QUOTIENT RINGS OF RINGS WITH ZERO SINGULAR IDEAL

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Many papers have been written recently (see [2]–[14] of bibliography) on extensions of rings to rings of quotients. In most of these papers, strong enough conditions are imposed on the given rings to insure that each has a vanishing singular ideal (first defined in [5]). It seems appropriate at this time to collect these results and present them in as general a form as possible. In this paper, it is assumed that each ring has a zero right singular ideal. A subsequent paper will give the quotient structure of a ring having a vanishing right and left singular ideal.

1. Introduction. If R is a ring and M is an R-module, then $L(R)$ and $L(M, R)$ will designate the lattices of right ideal of R and R-submodules of M, respectively. Superscripts "r" and "l" will be used in designating the right and left annihilators, respectively, of an element or subset of a ring or module. The context will always make it clear from what set the annihilators are to be chosen.

In a lattice L with 0 and I, an element B is called an *essential extension* of element A, and we write $A \subset' B$, if and only if $A \subset B$ and $C \cap A \neq 0$ for every C in L for which $C \cap B \neq 0$. An element A of L is called *large* if $A \subset' I$. The sublattice of L of all large elements is designated by $L^{\blacktriangle}$.

If R is a ring and M is a right R-module, then let

$$M^{\blacktriangle}(R) = \{x \mid x \in M, x^r \in L^{\blacktriangle}(R)\}, \quad R^{\blacktriangle} = \{x \mid x \in R, x^r \in L^{\blacktriangle}(R)\}.$$

It is easily shown that $M^{\blacktriangle}(R)$ is a submodule of M and $R^{\blacktriangle}$ is a (two-sided) ideal of R. The ideal $R^{\blacktriangle}$ is called the *singular ideal* [5; p. 894] of R.

A ring R with zero singular ideal has the unusual property, proved in [7; Section 6], that each $A \in L(R)$ has a unique maximal essential extension A^s in $L(R)$. The mapping $s: A \to A^s$ of $L(R)$ is shown there to be a closure operation on $L(R)$ having the following properties:

(1) $0^s = 0$,

(2) $(A \cap B)^s = A^s \cap B^s$ for each $A, B \in L(R)$, and

(3) $(x^{-1}A)^s = x^{-1}A^s$ for each $x \in R$ and $A \in L(R)$, where $x^{-1}B = \{y \mid y \in R, xy \in B\}$. The set $L^s(R)$ of closed right ideals (i.e., $A = A^s$) may be made into a lattice in the usual way by defining the union of a set of

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elements of $L^s(R)$ to be the least upper bound of the set. The resulting lattice $L^s(R)$, which is not in general a sublattice of $L(R)$, is proved *to be a complete complemented modular lattice in* [7; Section 6]. If M is a right R-module for which $M^\blacktriangle(R) = 0$, then the closure operation s may be defined in a similar way on $L(M, R)$. The resulting lattice $L^s(M, R)$ has similar properties to those of $L^s(R)$, as was shown in [7; Section 6].

For $A, B \in L(R), B$ is called a *complement* of A if $B \cap A = 0$ whereas $C \cap A \neq 0$ for every $C \supset B, C \neq B$. If B is a complement of A, then clearly $A + B \in L^\blacktriangle(R)$. Furthermore, if $R^\blacktriangle = 0$, then $B \in L^s(R)$.

If A is a two-sided ideal of R for which $A \cap A^l = 0$, then evidently A^l is the unique complement of A in $L(R)$. Since $(A + A^l)^l = A^l \cap A^{ll}$, clearly A^{ll} is the unique complement of A^l in case $R^\blacktriangle = 0$. In this case, both A^l and A^{ll} are in $L^s(R)$. By [7; 6.7], $C^s(R) = \{A \mid A$ ideal of $R, A \cap A^l = 0, A = A^{ll}\}$ is the *center of the* lattice $L^s(R)$. For each $A \in C^s(R)$, it is easily seen that $A^\blacktriangle = 0$, that $L^s(A) = \{B \cap A \mid B \in L^s(R)\}$, and that $C^s(A) = \{B \cap A \mid B \in C^s(R)\}$. Of course, $L^s(A) \subset L^s(R)$ and $C^s(A) \subset C^s(R)$.

Every regular ring R has a zero singular ideal. This is evident because $e^r \cap eR = 0$ for each idempotent $e \in R$. Since $R = eR + e^r$, evidently eR and e^r are complements of each other and each is in $L^s(R)$. Consequently, each principal right ideal $aR \in L^s(R)$.

A ring R for which $R^\blacktriangle = 0$ and $C^s(R) = \{0, R\}$ is called (right) *irreducible*. An irreducible ring need not be prime. For example, the ring of all $n \times n$ triangular matrices over the ring Z of integers is irreducible by [8; 3.5]. Clearly this ring has a nonzero nilpotent ideal. By [8; 2.1], an irreducible ring is prime if and only if it contains no nonzero nilpotent ideal.

If R is a subring of ring Q then Q is called a (right) *quotient ring* of R, and write $R \leqq Q$, if and only if $qR \cap R \neq 0$ each nonzero $q \in Q$. It was proved in [5] that each ring R for which $R^\blacktriangle = 0$ has a unique maximal quotient ring $\hat{R}$. By [5; Theorem 2], $\hat{R}$ is a regular ring with unity. Essentially, the definition of $\hat{R}$ in [5] was as follows:

$$\hat{R} = \bigcup_{A \in L^\blacktriangle(R)} \mathrm{Hom}_R(A, R) .$$

If $x, y \in \hat{R}$, then we take $x = y$ if and only if $xa = ya$ for every a in some large right ideal $A \subset \mathrm{Dom}\, x \cap \mathrm{Dom}\, y$.

In case R is a subring of a ring Q, then we may consider Q as a right R-module. If we do so, then the assumption $R \leqq Q$ implies that $R \subset' Q$, considering R and Q as right R-modules. It is easily verified

The more general definition of a quotient ring in [**12**] and [**2**] is equivalent to ours in case $R^\blacktriangle = 0$.

that if $R \leqq Q$ then $Q^{\blacktriangle}(R)=0$ if and only if $R^{\blacktriangle} = 0$.

2. Some basic lemmas. The rest of this paper will be concerned only with a ring R for which $R^{\blacktriangle} = 0$. We shall prove in this section that if Q is a quotient ring of such a ring R, then the lattices of closed right ideals of R and Q are isomorphic.

2.1 LEMMA. *If $R \leqq Q$ and $A \in L(Q)$, then $A \in L^{\blacktriangle}(Q)$ if and only if $A \cap R \in L^{\blacktriangle}(R)$.*

Proof. If $A \in L^{\blacktriangle}(Q)$ and $b \in R, b \neq 0$, then $A \cap bQ \neq 0$ and $a = bq \neq 0$ for some $a \in A$ and $q \in Q$. Now $qC \subset R$ for some $C \in L^{\blacktriangle}(R)$ by [7; 6.1]. Since $Q^{\blacktriangle}(R) = 0$, $bqC \neq 0$ and therefore $A \cap bR \neq 0$. Hence $(A \cap R) \cap bR \neq 0$ and $A \cap R \in L^{\blacktriangle}(R)$.

On the other hand, let us assume that $A \in L(Q)$ and $A \cap R \in L^{\blacktriangle}(R)$. For each nonzero $q \in Q$, $qC \subset R$ for some $C \in L^{\blacktriangle}(R)$. If we let $B = C \cap (A \cap R)$, then $B \in L^{\blacktriangle}(R)$ and $qB \neq 0$ since $Q^{\blacktriangle}(R) = 0$. Hence $qB \cap (A \cap R) \neq 0$ and we conclude that $qQ \subset A \neq 0$ for each nonzero $q \in Q$. Thus, $A \in L^{\blacktriangle}(Q)$.

2.2 LEMMA. *If $R \leqq Q$ and M is a right Q-module, then M is a right R-module and $M^{\blacktriangle}(R) = M^{\blacktriangle}(Q)$.*

Proof. If $x \in M$ and $A = x^r(\text{in } Q)$ then $A \in L^{\blacktriangle}(Q)$ if and only if $A \cap R \in L^{\blacktriangle}(R)$ by 2.1. Therefore, $M^{\blacktriangle}(R) = M^{\blacktriangle}(Q)$.

2.3 COROLLARY. *If $R \leqq Q$, then $Q^{\blacktriangle} = 0$.*

This follows from 2.2 if we let $M = Q$ and use the assumption that $R^{\blacktriangle} = 0$.

2.4 LEMMA. *If $R \leqq Q$ and M is a right Q-module such that $M^{\blacktriangle}(Q) = 0$, then $L^s(M, R) = L^s(M, Q)$.*

Proof. If $A \in L^s(M, R)$ and $q \in Q$, then $qB \subset R$ for some $B \in L^{\blacktriangle}(R)$. Therefore $(Aq)B \subset A$ and $Aq \subset A$ by [7; 6.4]. Hence, $A \in L(M, Q)$ and we conclude that $L^s(M, R) \subset L(M, Q)$.

If $A \in L(M, Q)$, $x \in M$ and $B_x = \{b \mid b \in Q, xb \in A\}$, then $x \in A^s$ if and only if $B_x \in L^{\blacktriangle}(Q)$ by [7; 6.4]. Therefore, in view of 2.1, the closure of A relative to Q is the same as its closure relative to R. Thus, $L^s(M, R) = L^s(M, Q)$.

2.5 THEOREM. *If $R \leqq Q$, if M is a right Q-module for which $M^{\blacktriangle}(Q) = 0$ and if $N \in L^{\blacktriangle}(M, R)$, the $L^s(M, Q) \cong L^s(N, R)$ under the* *correspondence* $A \to A \cap N$, $A \in L^s(M, Q)$.

Proof. By [7; 6.8], $L^s(M, R) \cong L^s(N, R)$. Thus 2.5 follows from 2.4.

2.6 COROLLARY. *If* $R \leqq Q$, *then* $L^s(Q) \cong L^s(R)$ *under the correspondence* $A \to A \cap R$, $A \in L^s(Q)$.

If R is an irreducible ring, so that $C^s(R) = \{0, R\}$, then $C^s(\hat{R}) = \{0, \hat{R}\}$ by 2.6. Hence $\hat{R}$ also is irreducible. Actually, since $\hat{R}$ is regular, $\hat{R}$ is a prime ring by [8; 2.1]. We state this result as follows.

2.7 THEOREM. *If R is an irreducible ring, then $\hat{R}$ is a prime ring.*

3. **$L^s(R)$ atomic.** Let us assume in this section that R is a ring for which $R^{\blacktriangle} = 0$ and the lattice $L^s(R)$ is *atomic*. We define this to mean that $L^s(R)$ has minimal nonzero elements, called atoms, and that each element of $L^s(R)$ contains at least one atom. It is proved in [7; 6.9] that a nonzero element x of R is contained in an atom if and only if x^r is a maximal element of $L^s(R)$. Incidentally, $(xR)^s$ is the atom containing x.

Two atoms A and B are said to be *perspective* [1; p. 118], and we write $A \sim B$, if and only if A and B have a common complement. It is easily shown in our case that $A \sim B$ if and only if $A \cup B$ contains a third atom [1; p. 120, Lemma 3]. We proved in [7; 6.10] that $A \sim B$ if and only if $a^r = b^r$ for some nonzero $a \in A$ and $b \in B$. If $A \sim B$ and $B \sim C$ then $a^r = b^r$ and $b_1^r = c^r$ for some nonzero $a \in A, b, b_1 \in B$ and $c \in C$. Since B is an atom, $bR \cap b_1R \neq 0$ and there exist $x, x_1 \in R$ such that $bx = b_1x_1 \neq 0$. Hence, $(ax)^r = (bx)^r = (b_1x_1)^r = (cx_1)^r$. It follows that perspectivity is an equivalence relation on the set of all atoms of $L^s(R)$. Clearly for a finite set $\{A_1, \cdots, A_n\}$ of perspective atoms, there exist nonzero $a_i \in A_i$ such that $a_i^r = a_j^r$ for each i and j.

For each atom A of $L^s(R)$, let A^* be the union in $L^s(R)$ of all atoms perspective to A. It is proved in [7] that A^* is an ideal of R [7; 6.7] and that A^* is an atom of $C^s(R)$ [7; 6.12]. Conversely, each atom of $C^s(R)$ is of the form A^* for some atom A of $L^s(R)$.

Since $C^s(R)$ is a Boolean algebra, R is the direct union of all atoms of $C^s(R)$. Hence, if $\{A_i^*; i \in \Delta\}$ is the set of all distinct atoms of $C^s(R)$, then the ring-union S of the atoms of $C^s(R)$ is a discrete direct sum of these atoms,

$$S = \sum_{i \in \Delta} A_i^* .$$

Since $S^l = 0$, evidently $S \leqq R$. Consequently, the maximal quotient ring of R is just the maximal quotient ring of S.

The following theorem characterizes $\hat{R}$ in terms of left full rings. We shall call a ring R a *left full ring* if there exists a division ring D and a right D-module M such that

$$R \cong \operatorname{Hom}_D(M, M) .$$

Evidently we may consider M as a (R, D)-module.

3.1 THEOREM. *If R is a right irreducible ring, then $\hat{R}$ is a left full ring. If R is right reducible, then $\hat{R}$ is a complete direct sum of left full rings.*

Proof. Consider first the case in which R is irreducible. Since $\hat{R}$ is regular and $L^s(R) \cong L^s(\hat{R})$, the lattice $L^s(\hat{R})$ is atomic and its atoms are principal and hence minimal right ideals of $\hat{R}$. Since $\hat{R}$ is prime and has minimal right ideals, it is primitive. Let e be an idempotent element of $\hat{R}$ such that $e\hat{R}$ is a minimal right ideal. Then $M = \hat{R}e$ is a minimal left ideal of $\hat{R}$ and $D = e\hat{R}e$ is a division ring. Since $x\hat{R}e \neq 0$ for each nonzero $x \in \hat{R}$ by the primeness of $\hat{R}$, evidently $\hat{R}$ is a right quotient ring of M. However, $\hat{R}$ is a maximal right quotient ring so that we must have $\hat{M} = \hat{R}$. Besides being a ring, M may be considered to be a $(\hat{R}, D)$-module. Clearly the right ideals of M are its D-submodules. Thus, M is the only large right ideal of M. Consequently,

$$\operatorname{Hom}_M(M, M) ,$$

considering M as a right M-module, is the maximal right quotient ring of M. Since $x(ae) = x(eae)$ for each $x \in M$ and $a \in \hat{R}$, evidently

$$\operatorname{Hom}_M(M, M) = \operatorname{Hom}_D(M, M) .$$

Since $\hat{M} = \hat{R}$, this proves that $\hat{R}$ is a left full ring.

If R is not irreducible, then there exists a set $\{R_i; i \in \Delta\}$ of irreducible rings, each having an atomic lattice of closed right ideals, such that

$$\sum_{i \in \Delta} R_i \leq R$$

by our previous results. We shall not give the details, but it is easily seen that if

$$S = \sum_{i \in \Delta} R_i , \quad \text{then } \hat{S} = \sum_{i \in \Delta}{}' \hat{R}_i$$

where $\sum'$ designates the complete direct sum. Since $\hat{S} = \hat{R}$, this proves the second part of 3.1.

The important special case of this theorem when R is a primitive ring was proved by Utumi [12; 5.1] and Wong [13; 4.1]. Both Utumi and Lambek [10] have independently proved the theorem if R is prime.

4. $L_s(R)$ **finite-dimensional.** The usual assumption that $R^\blacktriangle = 0$ is made for each ring R of this section. If either the a.c.c. or the d.c.c. holds for $L^s(R)$ then so does the other one. In fact, each is equivalent to the assumption that $L^s(R)$ contains a maximal chain of finite length. When this condition is satisfied, a *dimension function* d may be defined on $L^s(R)$ as follows [1; p. 67]: for each $A \in L^s(R)$, $d(A)$ is the length of the longest chain joining 0 to A. Incidentally, every maximal chain joining 0 to A has the same length $d(A)$. We shall assume in this section that such a dimension function d is defined on $L^s(R)$ and that $d(R)$ is finite. Since the lattice $L^s(R)$ is also complemented, each $A \in L^s(R)$ is a direct union of $d(A)$ atoms [1; p. 105].

It is proved in [9; 3.4] that if $d(R)$ is finite then for each $a \in R$, $aR \in L^\blacktriangle(R)$ if and only if $a^r = 0$. Of course, $a^l = 0$ whenever $aR \in L^\blacktriangle(R)$. Thus, $D(R) = \{a \mid a \in R, aR \in L^\blacktriangle(R)\}$ is the set of *regular* elements of R. Each $a \in D(R)$ has an inverse in $\hat{R}$. For, by the regularity or $\hat{R}$, $(ab-1)a = a(ba-1) = 0$ for some $b \in \hat{R}$. Since $(ab-1)^r \supset aR$, a large element of $L^\blacktriangle(R)$, $ab - 1 = 0$ in view of 2.1 and 2.3. Also, $ba - 1 = 0$ since $a^r = 0$ in $\hat{R}$ as well as in R. Consequently, $b = a^{-1}$.

4.1 THEOREM. *If R is irreducible and $d(R) = n$, then $\hat{R}$ is a full ring of dimension n.*

By a full ring of dimension n we mean a ring isomorphic to $\mathrm{Hom}_D(M, M)$ where D is a division ring and M is a right D-module of dimension n.

If we select $M = \hat{R}e$ as in the proof of 3.1, then $M \leqq \hat{R}$ and the lattices $L^s(R)$, $L^s(M)$ and $L^s(\hat{R})$ are isomorphic by 2.6. Since the right ideals of M are its D-submodules, M is an n-dimensional vector space over D. Hence 4.1 follows from 3.1.

A different proof of 4.1 was given in [9; 3.6].

If R is a prime ring for which $d(R)$ is finite, then it was proved in [3; Theorem 10] and in [9; 3.5] that every large right ideal of R contains a regular element. Since $B = \{b \mid b \in R, qb \in R\}$ is a large right ideal of R for each $q \in \hat{R}$, clearly $qb = a$ for some $b \in D(R)$ and $a \in R$; that is, $q = ab^{-1}$. This proves the following theorem of Goldie[2] [3] (also proved in [11] and [9]).

[2] That each ring considered by Goldie has a zero singular ideal is proved in [4; 3.2].

4.2 THEOREM. *If R is a prime ring for which $d(R) = n$, then not only is $\hat{R}$ the full ring of linear transformations of an n-dimensional vector space over a division ring but also* $R = \{ab^{-1} \mid a \in R, b \in D(R)\}$.

From 3.1 and 4.1, we easily deduce the following theorem.

4.3 THEOREM. *If R is a ring for which $d(R)$ is finite, then $\hat{R}$ is a direct sum of a finite number of finite-dimensional full rings.*

A ring R is called *semiprime* if it contains no nonzero nilpotent ideal. We recall that if S is the direct sum of the atoms of $C^s(R)$, then $S \leqq R$. Since each nonzero ideal of R has nonzero intersection with some atom of $C^s(R)$, evidently R is semiprime if and only if each atom of $C^s(R)$ is prime. The following theorem was recently proved by Goldie [4].

4.4 THEOREM. *If R is a semiprime ring for which $d(R)$ is finite, then not only is $\hat{R}$ a direct sum of a finite number of finite-dimensional full rings but also* $R = \{ab^{-1} \mid a \in R, b \in D(R)\}$.

The first part of 4.4 follows directly from 4.3. To prove the second part, let $S = R_1 \oplus \cdots \oplus R_k$ be the sum of the atoms of $C^s(R)$. Then $\hat{R} = \hat{S} = \hat{R}_1 \oplus \cdots \oplus \hat{R}_k$. If $q_i \in \hat{R}$, then $q_i = a_i b_i^{-1}$ for some $a_i \in R_i$ and $b_i \in D(R_i)$ by 4.2. Thus, if $q = q_1 + \cdots + q_k$, $a = a_1 + \cdots + a_k$, and $b = b_1 + \cdots + b_k$, $q = a\,b^{-1}$. This proves the second part of 4.4.

A converse of 4.4 has been given by Goldie [5; 4.4]. He proved that if R is a ring for which $d(\hat{R})$ is finite and $\hat{R} = \{ab^{-1} \mid a \in R, b \in D(R)\}$, then R is semiprime. Naturally, this implies the following converse of 4.2: If R is a ring for which $\hat{R}$ is a finite-dimensional full ring and $\hat{R} = \{ab^{-1} \mid a \in R, b \in D(R)\}$ then R is prime.

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