EXISTENCE OF A CLASS OF STEADY PLANE GRAVITY FLOWS

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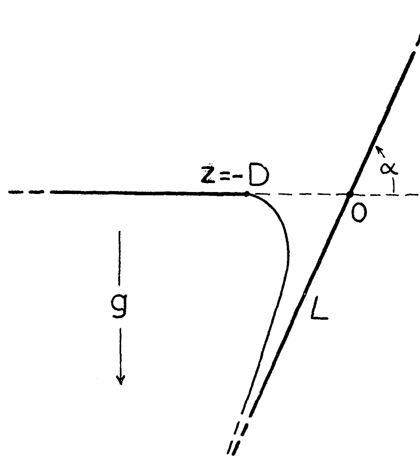
1. Introduction. A number of exact solutions representing free boundary flows of an incompressible fluid under gravity appear in the literature (see [1] and references given there). As pointed out in [1], however, these are obtained by "inverse" methods, and exact theoretical treatment of problems in the large, having prescribed fixed boundaries and singularities, appears to have been confined to the case of periodic gravity waves.

In this paper we consider the family of steady plane irrotational flows of an incompressible inviscid fluid in a uniform gravitational field, with geometric configurations as illustrated in Figure 1. The fluid is



supported by a semi-infinite horizontal plane, and is bounded on the right by an infinite plane inclined at an angle α with the horizontal. The flow is downward through an open slot in the horizontal plane into a jet with a free boundary extending to infinity. This family includes the case of the symmetric jet from a slot, obtained when $\alpha = \pi/2$ by reflecting the flow across the vertical boundary.

In addition to the angle α, the physical parameters entering the problem include the constant specific force of gravity g, the slot width D, the fluid velocity at the slot edge q, and the total flow rate A (cross-sectional area of fluid entering the jet per unit time). Our principal result, contained in Theorem 1, § 3, and Theorem 4, § 11, is that

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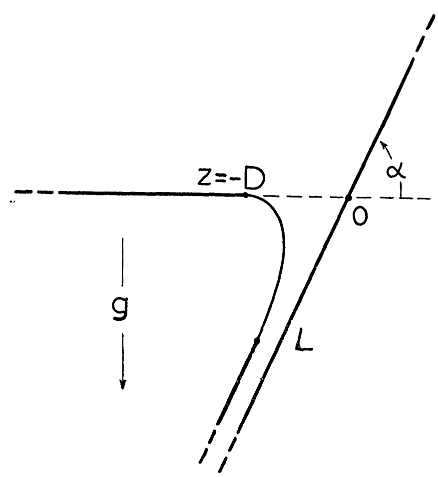
there is a similarity class of such flows for each choice of the dimensionless pair α, gA/q^3, such that $0 < \alpha < \pi$, $gA/q^3 > 0$. (Questions of uniqueness, continuity and monotonicity of the solutions will be discussed in another communication, where it will be shown that each pair α, gA/q^3 determines a unique similarity class).

A noteworthy property of the solutions is the convexity of the free boundaries. It is implicit in the method of proof that the inclination of each free boundary arc decreases monotonically from zero at the slot edge to $-\pi + \alpha$ as an asymptotic limit.

The problem is formulated in § 2, and reduced in § 3 to a nonlinear boundary value problem on the unit disk in an auxiliary complex plane (t-plane), involving the parameters $\beta = 2(\pi - \alpha)/\pi$, $\lambda = 6gA/\pi q^3$, and the function $u = (w/q)^{1/\beta}$, where w is the conjugate flow velocity. To circumvent a singularity due to to the infinite nature of the free boundary, a modified problem is introduced in § 4, with solutions corresponding to flows as illustrated in Figure 2, where the free jet extends only a finite



distance before flowing into a straight channel. The existence of these finite jet solutions (Theorem 2, § 4) is established in the next five sections by a combination of fixed point and conformal mapping techniques. In § 10 the analytic continuation of finite jet solutions beyond the unit disk is discussed, in preparation for the proof in § 11 that the infinite jet solutions exist as limits of normal families of finite jet solutions.

2. Mathematical formulation. Position in the complex plane of the flow is denoted by $z = x + iy$, the complex velocity potential by $W = U + iV$, and the conjugate velocity by $w = dW/dz$. For brevity, the line $\mathfrak{Im}\{e^{-i\alpha}z\} = 0$ is denoted by L. Then the mathematical construction corresponding to the flow consists of

(a) a simply-connected domain in the z-plane, bounded on the right by L, and bounded below and on the left by a simple curve consisting of a negative real segment $-\infty < z \leqq -D$ and a free boundary arc extending downward to $y = -\infty$.

(b) a function $W(z)$ which maps this domain conformally onto the strip $0 < V < A$ so that the horizontal boundary and the free boundary go into $V = 0$, and the inclined boundary L goes into $V = A$. The derivative $w = dW/dz$ is continuous on the closure of the flow domain, has the positive value $w(-D) = q$ at the slot edge, and satisfies the condition of constant pressure

$$|w|^2 = q^2 - 2gy \tag{1}$$

on the free boundary.

The solutions established below have the additional property

(c) The inclination of the free boundary is strictly decreasing from the value zero at the slot edge to the asymptotic limit $-\pi + \alpha$ near $y = -\infty$ where the free boundary is asymptotic to L.

3. Reduction to a problem on the unit disk. Following a standard hodograph method (outlined in [1]), the problem can be reduced to a nonlinear boundary value problem on the unit disk in an auxiliary plane. For purposes of the existence proof we reverse the usual derivation, and start with a direct formulation of the reduced problem:

Let β and λ be real numbers such that $0 < \beta < 2$, $\lambda > 0$. A function $u(t)$ is a solution of the reduced problem corresponding to the pair β, λ, provided u is

(A) continuous on the unit disk $|t| \leqq 1$, with $t = i$ and $t = -i$ excluded, and regular and univalent on the interior,

(B) real on the real diameter and pure imaginary on the imaginary diameter,

(C) a solution of the following integral equation on the quarter-circle $t = e^{i\sigma}$, $0 \leqq \sigma < \pi/2$:

The modulus $v(\sigma) = |u(e^{i\sigma})|$ and principal argument $\varphi(\sigma) = \arg u(e^{i\sigma})$ are related by

$$v(\sigma) = \left\{1 + \lambda \int_0^\sigma \tan\rho \sin[\beta\varphi(\rho)]\,d\rho\right\}^{1/3\beta} \tag{2}$$

(D) monotonic on the same quarter-circle in the sense that as σ varies from 0 to $\pi/2$, φ is strictly increasing between the same limits, and $v(\sigma)$ is also strictly increasing.

To interpret the reduced problem we have

THEOREM 1. *Let α, g, q, A be strictly positive numbers, where $\alpha < \pi$. Let $u(t)$ be a solution of the reduced problem corresponding to the pair $\beta = 2(\pi - \alpha)/\pi$, $\lambda = 6gA/\pi q^3$. Let*

$$(3) \qquad W(t) = (2A/\pi)\log[t/(1+t^2)]$$

$$(4) \qquad w(t) = qu^\beta(t)$$

$$(5) \qquad z(t) = \int^t [1/w(s)]dW(s) \ .$$

Then the constant of integration can be chosen so that $z(t)$ maps the first quadrant of the unit disk onto a flow domain as described above. The positive real radius goes into the horizontal part of the flow boundary, the positive imaginary radius goes into L, and the circumference goes into the free boundary. The function W, "transplanted" to the flow plane by the mapping $z(t)$, is the complex potential for a flow of the required type with the given flow parameters; and the function w, similarly transplanted, is the conjugate velocity.

The proof is contained in the following five propositions:

1. The function $w(t)$ maps the first quadrant of the unit disk into the sector $0 \leqq \arg w \leqq \pi - \alpha$ so that $w(1) = q$, the real radius goes into $\arg w = 0$, and the imaginary radius goes into $\arg w = \pi - \alpha$. The circumference goes into a monotonic arc in the sense that $\arg w(e^{i\sigma})$ is strictly increasing between the limits 0, $\pi - \alpha$, and $|w(e^{i\sigma})|$ is strictly increasing. Moreover w has an expansion of the form $t^\beta \sum_{n=0}^\infty a_{2n}t^{2n}$ with real coefficients, and with $a_0 > 0$, convergent for $|t| < 1$.

Proof. Conditions (C) and (D) on u imply $u(1) = v(0)e^{i\varphi(0)} = 1$. Since u is univalent, symmetric about the real and imaginary axes in the sense of (B), and positive at $t = 1$, the real and imaginary parts of u agree in sign with those of t. Therefore the Maclauren expansion of u has the form $t(b_0 + b_2t^2 + \cdots)$, in odd powers of t with real coefficients, and with $b_0 > 0$. The proposition follows with the help of condition (D), since $w = qu^\beta$, $\beta = 2(\pi - \alpha)/\pi$.

2. The constant of integration can be chosen so that $z(t)$ has an expansion of the form $t^{-\beta}(c_0 + c_1t^2 + \cdots)$ with real coefficients and with $c_0 < 0$, convergent for $|t| < 1$. Thus $z(t)$ maps the positive real radius onto a real segment extending to $x = -\infty$, and the positive imaginary radius goes onto a segment of L extending to $y = +\infty$.

Proof. From equations (3) and (4) we obtain

$$dz/dt = (1/w)(dW/dt) = 2A(1-t^2)/[t(1+t^2)\pi w] \ .$$

In view of **1**, therefore, dz/dt has an expansion $t^{-1-\beta}[d_0 + d_2t^2 + \cdots]$ with real coefficients and $d_0 > 0$. The proposition follows on taking the term-by-term integral $z = t^{-\beta}[-d_0/\beta + d_2t^2/(2-\beta) + \cdots]$.

Henceforth the integration constant is determined as in **2**, and the branch of the logarithm in equation (3) is chosen so that

3. The function $W(t)$ maps the first quadrant onto the strip $0 \leqq V \leqq A$, so that the real radius and circumference go into $V = 0$ and the imaginary radius into $V = A$.

4. The free boundary equation (1) is satisfied on the curve $z(e^{i\sigma})$, $0 \leqq \sigma < \pi/2$.

Proof. Equations (2) and (4) imply

$$|\, w(e^{i\sigma}) \,|^3 = q^3\left\{1 + \lambda \int_0^\sigma \tan \rho \sin [\arg w(e^{i\rho})] d\rho \right\} .$$

Setting $\lambda = 6gA/\pi q^3$, differentiating, and dividing by $3\,|\,w\,|$ gives $|\,w\,|\, d\,|\,w\,| = [2gA \tan \sigma \mathfrak{Im}\{w\}/\pi\, |\,w\,|^2] d\sigma$. Here

$$\mathfrak{Im}\{w\}/|\,w\,|^2 = -\mathfrak{Im}\{1/w\} = -\mathfrak{Im}\{dz/dW\} ,$$

and in view of (3), $dW = (2A/\pi) \tan \sigma d\sigma$. Hence $|\,w\,|\, d\,|\,w\,| = -g\mathfrak{Im}\{dz\} = -gdy$. Integrating gives (1), since $|\,w(1)\,| = q$.

5. The function $z(t)$ maps the first quadrant onto a flow domain as described above.

Proof. First, the free boundary arc $z(e^{i\sigma})$ extends to $y = -\infty$. This follows from (1) provided $w(e^{i\sigma})$ is unbounded. From the last paragraph we have $|\,w\,|\, d\,|\,w\,| = [2g\mathfrak{Im}\{w\}/|\,w\,|^2\} dW$. In view of **1**, $\mathfrak{Im}\{w\}$ is positive and bounded away from zero near $\sigma = \pi/2$. Therefore $|\,w\,|^4$ exceeds a positive constant times W, which is unbounded.

Secondly, the free boundary is convex in the sense that its inclination $\arg\{dz/dW\} = -\arg w(e^{i\sigma})$ is decreasing between the limits $0, \alpha - \pi$, in accordance with **1**.

Now let U denote the image under $z(t)$ of a curve $U = Re\{W(t)\} =$ constant, directed from the positive imaginary radius to the circumference near $t = i$. From (5) we have $d(e^{-i\alpha}z) = (i/e^{i\alpha}w)dV$ on U. Here V is decreasing, and in view of **1** $Re\{e^{i\alpha}w\} < 0$, $\mathfrak{Im}\{e^{i\alpha}w\} > 0$, so that $Re\{e^{-i\alpha}z\}$ is decreasing and $\mathfrak{Im}\{e^{-i\alpha}z\}$ is increasing on U. According to **2** the initial point of U is on L. The other endpoint is on the free boundary; and it follows in view of the convexity that the entire free boundary curve is in the half-plane $\mathfrak{Im}\{e^{-i\alpha}z\} > 0$, on the left of L. Moreover U does not cross either L or the free boundary. Finally, the length of U, $\int_0^A |\,w\,|^{-1} dV$, vanishes as $U \to +\infty$ (i.e., as the curve $U =$ constant "approaches" $t = i$); for we have observed that $|\,w(e^{i\sigma})\,|$ is unbounded near $\sigma = \pi/2$, so that $|\,w\,|^{-1}$ vanishes near $t = i$.

Together with 2, these results show that $z(t)$ maps the real radius and circumference onto the left boundary of a flow domain as described in § 2, condition (a), that the imaginary radius goes into the entire line L, and that the free boundary is asymptotic to L. To show that the interior of the first quadrant goes into the domain between these two boundaries, consider the image U' of a second curve $U =$ constant joining the real and imaginary radii near $t = 0$. From the expansion of $z(t)$ in 2 we see that U' joins the negative real axis with L, without crossing either. By a familiar application of the argument principle it follows that $z(t)$ is univalent and takes its values in the required domain.

Theorem 1 now follows, since composition of $W(t)$ with the inverse of $z(t)$ is the complex potential of a flow with conjugate velocity $w = dW/dz$.

4. The finite jet approximation. To avoid the singularity in the integral equation (2) due to the unbounded factor $\tan \rho$, we will replace $\tan \sigma$ by the bounded function

$$h_\gamma(\sigma) = 2 \sin 2\sigma/(2 \cos 2\sigma + \gamma^2 + \gamma^{-2}) \,, \quad 0 < \gamma < 1 \tag{6}$$

which approximates $\tan \sigma$ when γ is near 1. More precisely, we will consider the "finite jet problem" obtained by modifying the boundary value problem of § 3 as follows: First, u is required to be continuous for all $|t| \leqq 1$, and secondly, equation (2) is replaced by

$$v(\sigma) = \left\{1 + \lambda \int_0^\sigma h_\gamma(\rho) \sin [\beta\varphi(\rho)] d\rho \right\}^{1/3\beta} . \tag{7}$$

An appropriate modification of Theorem 1, in which $W(t)$ is replaced by $W_\gamma(t) = (A/\pi) \log [t^2(t^2 + \gamma^2)^{-1}(t^2 + \gamma^{-2})^{-1}]$, shows that solutions of this problem provide flows as illustrated in Figure 2. In the next five sections we will prove

THEOREM 2. *The finite jet problem has a solution for each triple* β, γ, λ, *such that* $0 < \beta < 2$, $0 < \gamma < 1$, *and* $\lambda > 0$.

5. The operator T. Let Φ be the family of all continuous real-valued functions $\varphi(\sigma)$ on the interval $[0, \pi/2]$, which are nondecreasing between the limits $\varphi(0) = 0$, $\varphi(\pi/2) = \pi/2$, and such that $\varphi(\sigma) \leqq \sigma$. Now we define an operator T on Φ which serves to reduce the finite jet approximation to a fixed point problem:

First, with each $\varphi \in \Phi$ *we associate the complex-valued function* $ve^{i\varphi}$, *where* $v(\sigma)$ *is determined by* φ *through equation* (7). Now the integrand in (7) vanishes wherever $\varphi(\rho) = 0$, but is strictly positive as soon as $\varphi(\rho)$ becomes positive. Hence $v = 1$ where $\varphi = 0$, and v is strictly increasing thereafter. Moreover v is bounded independently of φ by the number

$$M = \left[1 + \lambda\int_0^{\pi/2} h_\gamma(\rho)d\rho\right]^{1/3\beta} . \tag{8}$$

Combining this with the requirement $\varphi(\sigma) \leqq \sigma$, we have

LEMMA 1. *For each $\varphi \in \Phi$ the function $ve^{i\varphi}$ maps $[0, \pi/2]$ onto a simple Jordan arc C_I in the u-plane, in the intersection of the first quadrant with the annulus $1 \leqq |u| \leqq M$. C_I intersects the real and imaginary axes only in its endpoints $u = 1$, $iv(\pi/2)$. Moreover this mapping is a homeomorphism of $[\sigma_0, \pi/2]$ onto C_I, where $\sigma_0 =$* $\sup\{\sigma\colon \varphi(\sigma) = 0\}$.

Next we adjoin to C_I its reflections about the real and imaginary axes and through the origin. This provides a simple closed Jordan curve C, which encloses a simply-connected domain D, symmetric about both axes and containing the origin. Let $\hat{u}(t)$ be the function which is continuous for $|t| \leqq 1$ and which maps $|t| < 1$ conformally onto D so that $\hat{u}(0) = 0$ and the derivative $\hat{u}'(0)$ is positive. Let $\hat{v}(\sigma) = |\hat{u}(e^{i\sigma})|$ and $\hat{\varphi}(\sigma) = \arg \hat{u}(e^{i\sigma})$ for $0 \leqq \sigma \leqq \pi/2$. *The operator T is defined by $T\varphi = \hat{\varphi}$.*

Leaving aside for the moment the question of whether T does in fact map Φ into itself, we have

LEMMA 2. *Suppose that T has a "fixed point" φ which is also a strictly increasing member of Φ. Then the corresponding function $u = \hat{u}(t)$ is a solution of the finite jet approximation.*

Proof. We are to show that if φ is strictly increasing and $\varphi = \hat{\varphi}$, then $u = \hat{u}$ satisfies conditions (A) through (D) of § 3, as modified in § 4. Conditions (A) and (B), as well as the continuity of u for all $|t| \leqq 1$ follow from the definition of $\hat{u}$ together with the symmetry of D. The restriction of $\hat{u}$ to $|t| = 1$ is a homeomorphism of the unit circumference onto C, and consequently $\hat{v}(\sigma)e^{i\hat{\varphi}(\sigma)}$ is a homeomorphism of $[0, \pi/2]$ onto C_I. Since φ is strictly increasing it follows from Lemma 1 that $v(\sigma)e^{i\varphi(\sigma)}$ is a similar homeomorphism. Moreover there is only one value of φ for each point on C_I, and, since $\hat{\varphi} = \varphi$, the homeomorphisms are identical. Therefore $\hat{v} = v = |u(e^{i\sigma})|$ satisfies (7). Condition (D) is satisfied since φ and v are both strictly increasing; and the proof is complete.

It should be noticed that the strictly increasing property of φ plays an essential role in the proof, beyond the relatively unimportant fact that it is needed for condition (D). If φ were constant on some interval, we could not conclude that $v = \hat{v}$ there, and equation (7) would not follow.

Now the family Φ is clearly a closed, bounded, convex subset of the Banach space of continuous functions on $[0, \pi/2]$ with the "uniform norm"

$\|\varphi\| = \sup\{|\varphi(\sigma)| : 0 \leqq \sigma \leqq \pi/2\}$. In the next three sections it is shown that T is a continuous transformation of Φ into itself whose range has compact closure (i.e., T is a "completely continuous operator"). In accordance with a theorem of Schauder [2], it follows that T has a fixed point. To complete the proof of Theorem 2, it is shown in § 9 that every fixed point is strictly increasing.

6. Proof that T maps Φ into itself.

LEMMA 3. *Let $f(t)$ be regular on the unit disk, continuous on the closure, real on the real diameter, and pure imaginary on the imaginary diameter. Suppose further that f vanishes only at $t = 0$, that $f'(0)$ is positive, and that $F(\sigma) = |f(e^{i\sigma})|$ is of bounded variation. Then for $0 < |t| < 1$,*

$$\arg\{f(t)/t\} = \int_0^{\pi/2} k(t, \rho) d \log F(\rho)$$

where the kernel $k(t, \rho) = \pi^{-1} \log |(e^{2i\rho} - t^2)/(e^{-2i\rho} - t^2)|$ is negative for $0 < \arg t < \pi/2$.

Proof. Let $g(t) = \log[f(t)/t]$, where the branch of the logarithm is chosen so that $g(0) = \log f'(0)$ in real. Applying Poisson's formula we have

$$Re\{g(t)\} = Re\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} [(e^{i\rho} + t)/(e^{i\rho} - t)] \log F(\rho) d\rho\right\} ,$$

since $Re\{g(e^{i\sigma})\} = \log F(\sigma)$. Since the integral is regular for $|t| < 1$, it differs from $g(t)$ by an imaginary constant, which is zero since both functions are real at $t = 0$. Integration by parts gives

$$\arg\{f(t)/t\} = \mathfrak{Im}\{g(t)\} = \pi^{-1}\int_{-\pi}^{\pi} \log |e^{i\rho} - t| \, d \log F(\rho) .$$

This can be reduced to the required formula by means of the symmetry conditions $F(\sigma) = F(-\sigma) = F(\pi - \sigma)$. The proof is completed by observing that $|e^{2i\rho} - t^2| \leqq |e^{-2i\rho} - t^2|$ when $e^{i\rho}$ and t are both in the first quadrant, so that $k(t, \rho) \leqq 0$ there.

Returning to the operator T, it follows from the definition of $\hat{u}$ that $\hat{u}(e^{i\sigma}) = \hat{v}(\sigma)e^{i\hat{\varphi}(\sigma)}$ is a homeomorphism of $[0, \pi/2]$ onto C_I, and in the same direction as $v(\sigma)e^{i\varphi(\sigma)}$. Hence $\hat{\varphi}$, like φ, is nondecreasing between the limits $0, \pi/2$, and $\hat{v}$ is increasing. Taking $f(t) = \hat{u}(t)$ in Lemma 3, so that $F(\sigma) = \hat{v}(\sigma)$, it follows that $\arg\{\hat{u}(re^{i\sigma})\} - \sigma \leqq 0$ for $0 < r < 1$, $0 \leqq \sigma \leqq \pi/2$. In the limit $r = 1$ this gives $\hat{\varphi}(\sigma) \leqq \sigma$. Thus $T\varphi = \hat{\varphi}$ belongs to Φ.

7. Continuity of T. The proof of continuity is based on the following theorem of Radó [3][1]:

THEOREM 3. *Let D be a domain containing the origin and bounded by a closed Jordan curve C. Let $u(t)$ map $|t| < 1$ conformally onto D, subject to the conditions $u(0) = 0$, $u'(0) > 0$. If $\{D_n\}$ is a sequence of such regions "approaching" D, then the corresponding sequence of functions $\{u_n\}$ approaches u uniformly on $|t| \leqq 1$ if and only if the Fréchet distance between the boundaries C_n and C approaches zero.* (The Fréchet distance is the infimum, over all homeomorphisms between the curves, of the maximum distance between corresponding points.)

Now let φ and $\tilde{\varphi}$ be any two members of Φ, and let v, C_I, C, and $\tilde{v}, \tilde{C}_I, \tilde{C}$ be the functions and curves involved in the construction of $T\varphi$ and $T\tilde{\varphi}$, respectively. We will show that the Fréchet distance between C and $\tilde{C}$ is not greater than $k\,||\varphi - \tilde{\varphi}||$, where the number k is independent of φ and $\tilde{\varphi}$. Then it follows from Theorem 3 and the fact that C is bounded away from the origin (i.e., $|u| \geqq 1$ on C) that T is continuous.

Suppose that $||\varphi - \tilde{\varphi}|| > 0$; and let $[0, \sigma_0]$, $[0, \tilde{\sigma}_0]$ be the maximal intervals on which $\varphi, \tilde{\varphi}$ vanish, respectively. Let $[0, \sigma_1]$ be the maximal interval on which $\varphi \leqq 2\,||\varphi - \tilde{\varphi}||$. Clearly $\pi/2 \geqq \sigma_1 > \max\{\sigma_0, \tilde{\sigma}_0\}$. Now consider the correspondence $\sigma \rightarrow \sigma^*$ defined by

$$\sigma^* = \begin{cases} \tilde{\sigma}_0 + (\sigma - \sigma_0)(\sigma_1 - \tilde{\sigma}_0)/(\sigma_1 - \sigma_0) & \text{for } \sigma_0 \leqq \sigma \leqq \sigma_1 \\ \sigma & \text{for } \sigma \geqq \sigma_1 \end{cases} .$$

This is a homeomorphism of $[\sigma_0, \pi/2]$ onto $[\tilde{\sigma}_0, \pi/2]$; and in view of Lemma 1 the correspndence $v(\sigma)e^{i\varphi(\sigma)} \rightarrow \tilde{v}(\sigma^*)e^{i\tilde{\varphi}(\sigma^*)}$ is a homeomorphism of C_I onto $\tilde{C}_I$. The distance between corresponding points does not exceed

$$\begin{aligned} &|v(\sigma)e^{i\varphi(\sigma)} - v(\sigma^*)e^{i\varphi(\sigma^*)}| + |v(\sigma^*)e^{i\varphi(\sigma^*)} - \tilde{v}(\sigma^*)e^{i\tilde{\varphi}(\sigma^*)}| \\ &\quad \leqq |v(\sigma) - v(\sigma^*)| + M|\varphi(\sigma) - \varphi(\sigma^*)| + |v(\sigma^*) - \tilde{v}(\sigma^*)| \\ &\qquad + M|\varphi(\sigma^*) - \tilde{\varphi}(\sigma^*)| \end{aligned}$$

where M is the upper bound on v in (8). The first two terms vanish except when $\sigma, \sigma^* \in [0, \sigma_1]$. If K is a Lipschitz constant such that $|x^{1/3\beta} - y^{1/3\beta}| \leqq K|x - y|$ for $1 \leqq x \leqq y \leqq M^{3\beta}$, it follows that

$$|v(\sigma) - v(\sigma^*)| \leqq K\left|\lambda\int_\sigma^{\sigma^*} h_\gamma(\rho) \sin[\beta\varphi(\rho)d\rho\right| \leqq KM^{3\beta}\beta\cdot 2\,||\varphi - \tilde{\varphi}|| .$$

Moreover $M|\varphi(\sigma) - \varphi(\sigma^*)| \leqq 4M\,||\varphi - \tilde{\varphi}||$,

$$\begin{aligned} |v(\sigma^*) - \tilde{v}(\sigma^*)| &\leqq K\lambda\int_\sigma^{\sigma^*} h_\gamma(\rho)\,|\sin\beta\varphi(\rho) - \sin\beta\tilde{\varphi}(\rho)|\,d\rho \\ &\leqq KM^{3\beta}\beta\,||\varphi - \tilde{\varphi}|| , \end{aligned}$$

[1] See also the discussion in the opening paragraphs of [**4**].

and $M\,|\varphi(\sigma^*) - \tilde{\varphi}(\sigma^*)| \leqq M\,||\varphi - \tilde{\varphi}||$. Hence the Fréchet distance between C_I and $\tilde{C}_I$ is not greater than $(3\beta KM^{3\beta} + 5M)\,||\varphi - \tilde{\varphi}||$; and in view of the symmetry, this is also an upper bound on the Fréchet distance between C and $\tilde{C}$.

8. Complete continuity. To establish the complete continuity of T we will show that its range is an equicontinuous family, and hence has compact closure. The chief tool is[2].

LEMMA 4. *Let $|t| < 1$ be mapped conformally onto a simply connected domain D of finite area A. Let t_0 be a point on $|t| = 1$, and k_r be that part of the circle $|t - t_0| = r$ which is contained in $|t| < 1$. Then for every r, $0 < r < 1$, there is an s, $r \leqq s \leqq r^{1/2}$, such that the image of k_s is a crosscut of D of length $l_s \leqq (-2\pi A/\log r)^{1/2}$.*

Now let D be the domain in the u-plane corresponding to $\varphi \in \Phi$. The function $\hat{\varphi} = \arg \hat{u}(e^{i\sigma})$, which is equal to $T\varphi$ on $[0, \pi/2]$, is actually defined and monotonic for all real σ. To obtain a modulus of continuity for $\hat{\varphi}$, let $\delta_0 = e^{-(\pi M)^2/2}$ where M is given by (8), and let σ, σ', δ be any three numbers such that $0 < |\sigma - \sigma'| \leqq \delta \leqq \delta_0$. Then the points $e^{i\sigma}, e^{i\sigma'}$ are inside the circle of radius $r = \delta$ about $t_0 = e^{i\sigma}$. Applying Lemma 4, let $e^{i\tau}$ and $e^{i\tau'}$ be the endpoints of an arc k_s. Since $\hat{\varphi}$ is monotonic we have $|\hat{\varphi}(\sigma) - \hat{\varphi}(\sigma')| \leqq |\hat{\varphi}(\tau) - \hat{\varphi}(\tau')|$. Now D is contained in the disk $|u| \leqq M$, so the length of k_s is $l_s \leqq (-2\pi^2 M^2/\log \delta) = 2(\log \delta_0/\log \delta) \leqq 2$. Since the boundary of D is outside the unit circle, it follows that $|\hat{\varphi}(\tau) - \hat{\varphi}(\tau')| = |\arg \hat{u}(e^{i\tau}) - \arg \hat{u}(e^{i\tau'})|$ is not greater than the angle subtended at the origin by a chord of the unit circle of length l_s. Therefore the principal value of 2 arcsin $(\log \delta_0/\log \delta)$ is a modulus of continuity of $\hat{\varphi}$ for $\delta \leqq \delta_0$, which is independent of φ.

The proof that T has a fixed point is now complete.

9. Strictly increasing nature of the fixed points. For the proof that every fixed point of T is a strictly increasing function it is convenient to change variables from t to ζ by the transformation

$$\zeta(t) = \frac{\lambda}{2} \log \frac{Bt^2}{(t^2 + \gamma^2)(t^2 + \gamma^{-2})}\,, \qquad B > 0\,. \tag{9}$$

Taking the principal branch of the logarithm, we see that $\zeta(t)$ maps the first quadrant of the unit disk onto the strip $0 \leqq \mathfrak{Im}\{\zeta\} \leqq \lambda\pi/2$ so that the real radius, the circumference, and the imaginary segment $t = i\tau$, $\gamma < \tau < 1$, go into the real axis, and the rest of the imaginary radius goes into the line $\mathfrak{Im}\{\zeta\} = \lambda\pi/2$. At the same time we change variables

[2] The statement of Lemma 4 follows Warschawski [**5**], who gives a proof and refers to J. Wolff [**6**]. The essential idea is given earlier by Courant [**7**].

from σ to $\zeta(e^{i\sigma})$, so that φ, v, and $\hat{v}$ become functions of ζ, defined on the real segment $a \leqq \zeta \leqq b$ corresponding to $0 \leqq \sigma \leqq \pi/2$. The location of this segment can be chosen at will by adjusting the value of B in (9).

With these changes the relation (7) between φ and v becomes

$$v(\zeta) = \left[1 + \int_a^\zeta \sin\,[\beta\varphi(\xi)]d\xi\right]^{1/3\beta} \tag{10}$$

since $\lambda h_\gamma(\sigma) = d\zeta(e^{i\sigma})/d\sigma$; and the relation $\hat{v}(\sigma) = |\hat{u}(e^{i\sigma})|$ becomes $\hat{v}(\zeta) = |\hat{u}(\zeta)|$, $a \leqq \zeta \leqq b$.

Let $\varphi(\zeta)$ correspond in this way to a fixed point of T. In accordance with Lemma 1 and the construction of $\hat{u}$, both $ve^{i\varphi}$ and $\hat{v}e^{i\varphi}$ are homeomorphisms of $[a, b]$ onto C_I; where v and $\hat{v}$ are both strictly increasing. Since C_I intersects the real and imaginary axes only in its endpoints we have

1. $\varphi(\zeta)$ is strictly increasing at the endpoints a, b.

Now let $\mathscr{C}$ be the open real set consisting of all $\zeta \in [a, b]$ such that φ is constant on some neighborhood of ζ; and let $\mathscr{T}$ be the complement of $\mathscr{C}$ relative to $[a, b]$.

2. For all $\zeta \in \mathscr{T}$, $v(\zeta) = \hat{v}(\zeta)$.

Proof. We will show that $v(\zeta) \neq \hat{v}(\zeta)$ implies $\zeta \in \mathscr{C}$. If $v(\zeta) > \hat{v}(\zeta)$, there is a $\zeta_1 < \zeta$ such that $v(\zeta_1) = \hat{v}(\zeta)$ and a $\zeta_2 > \zeta$ such that $v(\zeta) = \hat{v}(\zeta_2)$. Since there is at most one point on C_I for each value of $|u|$, we have $v(\zeta_1)e^{i\varphi(\zeta_1)} = \hat{v}(\zeta)e^{i\varphi(\zeta)}$ and $v(\zeta)e^{i\varphi(\zeta)} = \hat{v}(\zeta_2)e^{i\varphi(\zeta_2)}$. Hence $\varphi(\zeta_1) = \varphi(\zeta_2)$, so that φ is constant on the interval (ζ_1, ζ_2), and $\zeta \in \mathscr{C}$. Similarly, $v(\zeta) < \hat{v}(\zeta)$ implies $\zeta \in \mathscr{C}$.

3. The function $\hat{u}(\zeta)$ is regular on the real set $\mathscr{C}$. Moreover the logarithmic derivative $\hat{u}'/\hat{u} = \hat{v}'/\hat{v}$ is strictly increasing on each open interval contained in $\mathscr{C}$.

The proof depends on the following result, which is essentially a corollary of a Lavrentieff-Serrin comparison theorem [8]:

LEMMA 5. *Let R_1 and R_2 be two closed simply-connected regions such that R_1 contains R_2. Suppose that there are continuous functions $f_1(\zeta)$ and $f_2(\zeta)$ which map the strip $0 \leqq \mathfrak{Im}\{\zeta\} \leqq Q$ onto R_1 and R_2, respectively, and which are conformal on the interior of the strip. Let the image of $\mathfrak{Im}\{\zeta\} = Q$ under f_2 be a subarc of the corresponding image under f_1; and let the images of $\mathfrak{Im}\{\zeta\} = 0$ have a common point $f_1(\zeta_1) = f_2(\zeta_2)$. If f_1 has a continuous nonvanishing derivative f_1' on a semi-neighborhood $|\zeta - \zeta_1| < \rho$, $\mathfrak{Im}\{\zeta\} \geqq 0$, of ζ_1 and f_2 has a continuous* *nonvanishing derivative f_2' on a similar semi-neighborhood of ζ_2, then $|f_1'(\zeta_1)| \leqq |f_2'(\zeta_2)|$. Equality holds only if R_1 and R_2 are identical.*

To prove 3, we note first that on any open interval (α, β) contained in $\mathscr{C}$, $e^{-i\varphi(\alpha)}\hat{u}(\zeta) = \hat{v}(\zeta)$ is real. Hence, by the reflection principle, $\hat{u}$ is regular there. Moreover its derivative does not vanish since $\hat{u}$ is univalent.

Let ζ_1, ζ_2 be points in (α, β) such that $\zeta_1 < \zeta_2$; and let $f_1(\zeta) = \hat{u}(\zeta)$, $f_2(\zeta) = \mu\hat{u}(\zeta)$, where $\mu = \hat{u}(\zeta_1)/\hat{u}(\zeta_2) < 1$. Since $\varphi = \arg \hat{u}$ is increasing on $[a, b]$, f_1 and f_2 map $0 \leqq \mathfrak{Im}\{\zeta\} \leqq \lambda\pi/2$ onto starlike regions R_1, R_2 of the first quadrant such that R_1 contains R_2. Moreover f_1 maps $\mathfrak{Im}\{\zeta\} = \lambda\pi/2$ (corresponding to the imaginary segment $0 < t/i < \gamma$ in the t-plane) onto an imaginary segment $0 < u/i < k$, and the corresponding image under f_2 is the subsegment $0 < u/i < \mu k$. Applying Lemma 5 we obtain $|\hat{u}'(\zeta_1)| < |\mu\hat{u}'(\zeta_2)|$. Proposition 3 follows since $\hat{u} = e^{i\varphi(\alpha)}\hat{v}$, $\hat{u}' = e^{i\varphi(\alpha)}\hat{v}'$ on (α, β), where $\hat{v}' > 0$.

4. For all $\zeta \in \mathscr{C}$, $\hat{v}(\zeta) < v(\zeta)$. Moreover, if (α, β) is any maximal open interval contained in $\mathscr{C}$, $\lim_{\zeta \to \alpha_+} \hat{v}'(\zeta)/\hat{v}(\zeta) < v'(\alpha)/v(\alpha)$.

Proof. If (α, β) is a maximal open interval contained in $\mathscr{C}$, then its endpoints belong to $\mathscr{T}$. In view of 2 and 3, $\log \hat{v}$ is a strictly "concave" function on (α, β), varying between the limits $\log v(\alpha)$, $\log v(\beta)$. According to (7), on the other hand, $\log v(\zeta)$ varies linearly there since φ is constant; and the proposition follows.

Now we restrict our attention to a fixed maximal open interval (α, β) of $\mathscr{C}$, and adjust the parameter B in (9) so that $\alpha = 0$. Let $\varDelta$ denote a neighborhood $|\zeta| < R$ of the origin, cut along the real segment $-R < \zeta < 0$. In view of 1, α is greater than a, and we can choose $R > 0$ so that the segment $-R < \zeta < 0$ is contained in $[a, b]$, the segment $0 < \zeta < R$ is contained in (α, β), and $R < \lambda\pi/2$. Finally we set $g(\zeta) = \log \hat{u}(\zeta)$, where $\mathfrak{Im}\{g(\zeta)\} = \varphi(0)$ for $0 \leqq \zeta \leqq R$.

5. For all $\zeta \in \varDelta$ we have

$$g(\zeta) = p(\zeta) - \pi^{-1}\int_{-R}^{0} \log(\zeta - \tau) d\varphi(\tau) \tag{11}$$

where $p(\zeta)$ is a convergent power series in ζ, and the branch of the logarithm is such that $\log(\zeta - \tau)$ is real for $\zeta - \tau > 0$.

Proof. Since $g(\zeta) - i\varphi(0)$ is real for $0 \leqq \zeta \leqq R$, the analytic continuation of g across this segment is given by $g(\zeta) = 2i\varphi(0) + \overline{g(\bar{\zeta})}$, so that g is continuous on the *cut* disk $\varDelta$. The discontinuity *across* the cut is $g(\zeta) - g(\bar{\zeta}) = 2i[\varphi(0) - \varphi(\zeta)]$ for $\arg \zeta = -\pi$, $\arg \bar{\zeta} = \pi$.

Now the integral in (11) is defined and regular except on the cut, so this equation serves to define $p(\zeta)$ at least for $-\pi < \arg\zeta < \pi$. Let $\zeta = \xi + i\eta$, $\eta < 0$, and consider the difference $p(\zeta) - p(\bar\zeta) = g(\zeta) - g(\bar\zeta) + \pi^{-1}\int_{-R}^{0}\log[(\zeta-\tau)/(\bar\zeta-\tau)]d\varphi(\tau)$. Here $\log[(\zeta-\tau)/(\bar\zeta-\tau)] = 2i\arg(\zeta-\tau)$ is bounded, and as $\eta \to 0$ it approaches $-2\pi i$ for $\xi < \tau$ and 0 for $\xi > \tau$. Therefore $p(\zeta) - p(\bar\zeta)$ is bounded and

$$\lim_{\eta\to 0}[p(\xi+i\eta) - p(\xi-i\eta)] = 2i[\varphi(0) - \varphi(\xi)] - 2i\int_{\xi}^{0}d\varphi(\tau) = 0\,. \tag{12}$$

To show that p is regular for all $|\zeta| < R$, let A be the circular arc $|\zeta| = R$, $\Im\mathrm{m}\{\zeta\} \geqq \varepsilon$, where $0 < \varepsilon < R$, and let B be the segment $\Im\mathrm{m}\{\zeta\} = \varepsilon$ joining the endpoints of A. Let Γ be the contour $A + B$, and $\bar\Gamma$ be its reflection across the real axis. Now take ζ_1 inside either Γ or $\bar\Gamma$, and consider the sum of the integrals of $p(\zeta)/(\zeta - \zeta_1)$ around Γ and $\bar\Gamma$. The total contribution of the horizontal segments is

$$\int_B\left[\frac{p(\zeta)}{\zeta-\zeta_1} - \frac{p(\bar\zeta)}{\bar\zeta-\zeta_1}\right]d\zeta$$
$$= \int\frac{[p(\xi+i\varepsilon) - p(\xi-i\varepsilon)](\xi-\zeta_1) - i\varepsilon[p(\xi+i\varepsilon)+p(\xi-i\varepsilon)]}{(\xi+i\varepsilon-\zeta_1)(\xi-i\varepsilon-\zeta_1)}d\xi\,.$$

In view of (11) and (12) this vanishes as $\varepsilon \to 0$, so that $2\pi i p(\zeta_1) = \int_{|\zeta|=R}p(\zeta)(\zeta-\zeta_1)^{-1}d\zeta$. Thus $p(\zeta)$ is regular and has a convergent Maclauren series for $|\zeta| < R$. Finally, (11) holds on the cut, since $g(\zeta) - p(\zeta)$ is continuous on each side and the unbounded part of the integrand, $\log|\zeta - t|$, converges monotonically as $\Im\mathrm{m}\{\zeta\}$ vanishes.

6. The fixed point φ is strictly increasing.

Proof. We will use the representation (11) for $g = \log\hat u$ to obtain

$$\lim_{\zeta\to 0_+}\hat v'(\zeta)/\hat v(\zeta) \geqq v'(0)/v(0)\,. \tag{13}$$

This contradicts 4 if the open set $\mathscr{C}$ contains an open interval. Therefore $\mathscr{C}$ is void and φ is strictly increasing.

For $0 < \zeta < R$ we have $g'(\zeta) = \hat v'(\zeta)/\hat v(\zeta) = p'(\zeta) - \pi^{-1}\int_{-R}^{0}(\zeta-\tau)^{-1}d\varphi(\tau)$. Here g' is positive, so the last term, being negative, converges monotonically to a finite limit. Hence

$$\lim_{\zeta\to 0_+}\hat v'(\zeta)/\hat v(\zeta) = p'(0) + \pi^{-1}\int_{-R}^{0}\tau^{-1}d\varphi(\tau)\,. \tag{14}$$

For $\arg\zeta = \pi$, $-R < \zeta < 0$, we have

$$g(0) - g(\zeta) = p(0) - p(\zeta) + \pi^{-1}\int_{-R}^{0}\log(1 - \zeta/\tau)d\varphi(\tau)\,.$$

The last term is

$$\pi^{-1}\int_{-R}^{0}\log|1-\zeta/\tau|\,d\varphi(\tau)+i[\varphi(0)-\varphi(\zeta)]\,.$$

Taking real parts and using the inequality $\log x \leqq x-1$ we obtain $\log\hat{v}(0)-\log\hat{v}(\zeta)\leqq p(0)-p(\zeta)-\pi^{-1}\int_{-R}^{\zeta}(\zeta/\tau)d\varphi(\tau)+\pi^{-1}\int_{\zeta}^{0}(\zeta/\tau)d\varphi(\tau)$. In view of 2 and 4, $\hat{v}(0)=v(0)$ and $\hat{v}(\zeta)\leqq v(\zeta)$, so that

$$\begin{aligned}&[\log v(0)-\log v(\zeta)]/(-\zeta)\\&\quad\leqq[p(0)-p(\zeta)]/(-\zeta)+\pi^{-1}\int_{-R}^{\zeta}\tau^{-1}d\varphi(\tau)-\pi^{-1}\int_{\zeta}^{0}\tau^{-1}d\varphi(\tau)\,.\end{aligned}$$

Taking the limit $\zeta\to 0_-$ and combining with (14) gives (13).

The proof of Theorem 2 is now complete.

10. Analytic continuation of finite jet solutions. To obtain the infinite free boundary solutions as limits of finite jet solutions we require

LEMMA 6. *For each pair λ,β such that $\lambda>0$, $0<\beta<2$, there is a domain Δ in the t-plane containing the closed unit disk except for $t=\pm 1,\pm i$, such that for all $0<\gamma<1$ the finite jet solutions can be continued analytically onto Δ, where they form a locally uniformly bounded family of regular functions.*

The fact that analytic continuation beyond the unit disk is possible follows from the work of Lewy [9]. To establish the required uniform estimates we will give a separate proof, based on the differential equation

$$(15)\qquad d\tilde{v}/dt=f(t,\tilde{v})\equiv t\zeta'(t)[(u/\tilde{v})^{\beta}-(\tilde{v}/u)^{\beta}]/[6\beta\tilde{v}^{3\beta-1}]$$

where $\zeta(t)$ is given by (9). This is obtained from the boundary condition (7) by setting $t=e^{i\sigma}$, $\tilde{v}(t)=v(\sigma)$, differentiating, and observing that $\lambda h_{\gamma}(\sigma)=ie^{i\sigma}\zeta'(e^{i\sigma})$ and $e^{i\beta\varphi}=(u/\tilde{v})^{\beta}$. The method[3] is to show that equation (15) has a solution $\tilde{v}$ which is regular on a domain of the form $t=re^{i\sigma}$, $R(\sigma)<r<1$, $0<\sigma<\pi/2$, continuous and nonvanishing on the closure of this domain, and equal to $v(\sigma)$ for $t=e^{i\sigma}$. Then $\tilde{\varphi}=-i\log u/\tilde{v}$ is regular on the same domain, continuous on the closure, and equal to $\varphi(\sigma)$ for $t=e^{i\sigma}$. Since both $\tilde{v}$ and $\tilde{\varphi}$ are equal on the circumference, they can be continued across this circular arc by reflection; and the required analytic continuation of u in the first quadrant is given by $u=\tilde{v}e^{i\tilde{\varphi}}$ for $t=re^{i\sigma}$, $1<r<1/R(\sigma)$. The continuation in the other quadrants is given by the symmetry of u. Lemma 6 is valid provided

[3] Suggested by the work of Vitousek [**10**], and illustrated for a closely related case in [**11**].

$R(\sigma)$, together with local bounds on $|u|$, can be chosen independently of γ. To this end we have

1. The solutions $u(t)$ satisfy $|t| \leqq |u(t)| \leqq [1 - \lambda \log \cos \theta]^{1/3\beta}$ on each sector $|t| \leqq 1$, $0 \leqq \arg t \leqq \theta < \pi/2$.

Proof. The first inequality follows by applying the maximum principle to $t/u(t)$, which is regular for $|t| < 1$ and whose maximum absolute value for $|t| = 1$ is $1/|u(1)| = 1$.

Since u maps $|t| < 1$ onto a starlike domain we have (see **[12]**, Ch. V) $Re\{tu'/u\} = \partial \log |u| / \partial \log |t| \geqq 0$ for $|t| < 1$, so that $|u|$ is increasing on each radius. Since $|u(e^{i\sigma})|$ is also increasing for $0 \leqq \sigma \leqq \pi/2$, it follows that $|u(t)| \leqq v(\theta)$ on the sector in question. Estimating $v(\theta)$ with the help of (6) and (7) yields the second inequality, since $0 \leqq h_\gamma(\sigma) \leqq \tan \sigma$.

The next proposition follows easily with the help of **1**:

2. For $0 < \theta < \pi/2$, let Δ_θ be the domain $\frac{1}{2} < |t| < 1$, $0 < \arg t < \theta$, and let D_θ be the annulus $\frac{1}{2} \leqq |\tilde{v}| \leqq \frac{1}{2} + [1 - \lambda \log \cos \theta]^{1/3\beta}$. Then the function $f(t, \tilde{v})$ of equation (15) is regular in both arguments on the product region $\Delta_\theta \times D_\theta$, and continuous on the closure $\bar{\Delta}_\theta \times D_\theta$. Moreover there are numbers M_θ and L_θ, independent of γ, such that for $t \in \bar{\Delta}_\theta$ and $\tilde{v}, \tilde{v}_1 \in D_\theta$, $|f(t, \tilde{v})| \leqq M_\theta$ and $|f(t, \tilde{v}) - f(t, \tilde{v}_1)| \leqq L_\theta |\tilde{v} - \tilde{v}_1|$.

To construct the required solution of (15) near $t_0 = e^{i\theta}$, let K_θ be the intersection of $\bar{\Delta}_\theta$ with the disk $|t - t_0| \leqq \min\{1/2M_\theta, 1/2L_\theta, 1/2\}$, and let V be the family of functions $\tilde{v}(t)$ which map K_θ continuously into D_θ and are regular on the interior of K_θ. Then V is a complete space with respect to the uniform metric. Now consider the operator J on V defined by $J\tilde{v} = v(\theta) + \int_{t_0}^{t} f(s, \tilde{v}(s))ds$, where the path of integration is in K_θ. It follows from **1**, **2**, and the definition of K_θ, that J maps V into itself and is a contraction operator. Therefore J has a unique fixed point $\tilde{v}_0$, which is a solution of (15) and equal to $v(\theta) = |u(t_0)|$ at t_0. Since (15) was chosen so that $|u(t)|$ is a solution on the arc $|t| = 1$, it follows from the contraction property of J that $\tilde{v}_0 = |u|$ there. Also because of the contraction property, solutions corresponding to two different values of θ are equal on the intersection of the corresponding regions K_θ. This provides a solution $\tilde{v}$ on the region $t = re^{i\sigma}$, $R(\sigma) < r \leqq 1$, $0 < \sigma < \pi/2$, with $R(\sigma) = 1 - \min\{1/2M_\sigma, 1/2L_\sigma, 1/2\}$, such that $|\tilde{v}(re^{i\sigma})| \leqq 1/2 + [1 - \lambda \log \cos \sigma]^{1/3\beta}$ and $|u| \geqq 1/2$. Lemma 6 follows in view of the remarks preceding **1**.

11. Existence of infinite jet solutions.

THEOREM 4. *For every pair λ, β such that $\lambda > 0$, $0 < \beta < 2$, the* *infinite free boundary problem described in* § 3 *has a solution.*

Proof. It follows from Lemma 6, in accordance with Montel's Theorem, that the family of all finite jet solutions for given λ, β, is *normal* on a domain $\varDelta$ containing the closed unit disk, with $t = \pm 1, \pm i$ excluded. Hence there is a sequence $\{u_n\}$ of such functions for which the corresponding sequence $\{\lambda_n\}$ approaches 1, and which converges uniformly on every closed subset of $\varDelta$. We will show that the limit, u, of this sequence satisfies conditions (A) through (D) of § 3.

First, u is clearly regular, not only for $|t| < 1$, but also for $|t| = 1$, $t \neq \pm 1, \pm i$. To show that u is continuous for $|t| \leqq 1$, $t \neq \pm i$, it is sufficient, in view of symmetry, to establish continuity in the first quadrant at $t = 1$, where $u(1) = \lim u_n(1) = 1$. According to Proposition 1 of § 9 and the last paragraph of § 6, each u_n satisfies $r \leqq |u_n(re^{i\sigma})| \leqq [1 - \lambda \log \cos \sigma]^{1/3\beta}$ and $0 \leqq \arg u_n(re^{i\sigma}) \leqq \sigma$ in the first quadrant. Therefore u satisfies the same estimates, and the continuity follows. Since each u_n is univalent for $|t| < 1$, u is either univalent or constant (see [12], Ch. IV); and the latter case is ruled out because $u(1) = 1$ and $u(0) = \lim u_n(0) = 0$. Thus u satisfies condition (A).

Next, u satisfies condition (B) along with each u_n; and condition (C) follows from equation (7) for the u_n, since $h_{\gamma_n}(\sigma)$ approaches $\tan \sigma$.

Now $\varphi(\sigma)$, like each $\varphi_n(\sigma) = \arg u_n(e^{i\sigma})$, is increasing, and $\varphi(0) = 0$. Since u is regular for $t = e^{i\sigma}$, $0 < \sigma < \pi/2$, φ is either strictly increasing or identically zero. In the latter case (2) implies $v(\sigma) \equiv 1$, so that $u \equiv 1$, which contradicts univalence. Therefore φ and v are both strictly increasing, and v is unbounded near $\sigma = \pi/2$.

It remains to show that φ approaches $\pi/2$ near $\sigma = \pi/2$. Applying Lemma 3 to $u_n(t)$ gives $\arg \{u_n(t)/t\} = (1/3\beta)\int_0^{\pi/2} k(t, \rho) d \log v_n^{3\beta}(\rho)$. Choosing an arbitrary number τ, $0 < \tau < \pi/2$, we have $d \log v_n^{3\beta}(\rho) = [\lambda h_{\gamma_n} v_n^{-3\beta} \sin \beta\varphi] d\rho \leqq \lambda v_n^{-3\beta}(\tau) h_{\gamma_n} d\rho$ for $\rho \geqq \tau$, and $d \log v_n^{3\beta}(\rho) \leqq \lambda \tan \rho d\rho$ for all ρ. When t is in the first quadrant, where $k(t, \rho)$ is negative, it follows that

$$0 \geqq \arg \{u_n(t)/t\} \geqq (\lambda/3\beta)\int_0^{\tau} k(t, \rho) \tan \rho d\rho + [\lambda/3\beta v^{3\beta}(\tau)]\int_0^{\pi/2} k(t, \rho) h_{\gamma_n}(\rho) d\rho \ .$$

The last integral can be evaluated by applying Lemma 3 to $f(t) = t/(1 + \gamma^2 t^2)$, which satisfies $d \log |f(e^{i\rho})| = h_\gamma(\rho) d\rho$; so that $\int_0^{\pi/2} k(t, \rho) h_\gamma(\rho) d\rho = -\arg(1 + \gamma^2 t^2)$. Taking the limit $t \to e^{i\sigma}$, $\tau < \sigma < \pi/2$, and then $n \to \infty$, gives $0 \geqq \varphi(\sigma) - \sigma \geqq (\lambda/3\beta)\int_0^{\tau} k(e^{i\sigma}, \rho) \tan \rho d\rho - [\lambda/3\beta v^3(\tau)] \arg(1 + e^{2i\sigma})$. In the limit $\sigma \to \pi/2$ the first integral vanishes, since $k(i, \rho) = \pi^{-1} \log |(e^{2i\rho} + 1)/(e^{-2i\rho} + 1)| = 0$, so that $0 \geqq \varphi(\pi/2) - \pi/2 \geqq -\pi\lambda/6\beta v^3(\tau)$. Since v is unbounded we have finally $\varphi(\pi/2) = \pi/2$; and the proof is complete.

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