

PREDICTION THEORY FOR MARKOFF PROCESSES

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In this paper we consider the least square prediction problem for Markoff processes with stationary transitions. The main result concerns the partial differential equation characterizing the prediction operator, and the conditions for the uniqueness of the solutions.

Introduction. Let $x(t)$ be a Markoff process with stationary transitions. It is well-known that the optimum mean square predictor of $g(x(s+t))$ given $x(\sigma)$ for $\sigma \leq s$ is given by the conditional expectation:

$$E[g(x(t+s)) | x(\sigma) \leq s].$$

For a Markoff process this becomes

$$(1.1) \quad E[g(x(t+s)) | x(s)]$$

and further, if the transitions are stationary, we need only to consider:

$$(1.2) \quad E[g(x(t)) | x(0)]$$

Let $p(t, \xi | x)$ be the distribution function (suitably normalized) of the conditional or transition probability of transition from x to ξ in time t . Then, of course, (1.2) becomes

$$(1.3) \quad \int g(\xi) d_{\xi} p(t, \xi | x).$$

Now if $g(\cdot)$ is in $C[\alpha, \beta]$, where $-\infty \leq \alpha < \beta \leq +\infty$ is the interval over which the transition probabilities are defined, we obtain a semigroup of linear operators over $C[\alpha, \beta]$ defined through (1.3). If now we know the infinitesimal generator of this semigroup, we obtain an abstract differential equation for (1.3):

$$(1.4) \quad \frac{du(t, g)}{dt} = Au(t, g)$$

where $u(t, g)$ represents (1.3) and A is the infinitesimal generator, provided $g(\cdot)$ is in the domain of A . If we know the representation of A , and if in particular, it turns out to be a partial differential operator, (1.4) offers an alternate way of determining the prediction functions (1.2) provided uniqueness of the solution can be proved. In what follows, we shall be concerned primarily with situations where such a reduction is possible, and the associated conditions for uniqueness.

Main Results:

2. Markoff processes of the diffusion type. A well-known set of sufficient conditions under which the reduction to a parabolic partial differential equation is possible are the Lindberg-Levy conditions which we state here in their weakest form due to Feller [3, 4]. Let

$$(i) \quad \frac{1}{t} \int_{|\xi-x|>\varepsilon} d_\varepsilon P(t, \xi | x) \rightarrow 0 \text{ as } t \rightarrow 0+$$

$$L_1 \quad (ii) \quad \frac{1}{t} \int_{|\xi-x|<\varepsilon} (\xi-x) d_\varepsilon P(t, \xi | x) \rightarrow b(x) \text{ as } t \rightarrow 0+$$

$$(iii) \quad \frac{1}{t} \int_{|\xi-x|<\varepsilon} (\xi-x)^2 d_\varepsilon P(t, \xi | x) \rightarrow 2a(x) \text{ as } t \rightarrow 0+$$

Then for each $g(\cdot)$ in $C[\alpha, \beta]$, if we set:

$$(2.1) \quad T(t)g(x) = \int_\alpha^\beta g(\xi) d_\varepsilon P(t, \xi | x).$$

$T(t)$ is a semigroup of linear bounded operators over $[\alpha, \beta]$ and moreover

- (i) $\|T(t)g\| \|g\|$ (contraction semigroup)
- (ii) $\|T(t)g - g\| \rightarrow 0$ as $t \rightarrow 0+$ (strongly continuous)
- (iii) $T(t)g$ is non-negative if g is nonnegative.
(positivity preserving)
- (vi) For $g(\xi) = \text{constant}$ $T(t)g = g$.

Properties (i), (iii) and (iv) are obvious from (2.1). That $T(t)g$ again belongs to $C[\alpha, \beta]$ follows from condition (i) of L_1 , and so does property (ii). Let \mathcal{A} be the infinitesimal generator of the semigroup. Then the most important property one would like to deduce from L_1 is that it coincides with a second-order differential operator. Unfortunately, however, this is not always entirely true. For example following Feller [4], suppose we define the transition density kernels,

$$P(t, \xi | x) = \frac{1}{\sqrt{2\pi t}} \left[\exp - \frac{(g(\xi) - g(x))^2}{2t} \right] g'(\xi), \quad \alpha = -\infty, \quad \beta = +\infty$$

where say $g(\cdot)$ is a polynomial which vanishes at the origin, and $g'(\xi) > 0$. Taking $g(\xi) = \xi^3$, we obtain for $\xi \neq 0$

$$b(\xi) = -\frac{2}{9}\xi^{-5}$$

$$a(\xi) = \frac{\xi^{-4}}{9}.$$

However, at $\xi = 0$,

$$\begin{aligned} a(0) &= 0 \\ b(0) &= 0 . \end{aligned}$$

Direct substitution into (2.1) shows that for $f(\cdot)$ in the domain of A , $f(0) = f'(0) = f''(0)$ and that

$$Af(0) = \frac{2}{6!} f^{(6)}(0) .$$

Although for any $\xi \neq 0$,

$$(2.2) \quad Af(\xi) = a(\xi) \frac{d^2 f}{d\xi^2} + b(\xi) \frac{df}{d\xi} .$$

Here it may be noted that the exceptional point zero is a point of discontinuity of the functions $a(\cdot)$ and $b(\cdot)$. One might then expect that this may be avoided if they are required to be continuous. However, it should be noted even in this case that A may not still coincide entirely with the differential operator on the right in (2.2)—in fact, it may only be a contraction of that operator. With some additional conditions on $a(\xi)$ and $b(\xi)$ we can nevertheless obtain a stronger result.

THEOREM 2.1. *Let $a(\xi)$, $b(\xi)$ given by L_1 be continuously twice differentiable in the open interval (α, β) and $a(\xi) > 0$ therein. Let the limits in L_1 hold uniformly in x in each compact sub-interval. Suppose in addition they satisfy:*

$$(2.3) \quad \int_0^\alpha q(\xi)w(\xi)d\xi = +\infty = \int_0^\beta q(\xi)w(\xi)d\xi$$

where $\alpha < 0 < \beta$

$$\begin{aligned} q(t) &= \int_0^t \frac{dt}{a(t)w(t)} \\ w(t) &= \exp - \int_0^t \frac{b(\xi)}{a(\xi)} d\xi . \end{aligned}$$

Then the infinitesimal generator A of the semigroup coincides with the differential operator C

$$(2.4) \quad C = a(\xi) \frac{d^2}{d\xi^2} + b(\xi) \frac{d}{d\xi}$$

where the domain of C consists of functions $f(\xi)$ with first and second derivatives such that

$$a(\xi) \frac{d^2 f}{d\xi^2} + b(\xi) \frac{df(\xi)}{d\xi}$$

belongs to $C[\alpha, \beta]$.

Conversely suppose the functions $a(\xi)$ and $b(\xi)$ are given, with $a(\xi)$ positive and continuous and $b(\xi)$ continuous in the open interval $[\alpha, \beta]$, and suppose (2.3) is satisfied. Then C generates a semigroup given by (2.1) where the kernels are Markoff transition probabilities which satisfy the conditions L_1 , the limits holding uniformly in x in each compact sub-interval.

Proof. Let us consider the converse statement first. Under the conditions (2.3) on the coefficients $a(\xi)$ and $b(\xi)$, Hille [5] has shown that C is the infinitesimal generator of a strongly continuous positive contraction semigroup. Denoting this semigroup by $S(t)$, we have, for any $f(\cdot)$ in $C[\alpha, \beta]$:

$$(2.5) \quad S(t)f(x) = \int_{\alpha}^{\beta} f(\xi)d_{\xi}P(t, \xi; x)$$

where the $P(t, \xi; x)$ are Markoff transition kernels. Moreover, it is readily shown that the kernels satisfy the conditions L_1 , with the necessary uniformity.

Suppose next that we are given transition probabilities satisfying L_1 where $a(\cdot)$ and $b(\cdot)$ satisfy (2.3). We know then (2.1) yields a strongly continuous semigroup, and we have to show that its infinitesimal generator A coincides with C . For this, suppose $f(\cdot)$ is in the domain of C^2 . Then $f(\cdot)$ has first and second derivatives. Further, suppose $f'(\cdot)$ vanishes outside a compact sub-interval, say $[r_1, r_2]$. Now because $a(\xi) > 0$ and continuous in $[\alpha, \beta]$, it follows that $f''(\cdot)$ is continuous in compact sub-intervals, and hence in particular in $[r_1, r_2]$. Now for each x in $[\alpha, \beta]$

$$\begin{aligned} \frac{S(t)f(x) - f(x)}{t} &= \frac{1}{t} \int_{|\xi-x|>\varepsilon} [f(x) - f(\xi)]dP(t, \xi | x) \\ &+ \frac{f'(x)}{t} \int_{|\xi-x|<\varepsilon} (\xi - x)dP(t, \xi | x) \\ &+ \frac{f''(x + \theta\varepsilon)}{2t} \int_{|\xi-x|<\varepsilon} (\xi - x)^2dP(t, \xi | x) \end{aligned}$$

where $0 < |\theta| < 1$.

In view of L_1 , it follows that

$$(2.6) \quad \lim_{t \rightarrow 0} \frac{T(t)f(x) - f(x)}{t} = a(x)f''(x) + b(x)f'(x)$$

and because of the asserted uniformity of the limits in L_1 and the conditions on $f(\cdot)$, it is clear the limit in (2.6) is uniform in x in $[\alpha, \beta]$.

Hence, for such $f(\cdot)$ it follows that

$$Af = Cf .$$

Moreover, for the same $f(\cdot)$, note that Cf again vanishes outside $[r_1, r_2]$. Also, Cf again belongs to the domain of C and hence has first and second derivatives. Hence the argument above can be repeated to yield that

$$ACf = C^2f$$

and, of course

$$CAf = C^2f$$

or

$$CAf = ACf .$$

Denoting the semigroup generated by C by $S(t)$ it follows readily that

$$S(t)T(t)f = T(t)S(t)f \qquad t > 0$$

and hence using the Dunford argument [See [7]]:

$$S(t)f - T(t)f = \int_0^t (d/d\sigma)S(\sigma)T(\sigma)f = \int_0^t S(\sigma)T(\sigma)(Af - Cf)d\sigma$$

it follows that

$$S(t) = T(t)f .$$

It only remains to show that the set of such functions $f(\cdot)$ is dense in $C[\alpha, \beta]$. Now the class of functions in $C[\alpha, \beta]$ whose derivatives vanish outside compact subsets is dense in the domain of C . Because of the postulated twice differentiability of the coefficients $a(x)$ and $b(x)$, it follows that this class automatically belongs to the domain of C^2 , proving the required denseness. It is quite probable the result holds without demanding differentiability of the functions $a(x)$ and $b(x)$.

This proves the theorem.

We note in passing that the conditions (2.3) do not imply uniqueness of solutions of the forward equation, as Hille [3] has shown. From our point of view, this lack of uniqueness is of no concern to us, thus avoiding problems associated with the duality between the backward and forward equations. In particular, Theorem 2.1 establishes that for $f(\cdot)$ in the domain of C ,

$$E[f(x(t)) | x(0)] = u(t, x)$$

is the unique solution of the Cauchy problem:

$$\frac{\partial u}{\partial t} = a(x) \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x}$$

with

$$u(0, x) = f(x) .$$

As an example, consider the situation of Gaussian white-noise input to a nonlinear system, the input-output processes being related by [See Doob [2, p. 273] for the notation]

$$(2.7) \quad dx(t) = \sqrt{2} (x(t)^2 + 1) d\zeta(t)$$

$\zeta(t)$ being the real Gaussian additive process with

$$E[|d\zeta(t)|^2] = dt$$

so that the output process is Markoffian and we have for the limits in L_1 :

$$a(x) = (x^2 + 1)^2$$

$$b(x) = 0 .$$

These clearly satisfy all the required conditions of Theorem 2.1 and the predication function is the solution of the equation:

$$(2.8) \quad \frac{\partial u}{\partial t} = (x^2 + 1)^2 \frac{d^2 u}{dx^2}$$

subject to the initial condition

$$u(0, x) = f(x)$$

where it is assumed that

$$(x^2 + 1)f''(x)$$

belongs to $C[-\infty, +\infty]$.

In this particular case, we can obtain the solution in terms of orthogonal functions:

$$(2.9) \quad u(t, x) = \sum_0^{\infty} a_n W_n(x) e^{-n(n+1)t}$$

where

$$a_n = \int_{-\infty}^{\infty} \frac{W_n(x)}{(x^2 + 1)^2} f(x) dx$$

$$W_n(x) = \sqrt{\frac{2}{\pi}} \sqrt{(x^2 + 1)} \sin(n+1) \left(\frac{\pi}{2} - \arctan x \right) .$$

[See Hille [5]] for this solution. The convergence of the series in (2.4)

is uniform in x in $[\alpha, \beta]$.

It may be noted that functions such as

$$f(x) = x$$

and

$$f(x) = \exp i\lambda x$$

are not in $C[-\infty, +\infty]$, so that we cannot obtain the prediction as the solution of the partial differential equation directly, in the sense in which we have stated that Cauchy problem. It may, however, be possible to consider a slightly different B -space such as the space of functions $f(x)$ continuous in $(-\infty, \infty)$ and such that

$$\lim_{|x| \rightarrow \infty} f(x) \exp -|x|^\rho, \quad 0 < \rho < 1$$

exist for some ρ , as Hille [6] does for the heat equation.

It should also be noted that in this example, the transition density kernel has the expansion

$$p(t; \xi | x) = (\xi^2 + 1)^{-2} \sum_0^\infty W_n(x) W_n(\xi) e^{-n(n+1)t}.$$

As $t \rightarrow \infty$, we obtain the density

$$(2.10) \quad p(\infty; \xi | x) = 2/\pi(\xi^2 + 1)^{-2}$$

and it should be noted that (2.9) for each t is an orthogonal expansion with respect to this density. Also (2.10) corresponds to the (unique) stationary first order distribution with respect to which the process is ergodic. A sufficient condition for the existence of such an expansion (which automatically also yield the corresponding limiting density) due to Hille [5] is that in addition to (2.3) the following

$$(2.11) \quad \int_0^\alpha q'(x) dx \int_0^x w(\xi) d\xi < +\infty \quad \text{and} \quad \int_0^\beta q'(x) dx \int_0^x w(x) dx < \infty$$

be also satisfied. In this case, the limiting density is simply

$$\frac{q'(x)}{q(\beta) - q(\alpha)}.$$

All transition probabilities are absolutely continuous.

3. Markoff processes not of the diffusion type. We shall next consider the prediction problem not of the diffusion type, i.e., whose transition kernels do not satisfy conditions L_1 , but rather an extended version of them, leading to elliptic partial differential equations. Thus,

let the transition density kernels satisfy:

- (i) $2p(t, \xi | y) - p(2t, \xi | y) \geq 0$
- (ii) $\frac{1}{t^2} \int_{|y-\xi|>\varepsilon} [2p(t, \xi | y) - p(2t, \xi | y)] d\xi \rightarrow 0$ as $t \rightarrow 0 + L_2$
- (iii) $\frac{1}{t^2} \int_{|y-\xi|<\varepsilon} (\xi - y)[2p(t, \xi | y) - p(2t, \xi | y)] d\xi \rightarrow b(y)$
- (iv) $\frac{1}{t^2} \int_{|y-\xi|<\varepsilon} (y - \xi)^2 [2p(t, \xi | y) - p(2t, \xi | y)] d\xi \rightarrow 2a(y)$.

Then the prediction function satisfies the elliptic partial differential equation:

$$(3.1) \quad \frac{\partial^2 u}{\partial t^2} + a(x) \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x} = 0.$$

As before, the main difficulty is in obtaining uniqueness of the solutions.

THEOREM 3.1. *Suppose $a(y)$, $b(y)$ are twice continuously differentiable in $[\alpha, \beta]$ and $a(y) > 0$ therein. Suppose further that the limits in L_1 hold uniformly in y in each compact subinterval. Further, suppose that $a(\xi)$, $b(\xi)$ satisfy (2.3). Then for each $f(\cdot)$ in the domain of C ,*

$$u(t, x) = E[f(x(t)) | x(0) = x]$$

satisfies the partial differential equation

$$(3.2) \quad \frac{\partial^2 u}{\partial t^2} + Cu(t, x) = 0$$

and is the only solution of it satisfying to the conditions:

- (a) $\|u(t, \cdot) - f(\cdot)\| \rightarrow 0$ as $t \rightarrow 0$
- (b) $\|2u(t, \cdot) - u(2t)\| \leq \|f\|$
- (c) $\text{Sup}_t \|u(t, \cdot)\| < \infty$.

Conversely, suppose $a(\xi)$, $b(\xi)$ are given such that they are continuous in $[\alpha, \beta]$ and $a(\xi) > 0$ therein, and such that they satisfy (3.1). Then the Cauchy problem for (3.1) has a unique solution satisfying (a), (b) and (c) for each $f(\cdot)$ in the domain C , the solution being given by

$$u(t, x) = \int_{\alpha}^{\beta} f(\xi) p(t, \xi | x) d\xi$$

where the $p(t, \xi | x)$ are Markoff transition densities satisfying L_2 , the limits existing uniformly in y in compact sub-intervals.

Proof. For a proof of the converse part [see [6]]. Since $a(\cdot)$, $b(\cdot)$

satisfy these conditions in the forward part as well, let us denote the corresponding semigroup by $T(t)$ with generator B . Then we know that

$$B^2 = -C .$$

For each $f(\cdot)$ in $C[\alpha, \beta]$ let us next let

$$u(t, x) = E[f(x(t)) | x(0) = x] .$$

Then the conditions L_2 on the transition kernels imply that $u(t, x)$ satisfies (a), (b) and (c), and moreover setting

$$u(t, x) = S(t)f(x)$$

$S(t)$ is a strongly continuous semigroup over $C[\alpha, \beta]$. Let us denote its generator by A . We have now to show that

$$(3.2) \quad A^2 = B^2 = -C .$$

For this, let $f(\cdot)$ belong to $C[\alpha, \beta]$ and let $f'(\cdot)$ vanish outside a compact sub-interval $[r_1, r_2]$. Then as in the proof of Theorem 2.1, we shall first show that $f(\cdot)$ belongs also to the domain of A^2 and that

$$A^2f = B^2f .$$

For this we note that

$$\begin{aligned} & \frac{T(2t) + I - 2T(t)}{t^2}f(x) \\ &= \frac{1}{t^2} \int_{-\infty}^{\infty} (f(\xi) - f(x))[p(2t, \xi | x) - 2p(t, \xi | x)]d\xi \end{aligned}$$

and as before, as $t \rightarrow 0$, by virtue of L_2 this goes to

$$-a(x)f''(x) - b(x)f'(x)$$

and the rest of the arguments go over similarly. Also we readily obtain that:

$$A^2B^2f = B^2A^2f .$$

This is enough to imply that

$$T(t)f = S(t)f$$

and the differentiability properties of $a(x)$ and $b(x)$ again imply that such functions $f(\cdot)$ are dense in the domain of C and hence (3.2) follows. This concludes the proof.

The simplest example of a process with transition kernels satisfying the conditions L_2 is the Cauchy additive process, with the independent increments having a Cauchy distribution:

$$E[\exp i s(\zeta(t + \Delta) - \zeta(t))] = \exp - |s| \Delta .$$

More generally, such a process arises as the output of a first-order system whose input is the Cauchy additive process:

$$dx(t) = b(x(t))dt + a(x(t))d\zeta(t)$$

in the notation of Doob (loc. cit.), $\zeta(t)$ being the input Cauchy additive process ('non-Gaussian white noise'). Now

$$x(t + \Delta) - x(t) = b[x(t)]\Delta + a[x(t)][\zeta(t + \Delta) - \zeta(t)]$$

where the right-side, for given $x(t)$ is specified in terms of $\zeta(t)$ whose statistics are known. The limits required in L_2 are then established by direct calculation. In the case of (i), we may note that we need only prove it for small t , since the semigroup property will then imply it for all values of t . We omit the details of these calculations. The differential equation is:

$$\frac{\partial^2 u}{\partial t^2} + \frac{a(x)^2}{2} \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x} = 0 .$$

As an example we may consider the case where: $a(\cdot)$ and $b(\cdot)$ are constants:

$$\begin{aligned} a(x) &= \sqrt{2} \\ b(x) &= -2x . \end{aligned}$$

The differential equation then is:

$$\frac{\partial^2 u}{\partial t^2} = -2x \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} .$$

The (unique) solution of this is the prescribed type for each initial function $f(\cdot)$ can be expanded in Hermite polynomials [See [6] for a general proof]

$$(3.3) \quad u(t, x) = \sum_0^\infty a_n H_n(x) \exp - \sqrt{2n} t$$

where the $H_n(\cdot)$ are the Hermite polynomials orthogonal with respect to the Gaussian density:

$$(1/\sqrt{\pi}) \exp - x^2$$

and

$$a_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) H_n(x) \exp - x^2 dx .$$

The series in (3.3) converges to the solution function uniformly in com-

pact subsets of $(-\infty, +\infty)$. The transition kernel density $p(t, \xi | x)$ is given by

$$p(t, \xi | x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{t}{\sqrt{\sigma^2(1 - \exp - 2\sigma)}} \exp \left[\frac{-[\xi - x \exp - 2\sigma]^2}{(1 - \exp - 2\sigma)} - \frac{t^2}{4\sigma} \right] d\sigma$$

as follows again from the theory in [6]. Alternately, it has the expansion:

$$p(t, \xi | x) = \sum_0^{\infty} H_n(x)H_n(\xi) \exp - \xi^2 - \sqrt{2n} t .$$

As $t \rightarrow \infty$, the limiting density is: Gaussian:

$$p(\xi) = \frac{1}{\sqrt{\pi}} \exp - \xi^2$$

with respect to which (as first-order density) the process becomes strictly stationary. It may be shown that the limiting density is again always given by

$$p'(x)/(q(\beta) - q(\alpha))$$

and is thus completely determined by the system, that is by $a(\cdot)$ and $b(\cdot)$ only. The expansion (3.3) is, of course, in terms of functions orthogonal with respect to this density. Thus, taking the example treated in § 2, with

$$\begin{aligned} a(x) &= (x^2 + 1)\sqrt{2} \\ b(x) &= 0 \end{aligned}$$

yielding the differential equation:

$$\frac{\partial^2 u}{\partial t^2} + (x^2 + 1)^2 \frac{\partial^2 u}{2x^2} = 0$$

we have the expansion:

$$(3.4) \quad u(t, x) = \sum_0^{\infty} a_n W_n(x) \exp - \sqrt{n(n-1)} t$$

with $W_n(x)$ and a_n as in (2.3). As before, a sufficient condition for the existence of such expansion, is that (2.3) and (2.11) be satisfied. However, this is not necessary as the previous example (3.3) shows.

Extensions. A generalization of the type of process treated in § 3 is got by replacing the kernels in L_2 by

$$\frac{1}{t^n} \left[\sum_1^n \binom{n}{r} (-1)^{r-1} p(rt, \xi | x) \right]$$

$\binom{n}{r}$ being the Binomial coefficients, leading to the equations

$$\frac{\partial^n u}{\partial t^n} = (-1)^{n+1} \left[a(x) \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x} \right].$$

However, we have been unable as yet to establish the conditions for uniqueness of the solutions.

We have only so far considered first-order Markoff processes. The extension to higher order processes is similar in principle although it entails partial differential equations in several space variables [see [8] for example], and the results on the related Cauchy problems are still incomplete to a large degree.

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