# DECOMPOSITION OF HOLOMORPHS

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Let $G$ be a group, and let $H$ be its holomorph. There are two situations in which $H$ is known to be decomposable into the direct product of two proper subgroups. If $G$ is the direct product of two of its proper characteristic subgroups, say $G_1$ and $G_2$, then $H$ is the direct product of the holomorphs of $G_1$ and $G_2$. If $G$ is a complete group, then $H$ is the direct product of $G$ and $G^*$, where $G^*$ is the centralizer of $G$ in $H$. In this paper we will show that if $G$ is not the direct product of two proper characteristic subgroups, and if $G$ is not complete, then $H$ is indecomposable. Thus we have a complete characterization of those groups whose holomorphs are indecomposable.

A decomposition of $H$ into the direct product of indecomposable factors is known for the case where $G$ is a finite abelian group [1]. Our present results enable us to generalize this and give a decomposition of $H$ into the direct product of indecomposable factors, whenever $G$ is the direct product of a finite number of characteristically indecomposable characteristic subgroups. In particular this gives a complete decomposition of $H$ whenever $G$ is a finite group.

Peremans [2] has shown that a necessary and sufficient condition for $G$ to be a direct factor of $H$ is that $G$ be either complete or the direct product of a group of order two and a complete group that has no subgroups of index two. This result is related to the present paper. In fact Peremans' result can be deduced from Lemma 1*.

**1. Preliminaries.** Let $G$ be a group, and let $A$ be the group of all automorphisms of $G$. Let $e$ and $I$ denote the identities of $G$ and $A$ respectively. The holomorph $H$ of $G$ can be regarded as the semi-direct product of $G$ and $A$, i.e., the set of all pairs $(g, \sigma)$, $g \in G, \sigma \in A$, with multiplication defined by

$$(g, \sigma)(h, \tau) = (g(\sigma h), \sigma\tau) .$$

We identify $g$ in $G$ with $(g, I)$ in $H$. Then $H$ is a group that contains $G$ as an invariant subgroup, and every automorphism of $G$ can be extended to an inner automorphism of $H$.

For all $a$ in $G$ we let $\lambda_a$ denote the inner automorphism of $G$ corresponding to the element $a$. Thus $\lambda_a g = aga^{-1}$.

All the results of this paper depend on the following lemma:

LEMMA 1. *Let $H = H_1 \times H_2$. Then $G \cap H_1$ and $G \cap H_2$ are characteristic subgroups of $G$ and*

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$$G = (G \cap H_1) \times (G \cap H_2) .$$

*Proof.* We note first that $G \cap H_1$ and $G \cap H_2$ are normal subgroups of $H$, and hence they are characteristic subgroups of $G$.

For $i = 1$ or 2, let $\varepsilon_i$ denote the projection of $H$ onto $H_i$ corresponding to the decomposition $H = H_1 \times H_2$. Thus if $\alpha \in H_1$ and $\beta \in H_2$, then $\varepsilon_1(\alpha\beta) = \alpha$ and $\varepsilon_2(\alpha\beta) = \beta$. Put $J_i = \varepsilon_i G$. Clearly $J_i \subseteqq H_i$ and $J_i$ is a normal subgroup of $H$. Let $F_i$ and $S_i$ denote the set of all first and second components respectively of elements of $J_i$. Thus $F_i \subseteqq G$ and $S_i \subseteqq A$.

Let $(a, \sigma)$ be an element of $J_1$. Then for some $g$ in $G$ we have $\varepsilon_1 g = (a, \sigma)$. Put $\varepsilon_2 g = (b, \tau)$. Then $g = (a, \sigma)(b, \tau)$. Therefore $\tau = \sigma^{-1}$ and $(\sigma b^{-1}, \sigma) = (b, \tau)^{-1} \in J_2$. Hence $\sigma \in S_2$. It follows that $S_1 \subseteqq S_2$. By symmetry $S_2 \subseteqq S_1$, and hence $S_1 = S_2$.

Let $\sigma$ be an element of $S_1$ and let $\xi$ be an element of $A$. Put $\varepsilon_i(e, \xi) = (g_i, \xi_i)$, $i = 1, 2$. For some $a$ and $c$ in $G$ we have $(a, \sigma) \in J_1$ and $(c, \sigma) \in J_2$. Now

$$(a, \sigma)(g_2, \xi_2) = (g_2, \xi_2)(a, \sigma)$$

and

$$(c, \sigma)(g_1, \xi_1) = (g_1, \xi_1)(c, \sigma) .$$

Comparing second components we see that $\sigma$ commutes with both $\xi_1$ and $\xi_2$. Since $\xi = \xi_1\xi_2$, we have $\sigma\xi = \xi\sigma$. It follows that $S_1$ is contained in the center of $A$.

Let $(a, \sigma)$ be an element of $J_1$ and let $(d, \mu)$ be an element of $J_2$. Since $\sigma$ is contained in the center of $A$ and since $(a, \sigma)^{-1} = (\sigma^{-1}a^{-1}, \sigma^{-1})$, it follows that

$$d(a, \sigma)d^{-1}(e, \lambda_{\sigma d})(a, \sigma)^{-1}(e, \lambda_{\sigma d})^{-1} = d(\sigma d)^{-1} .$$

Therefore $d(\sigma d)^{-1} \in H_1$. Moreover

$$d(\sigma d)^{-1} = (d, \mu)(e, \sigma)(d, \mu)^{-1}(e, \sigma)^{-1} \in H_2 .$$

Hence $d(\sigma d)^{-1} \in H_1 \cap H_2$. This gives us $d(\sigma d)^{-1} = e$ and $\sigma d = d$. Thus $\sigma$ leaves every element of $F_2$ fixed. By symmetry, since $\sigma \in S_1 = S_2$, it follows that $\sigma$ leaves every element of $F_1$ fixed. Now let $g$ be an arbitrary element of $G$. Then $g = (f, \nu)(h, \zeta)$ with $(f, \nu) \in J_1$ and $(h, \zeta) \in J_2$. Since $g = f(\nu h)$, $\sigma f = f$, and $\sigma\nu h = \nu\sigma h = \nu h$, it follows that $\sigma g = g$. Hence $\sigma = I$. Therefore $S_1$ and $S_2$ consist of the identity alone. It follows that $J_1 \subseteqq G \cap H_1$, $J_2 \subseteqq G \cap H_2$, and

$$G \subseteqq J_1 \times J_2 \subseteqq (G \cap H_1) \times (G \cap H_2) \subseteqq G .$$

Therefore $G = (G \cap H_1) \times (G \cap H_2)$ and the proof is complete.

**2. Some known results.** Suppose $G = G_1 \times G_2 \times \cdots \times G_n$, where the $G_i$ are characteristic subgroups of $G$. Let $A_i$ denote the group of all automorphisms of $G_i$. We identify $\sigma_i$ in $A_i$ with the element $\sigma'_i$ in $A$ such that

$$\sigma'_i g = \begin{cases} g & \text{if } g \in G_j, j \neq i \, , \\ \sigma_i g & \text{if } g \in G_i \, . \end{cases}$$

Then $A = A_1 \times A_2 \times \cdots \times A_n$. Moreover $H_i$, the holomorph of $G_i$, becomes a subgroup of $H$, and $H = H_1 \times H_2 \times \cdots \times H_n$.

The centralizer of a group in its holomorph is called its conjoint. The conjoint $G^*$ of $G$ consists of the elements $(g^{-1}, \lambda_g)$, $g \in G$. The mapping $\eta$ defined by

$$\eta(g, \sigma) = (g^{-1}, \lambda_g \sigma)$$

is an automorphism of $H$ that maps $G$ onto $G^*$ and maps $G^*$ onto $G$. Therefore $G$ and $G^*$ are isomorphic, and $G$ is the centralizer of $G^*$ in $H$. Furthermore Lemma 1 is equivalent to the following:

LEMMA 1*. *Let $H = H_1 \times H_2$. Then $G^* \cap H_1$ and $G^* \cap H_2$ are characteristic subgroups of $G^*$ and*

$$G^* = (G^* \cap H_1) \times (G^* \cap H_2) \, .$$

If $G$ is complete, i.e., if $G$ is a centerless group with only inner automorphisms, then $H = G \times G^*$.

**3. Decomposable and indecomposable holomorphs.** If $G$ is the direct product of two proper characteristic subgroups, then $G$ is said to be characteristically decomposable. If not, then $G$ is said to be characteristically indecomposable.

THEOREM 1. *Let $G$ be a group, and let $H$ be its holomorph. If $G$ is either characteristically decomposable or complete, then $H$ is decomposable. If $G$ is characteristically indecomposable and not complete, then $H$ is indecomposable.*

*Proof.* We have seen in § 2 that $H$ is decomposable if $G$ is either characteristically decomposable or complete. Suppose that $G$ is characteristically indecomposable and that $H = H_1 \times H_2$. It follows from Lemma 1 that either $G \cap H_1 = G$ or $G \cap H_2 = G$. Thus either $G \subseteqq H_1$ or $G \subseteqq H_2$. Similarly it follows from Lemma 1* that either $G^* \subseteqq H_1$ or $G^* \subseteqq H_2$. Without loss of generality suppose that $G \subseteqq H_1$. Then $H_2$ is contained in the centralizer of $G$, that is $H_2 \subseteqq G^*$. If $G^* \subseteqq H_1$ we have $H_2 \subseteqq H_1$ and $H = H_1$. Thus we need only consider the case $G^* \subseteqq H_2$.

Here $G^* = H_2$ and $H_1$ is contained in the centralizer of $G^*$. Thus $H_1 \leqq G$, and hence $H_1 = G$. Now $G \cap G^*$ is the center of $G$, and $G \cap G^* = H_1 \cap H_2$. Hence $G$ is centerless. Since $H = H_1 \times H_2 = G \times G^*$, it follows that $G$ has only inner automorphisms. Therefore $G$ is complete. This completes the proof of the theorem.

## 4. Decomposition of the holomorph into indecomposable subgroups.

To complete our discussion we need the following result:

LEMMA 2. *If a group is complete and characteristically indecomposable, then it is indecomposable.*

*Proof.* Let $G$ be a complete group and suppose $G = G_1 \times G_2$. Since every automorphism of $G$ is inner, it follows that every automorphism of $G$ maps $G_1$ and $G_2$ onto themselves. Hence $G_1$ and $G_2$ are characteristic subgroups of $G$. This establishes the lemma.

THEOREM 2. *Suppose $G$ is the direct product of a finite number of characteristically indecomposable characteristic subgroups: $G = G_1 \times G_2 \times \cdots \times G_n$. Suppose that $G_i$ is complete for $1 \leqq i \leqq r$, and that $G_j$ is not complete for $r + 1 \leqq j \leqq n$. Then a decomposition of $H$ into indecomposable subgroups is given by*

$$H = \prod_{i=1}^{r} G_i \times \prod_{i=1}^{r} G_i^* \times \prod_{i=r+1}^{n} H_i \,, \tag{1}$$

*where $G_i^*$ and $H_i$ are the conjoint and holomorph respectively of $G_i$, and where $\Pi$ denotes a direct product.*

*Proof.* It follows from § 2 that (1) is a decomposition of $H$. By Lemma 2 the groups $G_i$ and $G_i^*$ are indecomposable for $1 \leqq i \leqq r$, and by Theorem 1 the groups $H_i$ are indecomposable for $r + 1 \leqq i \leqq n$.

Since a characteristic subgroup of a characteristic subgroup of $G$ is itself a characteristic subgroup of $G$ it follows that $G$ satisfies the condition of Theorem 2 whenever the characteristic subgroups of $G$ satisfy the descending chain condition. In particular Theorem 2 gives us a decomposition of $H$ into indecomposable subgroups whenever $G$ is a finite group.

If $G$ is the direct product of an infinite number of characteristic subgroups, then $H$ is not the direct product of their holomorphs. Thus Theorem 2 does not hold in this case.

## BIBLIOGRAPHY

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