EXTENSIONS OF HOMOMORPHISMS

PAUL CIVIN

1. Introduction. A multiplication was introduced by R. Arens [1] [2] into the second conjugate space B^{**} of a Banach algebra, B, which made B^{**} into a Banach algebra. The algebra of the second conjugate space was studied by Civin and Yood [3], with particular attention given to the case where B was $L(\mathfrak{G})$, the group algebra of the locally compact abelian group $\mathfrak{G}$. Among the results they noted was that the algebra $M(\mathfrak{G})$ of finite regular Borel measures on $\mathfrak{G}$ was isomorphic as an algebra with a quotient algebra of $L^{**}(\mathfrak{G})$. With $\mathfrak{H}$ also a locally compact abelian group, P. J. Cohen showed [4, p. 220] that any homomorphism of $L(\mathfrak{G})$ into $M(\mathfrak{H})$ has an extension which was a homomorphism of $M(\mathfrak{G})$ into $M(\mathfrak{H})$.

In §3 we discuss the extensions of homomorphisms defined on a Banach algebra A into either the second conjugate algebra B^{**} of a Banach algebra B or certain of its quotient algebras. The result of Cohen quoted above is included in Theorem 3.7 when $\mathfrak{G}$ and $\mathfrak{H}$ are compact groups. In §4 we indicate, for compact $\mathfrak{H}$, a class of homomorphisms from $L(\mathfrak{G})$ into $M(\mathfrak{H})$, which are induced by homomorphisms of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{H})$.

2. Notation. The notation of Civin and Yood [3] is used throughout. If A is a Banach algebra, A^*, A^{**}, $\cdots$ denote the various conjugate spaces of A. For $f \in A^*$, $x \in A$, $\langle f, x\rangle \in A^*$ is defined by $\langle f, x\rangle(y) = f(xy)$, $y \in A$. For $F \in A^{**}$, $f \in A^*$, $[F,f] \in A^*$ is defined by $[F,f](x) = F(\langle f, x\rangle)$, $x \in A$. Also for $F \in A^{**}$, $G \in A^{**}$ the multiplication FG is defined in A^{**} by $FG(f) = F([G,f])$, $f \in A^*$.

For some purposes, Arens [2] also considers a second multiplication $F\cdot G$ defined for F and G in A^{**} in a manner similar to the above, except that at the first stage, $\langle f | x\rangle \in A^*$ is defined by $\langle f | x\rangle(y) = f(yx)$, $f \in A^*$, $x, y \in A$. Arens calls the multiplication in A *regular* provided that $F\cdot G = GF$ for all $F,G \in A^{**}$. Clearly, if A is commutative, then A^{**} is commutative if and only if the multiplication in A is regular. The same notation as above, in terms of bilinear functionals, is used in the sequel with respect to a multiplication in A^{****} which comes from the first of the above multiplications in A^{**}.

If π is the natural mapping of A into A^{**}, we say that a mapping φ defined on A^{**} into a set $\mathfrak{S}$ is an extension of a mapping ρ defined on A into $\mathfrak{S}$ if $\varphi(\pi x) = \rho(x)$ for $x \in A$.

For any subset $\mathfrak{J}$ in A^*, we use the notation $\mathfrak{J}^{\perp}$ for $\{F \in A^{**} | F(f) = 0, f \in \mathfrak{J}\}$.

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For a commutative Banach algebra A, we let $\mathfrak{Y}(A)$ denote the closed subspace of A^* generated by the multiplicative linear functionals. If $A = L(\mathfrak{G})$, the group algebra of the locally compact group $\mathfrak{G}$, we write $\mathfrak{Y}(\mathfrak{G})$ in place of $\mathfrak{Y}(L(\mathfrak{G}))$.

3. **Extension of homomorphisms.** We first consider the possibility of extending a bounded homomorphism of the Banach algebra A into the Banach algebra B^{**} to a w^*-continuous homomorphism of A^{**} into B^{**}. Throughout this section we adopt the notation π for the natural mapping of A into A^{**} and σ for the natural mapping of B^* into B^{***}.

3.1 THEOREM. *Let A and B be Banach algebras. Let φ be a bounded homomorphism of A into the center of B^{**}. Then there is a unique w^*-continuous homomorphism ψ of A^{**} into B^{**} which is the extension of φ.*

Proof. Let $f \in B^{**}$, and $x, y \in A$. Then $\langle\varphi^*\sigma f, x\rangle(y) = \varphi^*\sigma f(xy) = \varphi(xy)(f) = \varphi(y)\varphi(x)(f) = \varphi(y)([\varphi(x), f]) = \varphi^*\sigma[\varphi(x), f](y)$. Thus $\langle\varphi^*\sigma f, x\rangle = \varphi^*\sigma[\varphi(x), f]$. For any $G \in A^{**}$, $[G, \varphi^*\sigma f](x)$ $G(\langle\varphi^*\sigma f, x\rangle) = G(\varphi^*\sigma[\varphi(x), f] = \sigma^*\varphi^{**}G([\varphi(x), f]) = \sigma^*\varphi^{**}G\varphi(x)(f) = \varphi(x)\sigma^*\varphi^{**}G(f) = \varphi(x)([\sigma^*\varphi^{**}G, f]) = \varphi^*\sigma[\sigma^*\varphi^{**}G, f](x)$. Consequently, $[G, \varphi^*\sigma f] = \varphi^*\sigma[\sigma^*\varphi^{**}G, f]$. Therefore for any $F \in A^{**}$, $F([G, \varphi^*\sigma f]) = F(\varphi^*\sigma[\sigma^*\varphi^{**}G, f]) = \sigma^*\varphi^{**}F([\sigma^*\varphi^{**}G, f])$. Hence $\sigma^*\varphi^{**}(FG)(f) = FG(\varphi^*\sigma f) = F([G, \varphi^*\sigma f]) = \sigma^*\varphi^{**}F([\sigma^*\varphi^{**}G, f]) = \sigma^*\varphi^{**}F\sigma^*\varphi^{**}G(f)$. Thus $\sigma^*\varphi^{**}$ is a homomorphism of A^{**} into B^{**}.

For $x \in A$, and $f \in B^*$, $\sigma^*\varphi^{**}(\pi x)(f) = \pi x(\varphi^*\sigma f) = \varphi^*\sigma f(x) = \sigma f(\varphi(x)) = \varphi(x)(f)$. Thus $\sigma^*\varphi^{**}(\pi x) = \varphi(x)$ and $\sigma^*\varphi^{**}$ is an extension of φ.

Let $G \in A^{**}$, $G_\alpha \in A^{**}$ and suppose $G = w^*\text{-}\lim G_\alpha$. Then for any $f \in B^*$, $\lim \sigma^*\varphi^{**}G_\alpha(f) = \lim G_\alpha(\varphi^*\sigma f) = \sigma^*\varphi^{**}G(f)$, and so $\sigma^*\varphi^{**}$ is w^*-continuous.

The assertion of uniqueness follows from the following.

3.2 LEMMA. *Let A and B be Banach algebras, and let φ be any bounded linear transformation of A into B^{**}. Then $\sigma^*\varphi^{**}$ is the only w^*-continuous extension of φ to a transformation of A^{**} into B^{**}.*

Proof. That $\sigma^*\varphi^{**}$ is a w-continuous extension was given above. Suppose that ψ is a w^*-continuous extension of φ, so that $\psi(\pi x) = \varphi(x)$ for all $x \in A$. Let $G \in A^{**}$ and let $\{x_\alpha\}$ be a net in A such that w^*-lim $\pi x_\alpha = G$. Then for $f \in B^*$, $\psi(G)(f) = \lim \psi(\pi x_\alpha)f = \lim \varphi(x_\alpha)(f) = \lim \varphi^*\sigma f(x_\alpha) = \lim \pi x_\alpha(\varphi^*\sigma f) = G(\varphi^*\sigma f) = \sigma^*\varphi^{**}G(f)$. Hence $\psi(G) = \sigma^*\varphi^{**}G$.

If B is commutative with a regular multiplication, an alternative proof of Theorem 3.1 may be given on the basis of the following lemma and Theorem 6.1 of [3].

3.3 LEMMA. *If B is a commutative Banach algebra with a regular multiplication then σ^* is a homomorphism of B^{****} into B^{**}.*

Proof. Since multiplication in B is regular, B^{**} is [2] a commutative algebra. Let $U, V \in B^{****}$. For $f \in B^*$, and $F, G \in B^{**}$, $\langle \sigma f, F\rangle(G) = \sigma f(FG) = FG(f) = GF(f) = G([F,f]) = \sigma[F,f](G)$, and therefore $\langle \sigma f, F\rangle = \sigma[F,f]$. Also $[V, \sigma f](F) = V(\langle \sigma f, F\rangle) = V(\sigma[F,f]) = \sigma^* V[F,f] = (\sigma^* V)F(f) = F\sigma^* V(f) = F([\sigma^* V, f]) = \sigma[\sigma^* V, f](F)$. Thus $[V, \sigma f] = \sigma[\sigma^* V, f]$. Consequently $\sigma^*(UV)(f) = UV(\sigma f) = U([V, \sigma f]) = U(\sigma[\sigma^* V, f]) = \sigma^* U([\sigma^* V, f]) = \sigma^* U \sigma^* V(f)$ and σ^* is a homomorphism a claimed.

We note that it is impossible in general to conclude that the range of the extension of φ is in the center of B^{**} even though the range of φ is in the center. For let $A = B$ be a commutative algebra whose multiplication is not regular, and let $\varphi = \pi$. Then the w^*-continuous extension of π is the identity map and B^{**} is not commutative.

One further example is in order, to see that in general a bounded homomorphim φ from A into B^{**} does not admit a w^*-continuous extension as a homomorphism from A^{**} into B^{**}. For this purpose let A be the group algebra of the integers, $\mathfrak{G}$, and let $B = A$. Let t_γ, $\gamma \in \mathfrak{G}$ be the translation operator on A^*, defined by $t_\gamma f(\alpha) = f(\alpha + \gamma)$, $f \in A^*$, and α, $\gamma \in \mathfrak{G}$. Let $e \in A^*$ correspond to the function identically one on $\mathfrak{G}$. Let $\mathfrak{J} = \{F \in A^{**} \mid F(t_\gamma f) = F(f)$, for all $\gamma \in \mathfrak{G}, f \in A^*\}$. Then as noted in formula (3.2) of [3],

$$GF = G(e)F, \ F \in \mathfrak{J}, \ G \in A^{**}. \tag{3.1}$$

In particular any $F \in \mathfrak{J}$ with $F(e) = 1$ is an idempotent. As noted in [3], $\mathfrak{J}$ is a two sided ideal in A^{**} with only zero in common with the center of A^{**}. Since $\mathfrak{G}$ is a discrete group A has an identity and thus [3, Lemma 5.4] A^{**} has an identity E. Let F be a nonzero idempotent in $\mathfrak{J}$. Thus $E - F$ is also an idempotent. Let $\varphi(x) = \pi x(E - F)$. Since πA is in the center of A^{**}, $\varphi(x)$ is a homomorphism of A into A^{**}. If φ had a w^*-continuous extension as a homomorphism, the extension ψ would have the value $\psi(G) = G(E - F)$, $G \in A^{**}$. We now show that ψ is not a homomorphism. As noted above F is not in the center of A^{**}, so we may pick $H \in A^{**}$ such that $HF \neq FH$. Also pick $G \in A^{**}$ such that $G(e) = 1$. Then $\psi(GH) = GH(E - F) = GH - GHF = GH - (GH)(e)F$. Now e is a multiplicative linear functional on A, and so by Lemma 3.6 of [3], $(GH)(e) = G(e)H(e) = H(e)$. Thus $\psi(GH) = GH - H(e)F = GH - HF$. On the other hand $\psi(G)\psi(H) = (G - GF)(H - HF) = (G - F)(H - H(e)F) = GH - FH - H(e)GF + H(e)F = GH - FH$. Since $FH \neq HF$, $\psi(GH) \neq \psi(G)\psi(H)$ and ψ is not a homomorphism.

Before turning to other types of extensions we note one further item on the matter of w^*-continuity of homomorphisms.

3.4 LEMMA. *If A and B are Banach algebras and ψ is a bounded homomorphism of A^{**} into the center of B^{**}, then there is a w^*-continuous homomorphism ρ of A^{**} into B^{**} such that $\psi(\pi x) = \rho(\pi x)$ for $x \in A$.*

Proof. Since $\psi\pi$ is a homomorphism of A into the center of B^{**}, we may take $\rho = \sigma^*\psi^{**}\pi^{**}$ and apply Theorem 3.1.

Homomorphisms of A^{**} into B^{**} which are not w^*-continuous exist, as may be seen in the following example. Let $\mathfrak{G}$ be an infinite compact group and let $A = B$ be the group algebra of $\mathfrak{G}$. Then by Lemma 3.8 of [3], A^{**} has a right identity E which is not an identity. Define for $F \in A^{**}$, $\psi(F) = EF$. Then $\psi(FG) = EFG = EFEG = \psi(F)\psi(G)$. However ψ although bounded is not w^*-continuous. For let $G \in A^{**}$ and let $\{x_\alpha\}$ be a net such that $w^* - \lim \pi x_\alpha = G$. Then if ψ were w^*-continuous we would have $\psi(G) = \lim \psi(\pi x_\alpha) = \lim E\pi x_\alpha = \lim \pi x_\alpha = G$. However, $\psi(G) = EG$ and $EG \neq G$ for some $G \in A^{**}$.

We next turn to the question of extending homomorphisms from A into certain quotient algebras of B^{**} in the case in which both A and B are commutative. We must first characterize the w^*-closed ideals of a second conjugate algebra.

3.5 LEMMA. *Let A be a commutative Banach algebra. Let $\mathfrak{J}$ be a w^*-closed subspace of A^{**} and let $\mathfrak{J}_0 = \{f \in A^* \mid F(f) = 0, F \in \mathfrak{J}\}$. Then $\mathfrak{J}$ is an ideal of A^{**} if and only if $[G, f] \in \mathfrak{J}_0$ for all $G \in A^{**}, f \in \mathfrak{J}_0$.*

Proof. Since $\mathfrak{J}$ is w^*-closed, $\mathfrak{J} = \mathfrak{J}_0{}^\perp$. Suppose $\mathfrak{J}$ is an ideal of A^{**}. For any $F \in \mathfrak{J}, G \in A^{**}$, and $f \in \mathfrak{J}_0$, $FG \in \mathfrak{J}$ and $FG(f) = 0$. Therefore $F([G, f]) = 0$ for all $F \in \mathfrak{J}$, and so by definition $[G, f] \in \mathfrak{J}_0$. Suppose next that the stated condition holds. Let $F \in \mathfrak{J}$ and $G \in A^{**}$. For any $f \in \mathfrak{J}_0$, $[G, f] \in \mathfrak{J}_0$ and thus $FG(f) = F([G, f]) = 0$. Consequently $FG \in \mathfrak{J}_0{}^\perp = \mathfrak{J}$ and $\mathfrak{J}$ is a right ideal. For any $x \in A, \pi x$ is in the center of A^{**}, hence if $F \in \mathfrak{J}$, $\pi x F = F \pi x \in \mathfrak{J}$. Since πA is w^*-dense in A^{**} and left multiplication is w^*-continuous [2], we see that $GF \in \mathfrak{J}$ for any $G \in A^{**}$, and thus $\mathfrak{J}$ is an ideal of A^{**}.

3.6 THEOREM. *Let A and B be commutative Banach algebras. Let $\mathfrak{J}$ be a w^*-closed ideal of B^{**}. Suppose that φ is a bounded homomorphism of A into the center of $B^{**}/\mathfrak{J}$. Then there exists a w^*-closed ideal $\mathfrak{J}'$ of A^{**} and a homomorphism ψ of $A^{**}/\mathfrak{J}'$ into $B^{**}/\mathfrak{J}$ such that if π is the natural embedding of A into A^{**}, then $\psi(\pi x + \mathfrak{J}') = \varphi(x), x \in A$.*

Proof. Since $\mathfrak{J}$ is w^*-closed, $\mathfrak{J} = \mathfrak{J}_0{}^\perp$ where $\mathfrak{J}_0 = \{f \in B^* \mid F(f) = 0$ for all $F \in \mathfrak{J}\}$. Let β be the linear space isometric isomorphism of $\mathfrak{J}_0{}^*$ onto $B^{**}/\mathfrak{J}$ defined for $F_0 \in \mathfrak{J}_0{}^*$ by $\beta F_0 = F + \mathfrak{J}$ where $F \in B^{**}$ is an arbitrary extension of F_0. Define multiplication in $\mathfrak{J}_0{}^*$ so that β (and thus β^{-1}) is an algebra isomorphism. For $f \in \mathfrak{J}_0$, define $\varphi_* f$ by $\varphi_* f(x) = (\beta^{-1}\varphi(x))(f)$, $x \in A$. Then $\varphi_* f$ is linear and since φ is bounded $\|\varphi_* f(x)\| \leqq \|\varphi\| \, \|x\| \, \|f\|$, and $\varphi_* f \in A^*$.

Let $\mathfrak{J}_0{}'$ be the w^*-closure of the range of φ_*, and let $\mathfrak{J}' = \mathfrak{J}'_0{}^\perp$. Clearly $\mathfrak{J}'$ is w^*-closed. We next show that $\mathfrak{J}'$ is an ideal of A^{**}. Let $f \in \mathfrak{J}_0$. Then for any $x, y \in A$, $\langle \varphi_* f, x\rangle(y) = \varphi_* f(xy) = (\beta^{-1}\varphi(xy))f = (\beta^{-1}\varphi(yx))(f)$, since the range of φ is commutative. Suppose that $\varphi(y) = U + \mathfrak{J}$, and $\varphi(x) = V + \mathfrak{J}$ so that $\varphi(yx) = UV + \mathfrak{J}$. Then $(\beta^{-1}\varphi(xy))(f) = UV(f) = U([V, f])$. Since $f \in \mathfrak{J}_0$, and $\mathfrak{J} = \mathfrak{J}_0{}^\perp$ is an ideal, $g = [V, f] \in \mathfrak{J}_0$ by Lemma 3.5. Hence $(\beta^{-1}\varphi(yx))(f) = U(g) = (\beta^{-1}\varphi(y))(g) = \varphi_* g(y)$, for all $y \in A$. We therefore have $\langle \varphi_* f, x\rangle = \varphi_* g$ and so $\langle \varphi_* f, x\rangle \in \mathfrak{J}'_0$ for any $x \in A$ and $f \in \mathfrak{J}_0$. Suppose next that $g \in \mathfrak{J}_0{}'$, and $x \in A$. Say $g = w^*\text{-}\lim \varphi_* f_\alpha$ with $f_\alpha \in \mathfrak{J}_0$. Then for $y \in A$, $\langle g, x\rangle(y) = g(xy) = \lim \varphi_* f_\alpha(xy) = \lim \langle \varphi_* f_\alpha, x\rangle(y)$, and hence $\langle g, x\rangle = w^* - \lim \langle \varphi_* f_\alpha, x\rangle$. However, by the above, $\langle \varphi_* f_\alpha, x\rangle \in \mathfrak{J}_0{}'$, and $\mathfrak{J}_0{}'$ is w^*-closed so $\langle g, x\rangle \in \mathfrak{J}_0{}'$ for any $g \in \mathfrak{J}_0{}'$ and $x \in A$.

Let $G \in A^{**}$ and let $f \in \mathfrak{J}_0{}'$. Let $\{x_\alpha\}$ be a net in A such that $w^*\text{-}\lim \pi x_\alpha = G$. Then $[G, f](x) = G(\langle f, x\rangle) = \lim \pi x_\alpha(\langle f, x\rangle) = \lim \langle f, x\rangle(x_\alpha) = \lim f(x x_\alpha) = \lim f\langle x_\alpha x) = \lim \langle f, x_\alpha\rangle(x)$ for $x \in A$. Consequently $[G, f] = w^*\text{-}\lim \langle f, x_\alpha\rangle$, and is thus in $\mathfrak{J}_0{}'$ as $\mathfrak{J}_0{}'$ is w^*-closed. Hence, by Lemma 3.5, $\mathfrak{J}' = \mathfrak{J}_0{}'^\perp$ is a w^*closed ideal of A**.

For $F \in A^{**}$, define $\gamma F(f) = F(\varphi_* f)$ for $f \in \mathfrak{J}_0$. Clearly γF is a bounded linear functional on $\mathfrak{J}_0$, and so has an extension of the same norm which is an element of B^{**}. We again denote the extension by γF. Thus γ is a bounded linear map from A^{**} into B^{**}. Note that if $F_1 - F_2 \in \mathfrak{J}'$ and $f \in \mathfrak{J}_0$, then $\gamma(F_1 - F_2)(f) = (F_1 - F_2)(\varphi_* f)\, 0$, and thus $\gamma F_1 - \gamma F_2 \in \mathfrak{J}$. Thus for any $F \in F_0 + \mathfrak{J}$, $\|\gamma F_0 + \mathfrak{J}'\| = \|\gamma F + \mathfrak{J}\| \leqq \|\gamma F\| \leqq \|F\| \, \|\varphi_*\|$ and hence $\|\gamma F_0 + \mathfrak{J}\| \leqq \|F_0 + \mathfrak{J}'\| \, \|\varphi_*\|$.

Define ψ on $A^{**}/\mathfrak{J}'$ by $\psi(F + \mathfrak{J}') = \gamma F + \mathfrak{J}$. By the above, we see that ψ is a bounded linear mapping of $A^{**}/\mathfrak{J}'$ into $B^{**}/\mathfrak{J}$. Also for $x \in A$, $\psi(\pi x + \mathfrak{J}') = \gamma \pi x + \mathfrak{J}$. Since $\gamma \pi x(f) = \pi x(\varphi_* f) = \varphi_* f(x) = (\beta^{-1}\varphi(x))(f)$ for $f \in \mathfrak{J}_0$, $\gamma \pi x - \beta^{-1}\varphi(x) \in \mathfrak{J}$, and $\psi(\pi x + \mathfrak{J})' = \varphi(x)$.

Thus all that remains is to see that ψ satisfies the required multiplicative property of a homomorphism. Let $F, G \in A^{**}$. To see that $\psi(FG) = \psi(F)\psi(G)$, we must show that for $f \in \mathfrak{J}_0$, $\{\gamma(F)\gamma(G) - \gamma(FG)\}(f) = 0$. Since $\{\gamma(F)\gamma(G) - \gamma(FG)\}(f) = \gamma(F)([\gamma(G), f]) - FG(\varphi_* f) = F(\varphi_*[\gamma(G), f] - [G, \varphi_* f])$, it suffices if we show that $\varphi_*[\gamma(G), f] - [G, \varphi_* f] = 0$. Let $x, y \in A$ and suppose that $\varphi(x) = U + \mathfrak{J}$, $\varphi(y) = V + \mathfrak{J}$, and thus $\varphi(xy) = \varphi(yx) = VU + \mathfrak{J}$. It follows that $\langle \varphi_* f, x\rangle(y) = \varphi_* f(xy) = VU(f) = V([U, f])$. Now, since $f \in \mathfrak{J}_0$, $[U, f] \in \mathfrak{J}_0$ by Lemma 3.5. We therefore have $\langle \varphi_* f. x\rangle(y) =$

$\varphi_*[U, f](y)$ for all $y \in A$, and consequently $\langle \varphi_* f, x \rangle = \varphi_*[U, f]$. Thus $[G, \varphi_* f](x) = G(\langle \varphi_* f, x \rangle) = G(\varphi_*[U, f]) = \gamma G([U, f] = (\gamma G) U(f)$. On the other hand, $\varphi_*[\gamma G, f](x) = U([\gamma G, f]) = U\gamma G(f)$. Since under our hypothesis $\varphi(x) = U + \mathfrak{J}$ is in the center of $B^{**}/\mathfrak{J}$, $U\gamma G(f) = (\gamma G) U(f)$ for $f \in \mathfrak{J}_0$ and we have the desired result.

It should be noted that the ideal $\mathfrak{J}'$ in general is dependent on the homomorphism φ. Two instances should be noted where this is not the case. The first, when $\mathfrak{J}' = 0$, has already been treated in the discussion of w^*-continuous extensions of homomorphisms of A into the center of B^{**}. The other is the following.

3.7 THEOREM. *Let A and B be commutative Banach algebras. Let φ be a homomorphism of A into $B^{**}/\mathfrak{Y}^{\perp}(B)$. Then there is a homomorphism ψ of $A^{**}/\mathfrak{Y}^{\perp}(B)$ such that $\psi(\pi x + \mathfrak{Y}^{\perp}) = \varphi(x)$.*

Proof. If in the proof of Theorem 3.6, $\mathfrak{J}_0 = \mathfrak{Y}(B)$, it follows from Lemma 3.6 of [3] that for any $f \in \mathfrak{J}_0$ which is a multiplicative linear functional on B, that $\varphi_* f$ is a multiplicative linear functional on A. Hence, the norm closure of the range of φ_* is contained in $\mathfrak{Y}(A)$. In view of Lemma 3.6 of [3], the subspace $\mathfrak{Y}^{\perp}(A)$ is a w^*-closed ideal of A^{**}, and if used in the role of $\mathfrak{J}'$ affords the same conclusion. Note that the homomorphism φ is not postulated to be bounded or with range in the center of $B^{**}/\mathfrak{Y}^{\perp}(B)$. This is legitimate since in view of Theorem 3.7 of [3], $B^{**}/\mathfrak{Y}^{\perp}$ is automatically commutative and semi-simple, and thus φ is automatically bounded.

If A and B are the group algebras of the compact groups $\mathfrak{G}$ and $\mathfrak{H}$, then $A^{**}/\mathfrak{Y}^{\perp}(A)$ and $B^{**}/\mathfrak{Y}^{\perp}(B)$ may be identified with the measure algebras $M(\mathfrak{G})$ and $M(\mathfrak{H})$ respectively by Theorem 3.18 of [3]. Thus Theorem 3.7 includes in the case of compact groups, the result of P. J. Cohen [4] quoted in the introduction.

4. Group algebras. Let $\mathfrak{G}$ be a locally compact abelian group. As in §3, we denote the group algebra of $\mathfrak{G}$ by $L(\mathfrak{G})$ and the algebra of finite regular Borel measures on $\mathfrak{G}$ by $M(\mathfrak{G})$. For notational purposes, it is also convenient to identify the character group $\hat{\mathfrak{G}}$ of $\mathfrak{G}$ with the subset of $L^*(\mathfrak{G})$ consisting of the nonzero multiplicttive linear functional on $L(\mathfrak{G})$. The topology of $\hat{\mathfrak{G}}$ is then in agreement with the w^*-topology of $\hat{\mathfrak{G}}$ as a subset of $L^*(\mathfrak{G})$.

Suppose that $\mathfrak{H}$ is a locally compact abelian group. A continuous homomorphism ν of $\mathfrak{G}$ into $\mathfrak{H}$ is called *nonsingular* if for every Borel set E is $\mathfrak{H}$ with zero Haar measure, $\nu^{-1}(\mathfrak{E})$ is of zero Haar measure in $\mathfrak{G}$.

A complete characterization of all homomorphisms φ of $L(\mathfrak{G})$ into $M(\mathfrak{H})$ was given by P. J. Cohen [4]. He utilized the function φ_* from $\hat{\mathfrak{H}}$ into $\{\hat{\mathfrak{G}}, 0\}$ defined by $\varphi_* f(x) = \varphi(x)(f)$, $x \in L(\mathfrak{G})$, $f \in \hat{\mathfrak{H}}$.

4.1 THEOREM. (P. J. Cohen) *Let* $\mathfrak{G}$ *and* $\mathfrak{H}$ *be locally compact abelian groups,* φ *a homomorphism of* $L(\mathfrak{G})$ *into* $M(\mathfrak{H})$, φ_* *the induced map of* $\hat{\mathfrak{H}}$ *into,* $\{\hat{\mathfrak{G}}, 0\}$. *Then there are a finite number of sets* $\mathfrak{K}_i$, *which are cosets of open subgroups of* $\hat{\mathfrak{H}}$, *and continuous maps* ψ_i: $\hat{\mathfrak{K}}_i \to \mathfrak{G}$, *such that*

$$\psi_i(x + y - z) = \psi_i(x) + \psi_i(y) - \psi_i(z) \tag{4.1}$$

for all x, y *and* z *in* $\mathfrak{K}_i$, *with the following property: There is a decomposition of* $\hat{\mathfrak{H}}$ *into the disjoint union of sets* $\mathfrak{S}_j$, *each lying in the Boolean ring generated by the sets* $\mathfrak{K}_i$, *such that on each* $\mathfrak{S}_j$, φ_* *is either identically zero or agrees with some* ψ_i, *where* $\mathfrak{S}_j \subset \mathfrak{K}_i$.

Conversely, for any map of $\hat{\mathfrak{H}}$ *into* $\{\hat{\mathfrak{G}}, 0\}$, *there is a homomorphism of* $L(\mathfrak{G})$ *into* $M(\mathfrak{H})$ *which induces it. The map* φ *carries* $L(\mathfrak{G})$ *into* $L(\mathfrak{H})$ *if and only if* φ_*^{-1} *of every compact subset of* $\hat{\mathfrak{G}}$ *is compact.*

Suppse that the sets $\mathfrak{K}_i$ are cosets of the subgroups $\mathfrak{U}_i$ of $\hat{\mathfrak{H}}$. There is a closed subgroup $\mathfrak{H}_i$ of $\mathfrak{H}$, $\mathfrak{H}_i = \{h \in \mathfrak{H} \mid (h, \hat{h}) = 1, \hat{h} \in \mathfrak{U}_i\}$, such that $\mathfrak{U}_i$ may be viewed [6, p. 130] as the character group of $\mathfrak{H}/\mathfrak{H}_i$. Let $a_i \in \mathfrak{K}_i$, and define ψ_i': $\mathfrak{U}_i \to \mathfrak{G}$ by

$$\psi_i'(x) = \psi_i(a_i + x) - \psi_i(a_i), \qquad x \in \mathfrak{U}_i. \tag{4.2}$$

The condition (4.1) on ψ_i is then equivalent to the assertion that ψ_i' is a homomorphism of $\mathfrak{U}_i$ into $\hat{\mathfrak{G}}$, and ψ_i' is continuous along with ψ_i. We may also consider the dual homomorhism ρ_i: $\mathfrak{G} \to \hat{\mathfrak{U}}_i = \mathfrak{H}/\mathfrak{H}_i$, defined dy

$$(\psi_i'(x), g) = (x, \beta_i(g)), \qquad x \in \mathfrak{U}_i = (\mathfrak{H}/\mathfrak{H}_i)^\wedge, g \in \mathfrak{G}. \tag{4.3}$$

In view of the Cohen theorem, the homomorphism ψ is determined by the sets $\mathfrak{K}_i, \mathfrak{S}_j$ and the functions β_i. The notation introduced above will be used in the sequel without further comment. We also use the notation ρ_* as the mapping of $L^*(\mathfrak{H})$ into $L^*(\mathfrak{G})$ which is defined by $\rho_* f(x) = \rho(x)(f)$, $x \in L(\mathfrak{G})$, $f \in L^*(\mathfrak{H})$, whenever ρ is a bounded linear map of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{H})$.

4.2 LEMMA. *Let* λ *be a nonsingular homomorphism of* $\mathfrak{G}$ *into a locally compact abelian group* $\mathfrak{K}$. *Then* λ *induces a homomorphism* ρ *of* $L(\mathfrak{G})$ *into* $L^{**}(\mathfrak{K})$ *such that for* $f \in \hat{\mathfrak{K}}$, $\rho_*(f) = f \circ \lambda$.

Proof. For $k \in L^*(\mathfrak{K})$, define $\lambda_*(k)$ by

$$\lambda_*(k)(\alpha) = k \circ \lambda(\alpha), \qquad \alpha \in G.$$

We first must show that λ_* is a well-defined bounded linear mapping of $L^*(\mathfrak{K})$ into $L^*(\mathfrak{G})$. Suppose that K_1 andK_2 are two bounded Borel measurable functions on $\mathfrak{K}$ such that $k_1(\beta) = k_2(\beta)$ for almost all β in $\mathfrak{K}$. Let $\mathfrak{E} = \{\alpha \in \mathfrak{G} \mid k_1(\lambda(\alpha)) \neq k_2(\lambda(\alpha))\}$. Then $\mathfrak{E} = \lambda^{-1}(\lambda(\mathfrak{E}))$ and by the hypothesis of non-singularity $\mathfrak{E}$ has measure zero in $\mathfrak{G}$. Since it is now immediate that $|\lambda_*(k)(\alpha)| \leqq \|k\|$ for almost all α in $\mathfrak{G}$, it follows that λ_* is a bounded linear map of $L^*(\mathfrak{K})$ into $L^*(\mathfrak{G})$.

For $x \in L(\mathfrak{G})$, define $\rho(x)$ on $L^*(\mathfrak{K})$ by

$$\rho(x)(f) = \lambda_* f(x), \qquad f \in L^*(\mathfrak{K}).$$

Clearly $\rho(x) \in L^{**}(\mathfrak{K})$, and ρ is a bounded linear mapping from $L(\mathfrak{G})$ into $L^{**}(\mathfrak{K})$, and $\rho_* f = f \circ \lambda$.

We next show that ρ satisfies the multiplicative condition for a homomorphism. Let $x, y \in L(\mathfrak{G})$ and $f \in L^*(\mathfrak{K})$. Then

$$\begin{aligned}\rho(xy)(f) = \lambda_* f(xy) &= \int_{\mathfrak{G}} \lambda_* f(\alpha) \int_{\mathfrak{G}} x(\beta) y(\alpha - \beta)\, d\beta\, d\alpha \\ &= \int_{\mathfrak{G}} \int_{\mathfrak{G}} f(\lambda)(\alpha)) x(\beta) y(\alpha - \beta)\, d\beta\, d\alpha \\ &= \int_{\mathfrak{G}} \int_{\mathfrak{G}} f(\lambda(\alpha + \beta)) x(\beta) y(\alpha)\, d\beta\, d\alpha.\end{aligned}$$

For any $z \in L(\mathfrak{K})$, and $\delta \in \mathfrak{K}$, it is easily seen [3] that $\langle f, z\rangle(\delta) = \int_{\mathfrak{K}} f(z + \delta) z(\gamma)\, d\gamma$. Therefore,

$$\begin{aligned}[\rho(y), f](z) = \rho(y)(\langle f, z\rangle) = \lambda_* \langle f, z\rangle(y) &= \int_{\mathfrak{G}} \lambda_* \langle f, z\rangle(\alpha) y(\alpha)\, d\alpha \\ = \int_{\mathfrak{G}} \langle f, z\rangle(\lambda(\alpha)) y(\alpha) d\alpha &= \int_{\mathfrak{G}} \int_{\mathfrak{K}} f(\gamma + \lambda(\alpha)) z(\gamma) y(\alpha)\, d\gamma\, d\alpha.\end{aligned}$$

Since the order of integration may be reversed, we see that for $\gamma \in \mathfrak{K}$, $[\rho(y), f](\gamma) = \int_{\mathfrak{G}} f(\gamma + \lambda(\beta)) y(\beta)\, d\beta$. Hence,

$$\begin{aligned}\rho(x)\rho(y)(f) = \rho(x)([\rho(y), f]) = \lambda_*[\rho(y), f](x) &= \int_{\mathfrak{G}} \lambda_*[\rho(y), f](\alpha) x(\alpha)\, d\alpha \\ = \int_{\mathfrak{G}} [\rho(y), f](\lambda(\alpha)) x(\alpha)\, d\alpha &= \int_{\mathfrak{G}}\int_{\mathfrak{G}} f(\lambda(\alpha) + \lambda(\beta)) y(\beta) x(\alpha)\, d\beta\, d\alpha\,.\end{aligned}$$

Since we thus have $\rho(xy)(f) = \rho(x)\rho(y)(f)$, for all $f \in L^*(K)$, ρ is a homomorphism.

4.3 THEOREM. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be locally compact abelian groups, with $\mathfrak{H}$ compact. Let φ be a homomorphism of $L(\mathfrak{G})$ into $M(\mathfrak{H})$. Let $M(\mathfrak{H})$ be regarded as $L^{**}(\mathfrak{H})/\mathfrak{Y}^{\perp}(\mathfrak{H})$, and let θ be the natural mapping of $L^{**}(\mathfrak{H})$ onto $L^{**}(\mathfrak{H})/\mathfrak{Y}^{\perp}(\mathfrak{H})$. Then if each homomorphism β_i, determined by φ, is nonsingular, there is a homomorphism ρ of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{H})$ such that $\varphi = \theta \circ \rho$.*

Proof. The justification for considering $M(\mathfrak{H})$ as $L^{**}(\mathfrak{H})/\mathfrak{Y}^{\perp}(\mathfrak{H})$ is

Theorem 3,18 of [3].

If $\varphi_*(f) = 0$ for all $f \in \mathfrak{S}_j$, define ρ_j: $L(\mathfrak{G}) \to L^{**}(\mathfrak{H})$ by $\rho_j(x) = 0$, $x \in L(\mathfrak{G})$.

Suppose that $\mathfrak{S}_j \subset \mathfrak{K}_i \subset \hat{\mathfrak{H}}$, and $\varphi_*(f) = \psi_i(f)$ for $f \in \mathfrak{S}_j$. In view of (4.1), the homomorphism ψ_i' of U_i into $\hat{G}$ may be defined by $\psi_i'(k) = \psi_i(k + k_i) - \psi_i(k_i)$ for an arbitrary $k_i \in \mathfrak{S}_j$. The dual homomorphism β_i of $\mathfrak{G}$ into $\mathfrak{H}/\mathfrak{H}_i$ is by hypothesis nonsingular. Thus by Lemma 4.2, there is a homomorphism ρ_j' of $L(G)$ into $L^{**}(\mathfrak{H}/\mathfrak{H}_i)$ such that $\rho_{j*}'(k) = k \circ \beta_i$, for $k \in (\mathfrak{H}/\mathfrak{H}_i)^\wedge = \mathfrak{U}_i$.

For $f \in L(\mathfrak{H}/\mathfrak{H}_i)$ define $\theta_i(f)$ on $\mathfrak{H}$ by $\theta_i(f)(\beta) = f(\beta + \mathfrak{H}_i)$. Suppose that the Haar measure on $\mathfrak{H}_i$ is normalized so that the measure of $\mathfrak{H}_i$ is one. The formula relating integration on a group with that on a quotient group shows that θ_i is an isometric isomorphism of $L(\mathfrak{H}/\mathfrak{H}_i)$ into $L(\mathfrak{H})$. Thus by Theorem 6.1 of [3], θ_i^{**} is a homomorphism of $L^{**}(\mathfrak{H}/\mathfrak{H}_i)$ into $L^{**}(\mathfrak{H})$. Also for any $u \in L(\mathfrak{H}/\mathfrak{H}_i)$, and $f \in L^*(\mathfrak{H})$,

$$\begin{aligned}\theta_i^*f(u) &= f(\theta_i u) = \int_{\mathfrak{H}} f(\beta)\theta_i(u)(\beta)\,d\beta \\ &= \int_{\mathfrak{H}/\mathfrak{H}_i}\int_{\mathfrak{H}_i} f(\beta+\gamma)\theta_i(u)(\beta+\gamma)\,d\gamma\,d\dot{\beta},\end{aligned}$$

where $d\dot{\beta}$ is the Haar measure on $\mathfrak{H}/\mathfrak{H}_i$. Thus

$$\theta_i^*f(u) = \int_{\mathfrak{H}/\mathfrak{H}_i} u(\dot{\beta})\int_{\mathfrak{H}_i} f(\beta+\gamma)\,d\gamma\,d\dot{\beta}\ ,$$

and we conclude that $\theta_i^*f(\dot{\beta}) = \int_{\mathfrak{H}_i} f(\beta + \gamma)\,d\gamma$.

It is well known that in a group algebra the pointwise multiplication by a character is an automorphism of the algebra. We next show that the same situation prevails in the second conjugate algebra of a group algebra. Let $\mathfrak{T}$ be a locally compact abelian group and define, for $\eta \in \hat{\mathfrak{T}}$, $\eta\circ g$ and $\eta\circ g$ by pointwise multiplication on $\mathfrak{T}$ if $x \in L(\mathfrak{T})$ and $g \in L^*(\mathfrak{T})$. Define $\eta \circ G(g) = G(\eta \circ g)$ for $G \in L^{**}(\mathfrak{T})$. Clearly the map $G \to \eta \circ G$ is a one-to-one bounded linear map of $L^{**}(\mathfrak{T})$ onto itself. Let $F, G \in L^{**}(\mathfrak{T})$ and $g \in L^*(\mathfrak{T})$. It remains for us to show that $(\eta\circ F)(\eta\circ G)(g) = \eta\circ(FG)(g)$. Since $(\eta\circ F)(\eta\circ G)(g) = \eta\circ F([\eta\circ G, g]) = F(\eta\circ[\eta\circ G, g])$, while $\eta\circ(FG)(g) = FG(\eta\circ g) = F([G, \eta\circ g])$, it suffices if we show that for all $x \in L(\mathfrak{T})$, $\eta\circ[\eta\circ G, g](x) = [G, \eta\circ g](x)$. Now $\eta\circ[\eta\circ G, g](x) = [\eta\circ G, g](\eta\circ x) = \eta\circ G(\langle g, \eta\circ x\rangle) = G(\eta\circ\langle g, \eta\circ x\rangle)$, while $[G, \eta\circ g](x) = G(\langle \eta\circ g, x\rangle)$, so it suffices if we show that for all $y \in L(\mathfrak{T})$, $\eta\circ\langle g, \eta\circ x\rangle(y) = \langle \eta\circ g, x\rangle(y)$. Since $\eta\circ\langle g, \eta\circ x\rangle(y) = g((\eta\circ x)(\eta\circ y)) = g(\eta\circ xy) = \eta\circ g(xy) = \langle \eta\circ g, x\rangle(y)$, the original assertion follows.

Define the mapping ρ_j by

$$\rho_j(x) = k_i^{-1}\circ\theta_i^{**}\rho_j'(\psi_i(k_i)\circ x)\ , \qquad x \in L(\mathfrak{G})\ , \tag{4.4}$$

where the dot at each occurrence indicates multiplication of the appropriate functions. Since $k_i \in \hat{\mathfrak{H}}$, and $\psi_i(k_i) \in \hat{\mathfrak{G}}$, ρ_j is a composite of four homomorphisms and is thus a homomorphism of $L(\mathfrak{G})$ and $L^{**}(\mathfrak{H})$.

Suppose that $f \in \mathfrak{S}_j \subset \mathfrak{K}_i$, so that $\varphi_* f = \psi_i f$. Since $\mathfrak{K}_i$ is a coset of $\mathfrak{U}_i$, there is a $k \in \mathfrak{U}_i$ such that $f = k_i + k$. We use the same notation for k when it is viewed as a member of $(\mathfrak{H}/\mathfrak{H}_i)^\wedge$. For any $x \in L(\mathfrak{G})$, $\rho_{j*}f(x) = \rho_j(x)(f) = k_i^{-1}\circ\theta_i^{**}\rho_j'(\psi_i(k_i)\circ x)(f) = \theta_i^{**}\rho_j'(\psi_i(k_i)\circ x)(k) = \rho_j'(\psi_i(k_i)\circ x)\theta_i^*(k)$. From the formula obtained earlier for θ_i^*, it is immediate that θ_i^* simply transfers k from being viewed as a member of $\mathfrak{U}_i \subset \hat{\mathfrak{H}}$, to being viewed as a member of $(\mathfrak{H}/\mathfrak{H}_i)^\wedge \subset L^*(\mathfrak{H}/\mathfrak{H}_i)$. Thus

$$\rho_{j*}f(x) = \rho_j'(\psi_i(k_i)\circ x)(k) = \int_{\mathfrak{G}} \rho_j'(k)(\alpha)\psi_i(k_i)(\alpha)x(\alpha)\,d\alpha$$
$$= \int_{\mathfrak{G}} (k, \beta_i(\alpha))\psi_i(k_i)(\alpha)x(\alpha)\,d\alpha = \int_{\mathfrak{G}} (\psi_i'(k), \alpha)\psi_i(k_i)(\alpha)x(\alpha)\,d\alpha\ ,$$

by use of (4.3). Thus by use of the definition of ψ_i' in terms of k_i, we have

$$\rho_{j*}f(x) = \int_{\mathfrak{G}} (\psi_i(k + k_i) - \psi_i(k_i), \alpha)(\psi_i(k_i), \alpha)x(\alpha)\,d\alpha$$
$$= \int_{\mathfrak{G}} (\psi_i(f), \alpha)x(\alpha)\,d\alpha = \int_{\mathfrak{G}} \varphi_* f(\alpha)x(\alpha)\,d\alpha\ .$$

We therefore conclude that $\rho_{j*}f(x) = \varphi_* f(x)$ for all $x \in L(\mathfrak{G})$ or that $\rho_{i*}f = \varphi_* f$ for $f \in \mathfrak{S}_i$.

Now, by the Cohen theorem, $\hat{\mathfrak{H}}$ is the disjoint union of the sets $\mathfrak{S}_j$. The characteristic function of $\mathfrak{S}_j$ is then the Fourier transform of an idempotent measure in $M(\mathfrak{H}) = L^{**}(\mathfrak{H})/\mathfrak{Y}^\perp(\mathfrak{H})$. Let F_j be any member of $L^{**}(\mathfrak{H})$ such that θF_j is the Fourier transform of the characteristic function of $\mathfrak{S}_j$. Then $F_j^2 - F_j \in \mathfrak{Y}^\perp(\mathfrak{H})$. Now, Theorem 3.15 of [3] states that $\mathfrak{Y}^\perp(\mathfrak{H})$ is the radical of $L^{**}(\mathfrak{H})$, and therefore Theorem 2.3.9 of [5] yields $E_j \in L^{**}(\mathfrak{H})$ such that $E_j^2 = E_j$ and $\theta E_j = \theta F_j$.

We next show that if $i \neq j$, then $E_i F E_j = 0$ for any $F \in L^{**}(\mathfrak{H})$. Suppose that $f \in \hat{\mathfrak{H}}$, then Lemma 3.6 of [3] yields

$$E_i F E_j(f) = E_i(f)F(f)E_j(f)\ .$$

For $f \in \hat{\mathfrak{H}}$, $E_k(f) = F_k(f) = \chi(\mathfrak{S}_k)(f)$, where $\chi(\mathfrak{S}_k)$ is the characteristic function of $\mathfrak{S}_k$. Thus since S_i and S_j are disjoint $E_i F E_j(f) = 0$. Hence $E_i F E_j \in \mathfrak{Y}^\perp$, the radical of $L^{**}(\mathfrak{H})$. For a compact group $\mathfrak{H}$, the radical is also the right annihilator of $L^{**}(\mathfrak{H})$ by Theorem 3.5 of [3]. Thus since $E_i = E_i^2$, $E_i F E_j = E_i(E_i F E_j) = 0$.

Let ρ be defined on $L(\mathfrak{G})$ by

$$\rho(x) = E_1\rho_1(x)E_1 + \cdots + E_r\rho_r(x)E_r, \qquad x \in L(\mathfrak{G}),$$

where $\hat{\mathfrak{H}} = \mathfrak{S}_1 \cup \cdots \cup \mathfrak{S}_r$. Clearly ρ is a bounded linear transformation of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{H})$, and to see that ρ is a homomorphism it suffices if we show that $E_i\rho_i(xy)E_i = E_i\rho_i(x)E_i\rho_i(y)E_i$. The latter equality is ertablished by an identical argument to that used above to show $E_iFE_j = 0$ for $i \neq j$. Thus ρ is a homomorphism of $L(\mathfrak{G})$ into $L^{**}(\mathfrak{H})$.

To see that $\theta\circ\rho = \varphi$, it suffices if we show that $\varphi_*(f) = (\theta\circ\rho)_*(f)$ for $f \in \hat{\mathfrak{H}}$. Suppose that $f \in \mathfrak{S}_k$. Then for $x \in L(\mathfrak{G})$, $(\theta\circ\rho)_*(f)(x) = \theta\circ\rho(x)(f) = E_k\rho_k(x)E_k(f)$, since $E_i(f) = 0$ if $i \neq k$. Thus $(\theta\circ\rho)_*(f)(x) = \rho_k(x)(f) = \varphi_* f$ as was shown earlier.

References

1. Richard Arens, *Operations in function classes*, Monat. für Math. **55** (1951), 1-19.
2. ———, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc., **2** (1954), 839-848.
3. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math., **11** (1961), 820-847.
4. P. J. Cohen, *On homomorphisms of group algebras*, Amer. J. Math., **82** (1960), 213-226.
5. C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, New York, 1960.
6. André Weil, *L' intégration dans les groupes topologique et ses appicationes*, Herman et Cie, Paris, 1953.

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