if E_1 (in the sequence of subsets) equals $\boldsymbol{C}$ modulo null sets. $\boldsymbol{C}$ equals $\bigcup_{i<\infty} E_i$ modulo null sets if and only if there is no uniform ∞-multiplicity involved (see [5]). E_i is null if and only if there is no uniform multiplicity i involved (see [5]).

5. Projection valued measures. The classical spectral theorem (see [6]) states that for each bounded normal operator A on a Hilbert space h there exists a projection valued measure P from the Baire sets of C to projections in $\mathscr{L}(h)$ such that $A = \int_C x dP$ where x is the current variable (since we take A to be bounded the integral may be regarded as the *uniform* limit of the usual sums). Our purpose is to generalize the classical spectral theorem so as to apply it to a normal sequence. Given any normal sequence $\{A_n\}$ on a Hilbert space h we will find a projection valued measure P from the proper subsets of $\boldsymbol{C}$ to projections in the W^*-algebra generated by the A_n such that $A_n = \int_C f_n dP$ all n, where f_n is the nth coordinate function on $\boldsymbol{C}$ and the integral is understood to be the uniform limit of the usual sums.

In order to develop this let $\{A_n\}$ be a normal sequence on a separable Hilbert space h. Let M be the W^*-algebra generated by all the A_n. For each cardinal $i = \infty, 1, 2, \cdots$ there is a subspace h_i of h such that M_{h_i} is an i-fold copy of a masa. For each such cardinal i select a subspace k_i of h_i such that M_{k_i} is a masa algebraically *-isomorphic M_{h_i}. Let k be the subspace of h generated by all the k_i. M_k is the direct sum of the masas M_{k_i}; hence M_k is a masa. $\{A_n\}$ reduced to k is a *wss* sequence. Let ϕ be the mapping which carries each operator in M into its reduction to k. It suffices to solve the problem first on k, then apply ϕ^{-1} to the projections in M_k thus obtained.

In other words we may now assume that $\{A_n\}$ is *wss*. By § 3 we assign to $\{A_n\}$ a family $\mathscr{F}$ of equivalent finite proper measures concentrated on a compact subset of $\boldsymbol{C}$. We identify h with $L_2(\boldsymbol{C}, \mathscr{F})$ in such a way that A_n is multiplication on $L_2(\boldsymbol{C}, \mathscr{F})$ by the nth coordinate function. For each proper subset E in $\boldsymbol{C}$ let the projection measure of E be multiplication on $L_2(\boldsymbol{C}, \mathscr{F})$ by the characteristic function of E. By the definition of the integral the desired conclusion is trivial.

THEOREM 3. *Given any normal sequence $\{A_n\}$ on a separable Hilbert space, there exists a measure P from proper subsets of $\boldsymbol{C}$ to projections in M, the abelian W^*-algebra generated by all the A_n, such that $A_n = \int_C f_n dP$ simultaneously for all n where f_n is the* nth *coordinate function of $\boldsymbol{C}$ and where the integral is understood to be the uniform limit of the usual sums.*

In conclusion we show that the hypothesis in Theorem 3 that the underlying Hilbert space is separable can be discarded. Zorn's axiom shows that there exists a family of vectors (v_α) such that the manifolds $\{Mv_\alpha\}$ are multually orthogonal and together generate h. Each of these manifolds reduces all the A_n and A_n^*. It suffices to show that the closures of each of those manifolds is a separable subspace of h. Theorem 3 solves the problem on each separable subspace and we merely sum to get the projections we desire.

A Hilbert space is separable if there is a W^*-algebra with a cyclic vector on the space generated by countably many operators. The verification of this we leave to the reader.

6. Shift invariant measures. In § 2 we defined the topological space $\boldsymbol{C}$ to be the cartesian product of a countable infinity of copies of C, the complex plane. For each integer i, positive, negative or zero, let C_i be a copy of C. We can regard $\boldsymbol{C}$ as $\prod_{-\infty}^{\infty} C_i$. Let s be the shift mapping on $\boldsymbol{C}$ defined $s\{\cdots, x_i, \cdots\} = \{\cdots, y_i, \cdots\}$ where $\{\cdots, x_i, \cdots\} \in \boldsymbol{C}$ and $y_i = x_{i-1}$. Plainly s and s^{-1} carry proper sets onto proper sets. s also preserves all the set operations on proper sets. A proper measure m is said to be "shift invariant" if $m(E) = m(sE)$ for any proper set E.

Let m and m_1 be two shift invariant finite proper measures on $\boldsymbol{C}$ which are equivalent measures. Let dm_1/dm be the Radon-Nikodym derivative function. For any proper set E we have $\int_E dm_1/dm(.)dm = m_1(E) = m_1(sE) = \int_{sE} dm_1/dm(.)dm = \int_E dm_1/dm(s.)dm$. It follows that dm_1/dm is shift invariant; that is, $dm_1/dm(.) = dm_1/dm(s.)$ a.e. with respect to m and m_1. Let U, V be the respective unitaries on $L_2(\boldsymbol{C}, m)$, $L_2(\boldsymbol{C}, m_1)$ induced by the shift mapping s. Let W be the spacial isometry from $L_2(\boldsymbol{C}, m)$ onto $L_2(\boldsymbol{C}, m_1)$ defined as follows; $Wf = (dm_1/dm)^{\frac{1}{2}}f$ for $f \in L_2(\boldsymbol{C}, m)$. For $g \in L_2(\boldsymbol{C}, m_1)$,

$$\begin{aligned} WUW^{-1}g &= WU(dm_1/dm)^{-\frac{1}{2}}(.)g(.) = W(dm_1/dm)^{-\frac{1}{2}}(s.)g(s.) \\ &= (dm_1/dm)^{\frac{1}{2}}(.)(dm_1/dm)^{-\frac{1}{2}}(s.)g(s.) = Vg , \end{aligned}$$

a. e. $WUW^{-1} = V$ on $L_2(\boldsymbol{C}, m_1)$. Furthermore if L_{f_0}, T_{f_0} denote the respective multiplications on $L_2(\boldsymbol{C}, m)$, $L_2(\boldsymbol{C}, m_1)$ by the 0th coordinate function f_0, then $W(UL_{f_0})W^{-1} = VT_{f_0}$ on $L_2(\boldsymbol{C}, m_1)$. Consequently

LEMMA 7. *If m, m_1 are equivalent finite proper shift invariant measures on $\boldsymbol{C}$, then UL_{f_0} is spatially equivalent VT_{f_0} where U (respectively V) is the unitary on $L_2(\boldsymbol{C}, m)$ (respectively $L_2(\boldsymbol{C}, m_1)$) induced by the shift mapping s on $\boldsymbol{C}$, and L_{f_0} (respectively T_{f_0}) is multiplication on $L_2(\boldsymbol{C}, m)$ (respectively $L_2(\boldsymbol{C}, m_1)$) by the 0th coordinate function f_0.*

We now apply our preceding work to the spatial invariants theory of certain individual operators. Let t be a measure preserving one-to-one point transformation of a finite separable measure space (X, λ) onto itself. Let f be a nonnegative bounded measurable function on X which is positive a.e., such that the operators $L_{f(t^i.)}$, $-\infty < i < \infty$, generate the multiplcation algebra of (X, λ). Let U be the unitary on $L_2(X, \lambda)$ implemented by the mapping t. Observe that $U^{-i}L_{f(.)}U^i = L_{f(t^i.)}$, $-\infty < i < \infty$. Also the polar decomposition of UL_f is $(U)(L_f)$ because U is unitary and L_f is nonsingular.

DEFINITION. *An operator UL_f on separable Hilbert space which* *arises as in the preceding paragraph (and any operator spatially equivalent such an operator) is said to be amenable* (the author is indebted to H. A. Dye for this concept).

In particular, a nonnegative nonsingular *wss* operator on separable Hilbert space is amenable; just let t be the identity. And the spatial invariants theory for bounded self adjoint operators on separable Hilbert space reduces to that of operators of this kind (see [5]).

Now suppose UL_f is amenable, (X, λ) the measure space as above. Express $\boldsymbol{C}$ as $\prod_{-\infty}^{\infty} C_i$. Let $R \subset C$ denote the product of the copies of the positive real axis. There exists a family $\mathscr{F}$ of equivalent finite proper measures concentrated on a compact subset of R and a spatial isometry W of $L_2(X, \lambda)$ onto $L_2(\boldsymbol{C}, \mathscr{F})$ such that $W(.)W^{-1}$ carries all $L_{f(t^i.)}$ on $L_2(X, \lambda)$ into multiplication by the ith coordinate function on $L_2(\boldsymbol{C}, \mathscr{F})$, $-\infty < i < \infty$. This was established in § 3—the indexing was different but that does not matter. $W(.)W^{-1}$ implements a Boolean isomorphism between the projections in the multiplication algebras of (X, λ) and $(\boldsymbol{C}, \mathscr{F})$, and hence a correspondence σ between measurable subsets of X (modulo λ null sets) and proper subsets of $\boldsymbol{C}$ (modulo null sets with respect to $\mathscr{F}$). λ and σ implement in an obvious manner a finite proper measure m concentrated on a compact subset of R. It is evident that a proper set is $\mathscr{F}$ null if and only if it is m null. Consequently $m \in \mathscr{F}$. $\sigma t \sigma^{-1}$ is an m preserving set transformation modulo m null sets of the proper subsets of $\boldsymbol{C}$ which preserves set operations; to save notation call it t also. On $L_2(\boldsymbol{C}, m)$, $L_{f_i(t.)} = L_{f_{i+1}(.)}$, $-\infty < i < \infty$, by construction where f_i is the ith coordinate function. It follows that $tE = sE$ modulo m null sets for proper subsets E in $\boldsymbol{C}$. Hence m is shift invariant. σ extends uniquely to a spatial isometry of $L_2(X, \lambda)$ onto $L_2(\boldsymbol{C}, m)$. Let V be the unitary on $L_2(\boldsymbol{C}, m)$ induced by the shift mapping s. $\sigma U \sigma^{-1} = V$ on $L_2(\boldsymbol{C}, m)$. $\sigma L_f \sigma^{-1} = L_{f_0}$ on $L_2(\boldsymbol{C}, m)$ where f_0 is the 0th coordinate function and f on X is given as in the definition of amenable operator. Hence $\sigma(UL_f)\sigma^{-1} = VL_{f_0}$ on $L_2(\boldsymbol{C}, m)$. We state

LEMMA 8. *If UL_f is an amenable operator arising from a finite measure space (X, λ), measurable function f and transformation t (as in the definition of amenable operator), there exists a finite shift invariant proper measure m concentrated on a compact subset of R and a spatial isometry σ of $L_2(X, \lambda)$ onto $L_2(\boldsymbol{C}, m)$ such that $\sigma L_{f(t^i.)}\sigma^{-1} = L_{f_i}$, $-\infty < i < \infty$, on $L_2(\boldsymbol{C}, m)$ where f_i is the* ith *coordinate function on $\boldsymbol{C}$, and $\sigma U \sigma^{-1} = V$ on $L_2(\boldsymbol{C}, m)$ where U (respectively V) is the unitary on $L_2(X, \lambda)$ (respectively $L_2(\boldsymbol{C}, m)$) implemented by the transformation t (respectively s).*

Thus Lemma 8 permits us to associate with each amenable operator UL_f a family of equivalent finite proper measures concentrated on a compact subset of $\boldsymbol{C}$ which contains at least one shift invariant member.

Lemma 7 proves that two amenable operators are assigned the same family only if they are spatially equivalent.

Now suppose UL_f and VL_g are spatially equivalent amenable operators on separable Hilbert spaces. There is a spatial isometry W such that $W(UL_f)W^{-1} = VL_g$. By the uniquesness of the polar decompositions of these amenable operators we have $WUW^{-1} = V$ and $WL_fW^{-1} = L_g$. Hence $W(U^iL_fU^{-i})W^{-1} = V^iL_gV^{-i}$, $-\infty < i < \infty$. The *wss* sequences $\{U^iL_fU^{-i}\}$ and $\{V^iL_gV^{-i}\}$ are spatially equivalent. By § 3 we have assigned UL_f and VL_g the same family.

It is trivial that family of equivalent finite proper measures concentrated on a compact subset of R which contains at least one shift invariant member, is assigned to an appropriate amenable operator (see Lemma 7). We recapitulate § 6 in

THEOREM 4. *To each amenable operator we have assigned a family of equivalent finite proper measures concentrated on a compact subset of R which contains at least one shift invariant member. Two amenable operators are spatially equivalent if and only if they are assigned the same family. A family of equivalent finite proper measures concentrated on a compact subset of R is assigned to some amenable operator if and only if it contains at least one shift invariant member, and f_0 is positive a.e.*

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