ON THE PROJECTIVE COVER OF A MODULE AND RELATED RESULTS

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Introduction. The concept of the "injective envelope" of a module was first given by Eckmann and Schopf [2], although this terminology was first employed by Matlis [5]. The injective envelope of a module always exists and is unique in a certain sense. The dual concept of the "projective cover" of a module has been given by Bass [1], and the concept of "minimal epimorphism" in a "perfect category" as defined by Eilenberg [3] is a particular case of this concept. The projective cover of a module does not always exist but is unique whenever it exists. Eilenberg [3] has proved that every module in a perfect category possesses a projective cover. Bass [1] calls a ring "perfect" if every module over the ring possesses a projective cover, and he gives several characterizations of a perfect ring. We shall call a module "perfect" if it possesses a projective cover. It would be natural to try to characterize a perfect module, but it seems likely that such attempts may result in obtaining equivalent definitions of the projective cover of a module. One might instead consider specific types of modules and try to obtain necessary and sufficient conditions so that they may be perfect. In §1 we first define a category of perfect modules and then give a necessary and sufficient condition for a finitely generated module over a Noetherian ring to be perfect. In §2 we give some results on "essential monomorphism" and "minimal epimorphism" [1, 2]. In §3 we give some results on modules over perfect rings. In §4 we give new proofs of some known results to show how the concepts of the injective envelope and the projective cover of a module simplify the proofs considerably.

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1. A category of perfect modules. Let Λ be a ring with unit element $1 \neq 0$. Throughout this paper we shall be concerned with unitary left Λ-modules and so we shall call them simply modules. We recall some definitions. Let $f\colon L \to M$ be a homomorphism of modules. If $H \cap Im\ f = 0$ implies $H = 0$, where H is a submodule of M, f is called an *essential* homomorphism; moreover, if f is a monomorphism and M is an injective module, then M is called the *injective envelope* of L and is denoted by $E(L)$. If, however, $K + \operatorname{Ker} f = L$ implies $K = L$, where

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K is a submodule of L, f is called a *minimal* homomorphism; moreover, if f is an epimorphism and L is a projective module, then L is called a *projective cover* of M and is denoted by $P(M)$. The ring Λ is said to be *perfect* if every Λ-module possesses a projective cover. A module is said to be *perfect* if it possesses a projective cover.

Let $\mathscr{C}$ be a category of modules which satisfies the following axioms:

Axiom I. If L, $M \in \mathscr{C}$ and $f: L \to M$, then $f \in \mathscr{C}$.

Axiom II. If $M \in \mathscr{C}$ and $R \subset M$, then $R \in \mathscr{C}$.

Axiom III. If $M \in \mathscr{C}$, there exists a projective module $P \in \mathscr{C}$ and an epimorphism $h: P \to M$.

Axiom IV. If $g: M \to M$ is an epimorphism without being an automorphism, then there exists a proper submodule R of M such that $R + g^{-1}(R) = M$.

Axiom V. If f: $L \to M$ is an epimorphism in $\mathscr{C}$, then L possesses a submodule S which is minimal for the relation $f(S) = M$.

PROPOSITION 1. If $M \in \mathscr{C}$, then M is perfect.

Proof. Let $h: P \to M$ be an epimorphism where P is a projective module in $\mathscr{C}$. Then P possesses a submodule $\bar{P}$ which is minimal for the relation $h(\bar{P}) = M$. Let $\bar{h}: \bar{P} \to M$ be the restriction of h on $\bar{P}$. Then $\bar{h}$ is a minimal epimorphism. We shall swow that $\bar{P}$ is a direct summand of P and so projective. Let $i: \bar{P} \to P$ be the inclusion map, so that $\bar{h} = hi$. Since P is projective, there exists a homomorphism $j: P \to \bar{P}$ such that $\bar{h}j = h$, and so $\bar{h}ji = hi = \bar{h}$. We have the sequence of homomorphisms

$$\bar{P} \xrightarrow{i} P \xrightarrow{j} \bar{P} \xrightarrow{\bar{h}} M .$$

Since $\bar{h}ji$ is an epimorphism and $\bar{h}$ is minimal, ji is an epimorphism. We shall show that ji is an automorphism. For, if ji is not an automorphism, there exists a proper submodule Q of $\bar{P}$ such that $Q+(ji)^{-1}Q=\bar{P}$. Then $\bar{h}jiQ + \bar{h}ji(ji)^{-1}Q = \bar{h}ji\bar{P} = M$. Since $\bar{h}ji = \bar{h}$ and $ji(ji)^{-1}Q \subset Q$, we have $\bar{h}Q = M$, which is impossible as $\bar{P}$ is minimal for the relation $\bar{h}\bar{P} = M$. Hence $(ji)^{-1}j$ is an epimorphism such that $(ji)^{-1}ji$ is the identity map and so $\bar{P}$ is a direct summand of P.

PROPOSITION 2. Let Λ be a Noetherian ring and let M be a finitely generated module. Then M is perfect if and only if the following condition is satisfied:

(C) If $f: L \rightarrow M$ is an epimorphism, then, L possesses a submodule S which is minimal for the relation $f(S) = M$.

Proof. Suppose M is perfect and has P as projective cover together with a minimal epimorphism $j: P \rightarrow M$. Since f is an epimorphism, there exists a homomorphism $g: P \rightarrow L$ such that $fg = j$. Take $S = Im\ g$; then S is a submodule of L which is minimal for the relation $f(S) = M$, since the induced map $\bar{f}: S \rightarrow M$ is a minimal epimorphism. Note that this part of the proposition does not require Λ to be Noetherian. The other part of the proposition results from the proof of the Proposition 1 and the following.

LEMMA. *If M is a finitely generated module over a Noetherian ring, then every epimorphism $g: M \rightarrow M$ is an automorphism.*

Proof. Suppose Ker $g \neq 0$. If we write Ker $g = R$, then the complete inverse image of R in M under g properly contains R. Again, the complete inverse of $g^{-1}(R)$ by g properly contains $g^{-1}(R)$; for if $g^{-1}(g^{-1}(R)) = g^{-1}(R)$, then $g^{-1}(R) = R$. Proceeding in this way and writing $g^{-p-1}(R) = g^{-1}(g^{-p}(R))$, we get a strictly ascending sequence of submodules of M:

$$R \subsetneqq g^{-1}(R) \subsetneqq \cdots \subsetneqq g^{-p}(R) \subsetneqq g^{-p-1}(R) \subsetneqq \cdots ;$$

which is impossible since M is a Noetherian module.

Proposition 1 has been formulated after a theorem of Rainwater [7].

2. Essential monomorphism and minimal epimorphism.

The following proposition relates the essential monomorphism and the minimal epimorphism.

PROPOSITION 3. In the exact sequence of modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \qquad (L \neq 0,\ N \neq 0)$$

(i) if L is simple and f is essential, then g is minimal; and

(ii) if N is simple and g is minimal, then f is essential.

Proof. (i) Let us identify L with its image in M. Suppose K is a submodule of M such that $K + L = M$. Since $K \cap L \neq 0$ (otherwise $K = 0$) and since L is simple, we have $K \supset L$ and so $M = K + L = K$.

(ii) Let H be a submodule of M such that $H \cap L = 0$. Since N is simple, L is a maximal submodule of M. If $H \neq 0$, then we shall have

$H+L=M$, which would mean that $H=M$; but this is impossible as $H \cap L=0$.

PROPOSITION 4. In the exact sequence of modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \qquad (L \neq 0,\ N \neq 0)$$

let M be indecomposable. Then

(i) if L is simple, g is minimal; and

(ii) is N is simple, f is essential.

Proof. (i) Let K be a submodule of such that $K+L=M$. Since M is indecomposable, $K \cap L \neq 0$; and so as in Prop. 3, $K=M$.

(ii) Let H be a submodule of M such that $H \cap L=0$. If $H \neq 0$, we shall have $H+L=M$, as L is a maximal submodule of M; but this would mean that M is a direct sum of the submodules H and L, which is impossible.

3. Modules over perfect rings. The following proposition is dual to [5, Prop. 2.2].

PROPOSITION 5. Let Λ be a perfect ring and let P be a projective module. Then the following conditions are equivalent:

(a) the natural epimorphism $P \rightarrow P/M$ is minimal for every proper submodule M of P;

(b) P is indecomposable.

Proof. (a)$\Rightarrow$(b). If $P=Q \oplus R$, $Q \neq 0$, $R \neq 0$, then the natural epimorphism $P \rightarrow P/R$ cannot be minimal.

(b)$\Rightarrow$(a). Let M be a proper submodule of P. Let P' be the projective cover of P/M and let $f: P' \rightarrow P/M$ be the minimal epimorphism. Since P is projective the natural epimorphism $P \rightarrow P/M$ can be factorized into $P \xrightarrow{g} P' \xrightarrow{f} P/M$, where g is an epimorphism since f is minimal. Since P' is projective, there exists a module S such that $P=P' \oplus S$; but since P is indecomposable, $S=0$ and $P=P'$.

PROPOSITION 6. If Λ is a perfect ring, then a finitely generated projective module which is indecomposable is generated by one element.

Proof. Let $\{m_1, \cdots, m_n\}$, $(n>1)$ be a minimal system of generators of a projective module P which is indecomposable. Let Q be the submodule of P generated by the elements $\{m_2, \cdots, m_n\}$ and let R be the submodule generated by m_1. Then, by Prop. 5, the natural epimorphism $P \rightarrow P/R$ is minimal. Since $Q+R=P$, it follows that $Q=P$; but this is impossible. This proves the proposition.

COROLLARY. In an Artinian ring, every indecomposable projective ideal is principal.

Proof. For an Artinian ring is perfect [1] as well as Noetherian.

4. Applications. Let N be a two-sided ideal of the ring Λ and let $\Gamma = \Lambda/N$ be the factor ring. Let M be a module and let $S(M)$ and $T(M)$ be respectively a submodule and a quotient module of M given by

$$S(M) = \{m \mid Nm = 0,\ m \in M\} \approx \operatorname{Hom}_\Lambda(\Gamma, M)$$

and

$$T(M) = M/NM \approx \Gamma \oplus_\Lambda M.$$

PROPOSITION 7. The following conditions are equivalent:

(i) $S(M) = 0$ implies $M = 0$;

(ii) the inclusion map $S(M) \to M$ is an essential monomorphism for every module M.

Proof. (i) $\Rightarrow$ (ii). Let H be a submodule of M such that $H \cap S(M) = 0$. Then $S(H) \cap S(M) = 0$; also $S(H) \subset S(M)$. This means $S(H) = 0$, which implies $H = 0$.

(ii) $\Rightarrow$ (i). Evident.

PROPOSITION 8. The following conditions are equivalent:

(i′) $T(M) = 0$ implies $M = 0$;

(ii′) the natural epimorphism $M \to T(M)$ is minimal for every module M.

Proof. (i′) $\Rightarrow$ (ii′). Let K be a submodule of M such that $K + NM = M$. Then $N(M/K) = M/K$, which implies $M/K = 0$, that is, $K = M$.

(ii′) $\Rightarrow$ (i′). Evident.

The Prop. 8 remains true if we confine ourselves to finitely generated modules only, since the quotient module of a finitely generated module is finitely generated.

PROPOSITION 9. If N is nilpotent, then the conditions (i) and (i′) of the Prop. 7 and 8 respectively are satisfied. Moreover, if $S(M)$ is simple, either M is simple or $S(M) \subset NM$.

Proof. $S(M) = 0$ means that if $Nm = 0$, $m \in M$, then $m = 0$. Let

m be any element of M. Since N is nilpotent, there exists an integer $P(\geqq 0)$ which is minimal for the relation $N^p m = 0$, it being understood that $N^0 = \Lambda$. If $p > 0$, then the relation $N^p m = N.N^{p-1}m = 0$ implies that $N^{p-1}m = 0$, which contradicts the minimality of p for the relation $N^p m = 0$. Therefore $p = 0$ and so $m = 0$.

Again $T(M) = 0$ means $M = NM$. Since N is nilpotent, $N^p = 0$ for some positive integer p, and so $M = NM = N^2M = \cdots = N^pM = 0$.

If $S(M)$ is simple, consider the submodule $S(M) \cap NM$. There are then two possibilities:

(a) $S(M) \cap NM = 0$ which implies $NM = 0$, since the inclusion map $S(M) \to M$ is essential. The relation $NM = 0$ means $S(M) = M$ and so M is simple.

(b) $S(M) \cap NM \neq 0$ which means$S(M) \subset NM$, since $S(M)$ is simple.

From now onwards we suppose that Λ is a semi-primary ring and that N is the radical of Λ. This means that N is nilpotent and that the factor ring $\Gamma = \Lambda/N$ is semi-simple. Then Λ is perfect [1]. The submodule $S(M)$ and the quotient module $T(M)$ are semi-simple modules. We shall call them the "semi-simple part" and the "semi-simple quotient" of M respectively. $S(M)$ is also called the "socle" of M.

PROPOSITION 10. A monomorphism $f: L \to M$ is essential if and only if the restriction of f over $S(L)$ is an isomorphism of $S(L)$ onto $S(M)$.

Proof. We identify L with its image in M. Suppose $S(L) = S(M) = R$. Then the natural monomorphism $R \to M$ is essential; but this map can be factorized into $R \xrightarrow{i} L \xrightarrow{f} M$ where i is the inclusion map, and so f is essential.

Conversely, suppose f is essential. Then the following diagram is commutative:

$$\begin{array}{ccc} S(L) & \xrightarrow{\bar{f}} & S(M) \\ \downarrow & & \downarrow \\ L & \xrightarrow[f]{} & M \end{array}$$

where the vertical maps are essential monomorphisms. Therefore $\bar{f}: S(L) \to S(M)$ is an essential monomorphism. Since $S(M)$ is semi-simple, $S(M) = S(L) \oplus G$, where G is a submodule of $S(M)$. Since $G \cap S(L) = 0$, $G = 0$.

COROLLARY. *An injective module I is the injective envelope of all its submodules which have $S(I)$ as their semi-simple part.*

The following results due to Morita, Kawada and Tachikawa [6] can now be very easily prove:

(1) Let M be an injective module and let R be a semi-simple

submodule of M. Then a submodule L of M which is maximal for the relation $S(L) = R$ is injective.

For the inclusion map $R \to L$ is essential and since L is maximal for this property, L is the injective envelope of R.

(2) If R is a semi-simple module, there exists an injective module I such that $S(I) = R$.

Take I to be the injective envelope of R.

(3) Let M be a module and let I be an injective module. If $S(M) \approx S(I)$, then there exists a monomorphism $f: M \to I$; if M is also injective, f is an isomorphism. Thus an injective module is uniquely determined upto an isomorphism by its semi-simple part.

The first part follows at once from the properties an essential monomorphism [2]. Moreover, by Prop. 10, f is essential. If M is also injective, then M is a direct summand of I and so $M \approx I$.

(4) An injective module is indecomposable if and only if its semi-simple part is simple.

Let I be an indecomposable injective module. Then I is the injective envelope of everyone of its sumbodules [5, Prop. 2.2]. Hence $S(I)$ is simple. For, if J is a nonzero proper submodule of $S(I)$, then the inclusion map $J \to S(I)$ is essential, which is impossible since $S(I)$ is semi-simple.

Conversely, suppose $S(I)$ is simple. Let $I = L \oplus M$, $L \neq 0$, $M \neq 0$. Then $S(I) = S(L) \oplus S(M)$, $S(L) \neq 0$, $S(M) \neq 0$; but this contradicts the simplicity of $S(I)$.

We state without proofs the duals of the Prop. 10 and its corollary.

PROPOSITION 11. An epimorphism $f: L \to M$ is minimal if and only if the induced epimorphism $\bar{f}: T(L) \to T(M)$ is an isomorphism.

COROLLARY. *A projective module P is the projective cover of all its quotients which have $T(P)$ as their semi-simple quotient.*

The following results are the duals of the results (2), (3) and (4) given above and can be proved easily:

(2′) If R is a semi-simple module, there exists a projective module P such that $T(P) = R$.

(3′) Let M be a module and let P be a projective module. If $T(M) \approx T(P)$, then there exists an epimorphism $f: P \to M$; if M is also projective, f is an isomorphism. Thus a projective module is determined uniquely up to an isomorphism by its semi-simple quotient.

(4′) A projective module is indecomposable if and only if its semi-simple quotient is simple.

Morita, Kawada and Tachikawa [6] have also proved:

(5) If Λ is an *Artinian* ring, then every injective module can be

expressed as the direct sum of indecomposable injective modules.

Matlis [5, Prop. 2.5] has generalized this result to the case when Λ is a *Noetherian* ring. The following result which is dual to (5) is true when Λ is only a *semi-primary* ring:

(5′) Every projective module can be expressed as the direct sum of indecomposable projective modules.

Let P be a projective module. Let $A, B, C, \cdots$ be the projective covers of the simple factors of $T(P)$. Then $T(P) = T(A) \oplus T(B) \oplus T(C) \oplus \cdots$. Also

$$\begin{aligned} T(A \oplus B \oplus C \oplus \cdots) &= \Gamma \otimes_\Lambda (A \oplus B \oplus C \oplus \cdots) \\ &= (\Gamma \otimes_\Lambda A) \oplus (\Gamma \otimes_\Lambda B) \oplus (\Gamma \otimes_\Lambda C) \oplus \cdots \\ &= T(A) \oplus T(B) \oplus T(C) \oplus \cdots . \end{aligned}$$

Thus $T(P) = T(A \oplus B \oplus C \oplus \cdots)$. Then (3′) shows that $P \approx A \oplus B \oplus C \oplus \cdots$, since $A \oplus B \oplus C \oplus \cdots$ is projective. Moreover, since $T(A)$, $T(B)$, $T(C), \cdots$ are simple, $A, B, C, \cdots$ are indecomposable.

Eilenberg [3] has shown that every projective module in a perfect category is the direct sum of singly generated projective modules.

We now give simple proofs of three lemmas proved by Eilenberg and Nakayama [4].

Let a be a subset of Λ. The orthogonality relation $a \perp A$ is defined by the condition $aP = 0$, where P is the projective cover of the module A.

LEMMA 1. *If $B \subset NA$, then the relations $a \perp A$ and $a \perp A/B$ are equivalent.*

Since the natural epimorphism $A \to A/NA$ is minimal and $B \subset NA$, the two maps in $A \to A/B \to A/NA$ are minimal. Hence A, A/B and A/NA have the same projective cover and so the relations $a \perp A$, $a \perp A/B$ and $a \perp A/NA$ are equivalent.

LEMMA 2. *If $B \subset NA$ and A/B is projective, then $B = 0$.*

For, then the map $A \to A/B$ is an isomorphism and so $B = 0$.

LEMMA 3. *Let a be a two-sided ideal of Λ, A a Λ-module and B a submodule of A such that* (i) $aA \subset B \subset NA$, (ii) *A/B is Λ/a-projective. Then $aA = B$.*

The minimal epimorphism $A \to A/NA$ can be factorized into

$$A \to A/aA \to A/B \to A/NA$$

in which each map is a minimal epimorphism. Since the map $A/aA \to A/B$

is minimal as a Λ-epimorphism, it is also minimal as a $\Lambda/\mathfrak{a}$-epimorphism. Since A/B is $\Lambda/\mathfrak{a}$-projective, the map $A/\mathfrak{a}A \to A/B$ is a $\Lambda/\mathfrak{a}$-isomorphism and so the kernel $B/\mathfrak{a}A = 0$, that is, $\mathfrak{a}A = B$.

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