MEAN CROSS-SECTION MEASURES OF HARMONIC MEANS OF CONVEX BODIES

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1. In [2] the notion of p-dot means of two convex bodies in Euclidean n-space was introduced and certain properties of these means investigated. For $p = 1$, the mean is more appropriately called the harmonic mean; here we restrict the discussion to this case. The harmonic mean of two convex bodies K_0 and K_1, which will always be assumed to share a common interior point Q, is defined as follows. Let $\hat{K}$ denote the polar reciprocal of K with respect to the unit sphere E centred at Q; let $(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1$, with $0 \leqq \vartheta \leqq 1$, be the usual arithmetic or Minkowski mean of $\hat{K}_0$ and $\hat{K}_1$. The harmonic mean of K_0, K_1 is the convex body $[(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge$. In more analytic terms, if $F_i(x)$ are the distance functions with respect to Q of K_i, for $i = 0, 1$, then the body whose distance function with respect to Q is $(1 - \vartheta)F_0(x) + \vartheta F_1(x)$ is the harmonic mean of K_0 and K_1.

In the paper mentioned, a dual Brunn-Minkowski theorem was established, namely

$$(1) \qquad V^{1/n}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leqq 1\Big/\left[\frac{(1 - \vartheta)}{V^{1/n}(K_0)} + \frac{\vartheta}{V^{1/n}(K_1)}\right]$$

where $V(K)$ means the volume of K. There is equality if and only if K_0 and K_1 are homothetic with the centre of magnification at Q.

Here we develop a more inclusive theorem regarding the behaviour of each mean cross-section measure, ("Quermassintegral") $W_\nu(K)$, $\nu = 0, 1, \cdots, n - 1$, cf. [1]. The result is

$$(2) \qquad W_\nu^{1/(n-\nu)}([(1 - \vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \leqq 1\Big/\left[\frac{(1 - \vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)}\right].$$

The cases of equality are just those of the dual Brunn-Minkowski theorem, ($\nu = 0$).

2. We first list some preliminary items used in the proof of (2). We shall use Minkowski's inequality in the form

$$(3) \qquad \int[(1 - \vartheta)f_0^p + \vartheta f_1^p]^{1/p}dx \leqq \left[(1 - \vartheta)\left(\int f_0 dx\right)^p + \vartheta\left(\int f_1 dx\right)^p\right]^{1/p}.$$

Here the functions f_i are assumed to be positive and continuous over the closed and bounded domain of integration common to all the integrals,

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and, for our puposes, p satisfies $-1 \leqq p < 0$. There is equality if and only if $f_0(x) \equiv \lambda f_1(x)$ for some constant λ. See [3], Theorem 201, coupled with the remark preceding Theorem 200.

Our second tool, which we shall refer to as the projection lemma, was established in [2]. Let K^* denote the projection of K onto a fixed, m-dimensional, linear subspace E_m through Q for $1 \leqq m < n$. We have

$$(4)\qquad [(1-\vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge \supseteqq \{[(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge\}^* .$$

Since E_m contains Q and the polar reciprocation is with respect to sphere E centred at Q, in forming $\hat{K}^*$ the order of operations is immaterial. This result is proved by a polar reciprocation argument from

$$(1-\vartheta)(\hat{K}_0 \cap E_m) + \vartheta(\hat{K}_1 \cap E_m) \subseteqq [(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1] \cap E_m .$$

There is equality in either inclusion if K_0 and K_1 are homothetic with centre of magnification at Q.

The dual Brunn-Minkowski theorem (1) will be used.

Finally we shall make use of Kubota's formula and some of its consequences. This material is covered in [1]. An $(n-\nu)$ dimensional cross-section measure ("Quermass") of K is the $(n-\nu)$ dimensional volume of that convex body which is the vertical projection of K onto an $E_{n-\nu}$. The mean cross-section measures are usually defined as the coefficients in Steiner's polynomial which describes $V(K+\lambda E)$, that is

$$(5)\qquad V(K+\lambda E) = \sum_{\nu=0}^{n} \binom{n}{\nu} W_\nu(K)\lambda^\nu .$$

If we denote the $(\nu-1)^{\text{th}}$ mean cross-section measure of the projection of K onto that E_{n-1} through Q which is orthogonal to the vector u_1 by $W'_{\nu-1}(K, u_1)$, then Kubota's formula is

$$W_\nu(K) = \frac{1}{\kappa_{n-1}} \int_{\Omega_n} W'_{\nu-1}(K, u_1)\, d\omega_n , \qquad \nu = 1, 2, \cdots, \nu-1 .$$

Here the integration with respect to the direction u_1 is extended over the surface Ω_n of E, $d\omega_n$ is the element of surface area on Ω_n and κ_{n-1} is the volume of the $n-1$ dimensional unit sphere.

Kubota's formula can be applied to the mean cross-section measure $W'_{\nu-1}(K, u_1)$ for fixed u_1:

$$W'_{\nu-1}(K, u_1) = \frac{1}{\kappa_{n-2}} \int_{\Omega_{n-1}} W''_{\nu-2}(K, u_1, u_2)\, d\omega_{n-1}$$

where $W''_{\nu-2}$ is the $(\nu-2)$th mean cross-section measure of the projection of κ onto the E_{n-2} through Q orthogonal to u_1 and u_2 with u_2 orthogonal to u_1. After ν such steps we have as the extended form of Kubota's formula:

$$W_\nu(K) = \frac{1}{\kappa_{n-1}\kappa_{n-2}\cdots\kappa_{n-\nu}}\int_{\Omega_n}\int_{\Omega_{n-1}}\cdots\int_{\Omega_{n-\nu}} W_0^{(\nu)}(K, u_1, u_2, \cdots, u_\nu)d\omega_{n-\nu}\cdots d\omega_{n-1}d\omega_n .$$

Each vector u_p is orthogonal to u_q for $q < p$ and $W_0^{(\nu)}(K, u_1, u_2, \cdots, u_\nu)$ is the 0th mean cross-section measure of the projection of K onto that $E_{n-\nu}$ through Q which is the orthogonal complement of the subspace spanned by $u_1, u_2, \cdots, u_\nu$.

Steiner's formula (5) with $\lambda = 0$ shows that $W_0(K)$ is the volume of K and so $W_0^{(\nu)}$ is an $(n-\nu)$ dimensional cross-section measure of K. Thus, to within a numerical factor depending on n and ν, $W_\nu(K)$ is the arithmetic mean of the $(n-\nu)$ dimensional cross-section measures.

In § 3 we shall use the following abbreviations: for $d\omega_{n-\nu}\cdots d\omega_{n-1}d\omega_n$ we write $d\bar{\omega}$ with sign of integration and omit reference to the domains of integration; for one $1/\kappa_{n-1}\kappa_{n-2}\cdots\kappa_{n-\nu}$ we write k; finally for $W_0^{(\nu)}(K, u_1, u_2, \cdots, u_\nu)$ we write $\sigma(K^*)$. In this notation the extended Kubota formula reads

$$W(K) = k\int\sigma(K^*)d\bar{\omega} .$$

3. We now prove (2). By the extended form of Kubota's formula

$$(6)\quad \begin{aligned} W_\nu^{1/(n-\nu)}([(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) &= \left[k\int\sigma(\{[(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge\}^*)d\bar{\omega}\right]^{1/(n-\nu)} \\ &\leqq \left[k\int\sigma([(1-\vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge)d\bar{\omega}\right]^{1/(n-\nu)} \end{aligned}$$

in virtue of the projection lemma and the set monotonicity of σ i.e., $\sigma(K^*) \leqq \sigma(\bar{K}^*)$ if $K^* \subseteqq \bar{K}^*$ with equality in the latter relation implying that in the former. We now apply (1), in $E_{n-\nu}$, to the integrand to obtain

$$\sigma([(1-\vartheta)\hat{K}_0^* + \vartheta\hat{K}_1^*]^\wedge) \leqq \left\{1\Big/\left[\frac{(1-\vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)}\right]\right\}^{(n-\nu)} .$$

Here we take advantage of the fact that

$$(\hat{K})^* = (K^*)^\wedge .$$

This gives

$$(7)\quad \begin{aligned} &W_\nu^{1/(n-\nu)}([(1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1]^\wedge) \\ &\leqq \left[k\int\left\{1\Big/\left[\frac{(1-\vartheta)}{\sigma^{1/(n-\nu)}(K_0^*)} + \frac{\vartheta}{\sigma^{1/(n-\nu)}(K_1^*)}\right]\right\}^{(n-\nu)}d\bar{\omega}\right]^{1/(n-\nu)} . \end{aligned}$$

There is equality if and only if all the projections K_0^* and K_1^* are homothetic with the centre of magnification at Q. This condition is sufficient for equality in (6); it is necessary and sufficient for (7).

We now use Minkowski's inequality (3) with $p = -1/n-\nu$. This yields

$$W_\nu^{1/(n-\nu)}(([1-\vartheta)\hat{K}_0 + \vartheta\hat{K}_1)$$
$$\leqq 1\Bigg/\left[\frac{(1-\vartheta)}{\left(k\int\sigma(K_0^*)d\bar{\omega}\right)^{1/(n-\nu)}} + \frac{\vartheta}{\left(k\int\sigma(K_1^*)d\bar{\omega}\right)^{1/(n-\nu)}}\right]$$
$$= 1\Bigg/\left[\frac{(1-\vartheta)}{W_\nu^{1/(n-\nu)}(K_0)} + \frac{\vartheta}{W_\nu^{1/(n-\nu)}(K_1)}\right].$$

The necessary and sufficient conditions for equality in (7) are sufficient for equality in (3) since $K_0 = \lambda K_1$ implies $\sigma(K_0^*) = \lambda^{n-\nu}\sigma(K_1^*)$. This establishes (2).

References

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