ANNIHILATORS IN THE SECOND CONJUGATE ALGEBRA OF A GROUP ALGEBRA

PAUL CIVIN

1. Introduction. Let $\mathfrak{G}$ denote an infinite locally compact abelian group, and let $L(\mathfrak{G})$ be its group algebra. The second conjugate space $L^{**}(\mathfrak{G})$ of the group algebra can also be considered as an algebra by the use of Arens multiplication [1] [2]. Civin and Yood [3, p. 857] have shown that $L^{**}(\mathfrak{G})$ is an algebra which is not commutative and has a nonzero radical $\mathfrak{R}^{**}$. They have also shown [3, p. 856] that if $\mathfrak{G}$ is not discrete, then the algebra $L^{**}(\mathfrak{G})$ has a nonzero right annihilator.

The object of the present note is the study of the nature of the left and right annihilators of the maximal modular left ideals in $L^{**}(\mathfrak{G})$. It is shown that such annihilators are either nilpotent two-sided or right ideals, respectively, or else the maximal modular left ideal in question must have the form $\{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$ where μ is some multiplicative linear functional on $L(\mathfrak{G})$. If $\mathfrak{G}$ is compact it is seen that all maximal modular left ideals of the latter form have a nonzero left annihilator and a right annihilator which properly contains the right annihilator of $L^{**}(\mathfrak{G})$.

It should be noted that the choice of the maximal modular left ideals as the subject of investigation is not simply for definiteness. At the present stage of available information concerning $L^{**}(\mathfrak{G})$, the maximal modular left ideals are more tractable than the corresponding right ideals.

2. Notation. Throughout the note we shall use the notation introduced above as well as other notation introduced by Civin and Yood [3]. In particular $\mathfrak{R}^{**}$ will denote the radical of $L^{**}(\mathfrak{G})$ and $\mathfrak{Y}$ will denote the closed subspace of $L^{*}(\mathfrak{G})$ generated by the multiplicative linear functionals on $L(\mathfrak{G})$. We shall write $\mathfrak{L}(I)(\mathfrak{R}(I))$ for the left (right) annihilators in the algebra $L^{**}(\mathfrak{G})$ of the subset I of $L^{**}(\mathfrak{G})$. We also use the notation $I^{\perp}(I^{\top})$ for the linear space annihilator in $B^{*}(B)$ of the linear manifold I in the Banach space B (the conjugate space B^{*}). Throughout π will be used for the natural embedding of a Banach space B into its second conjugate space B^{**}. It should be recalled [1] that when B is a Banach algebra, π is an algebra homomorphism, and if B is commutative then [3, p. 855] πB is in the center of B^{**}.

3. Left annihilators. Throughout this section we let $\mathfrak{M}$ denote a maximal modular left ideal in $L^{**}(\mathfrak{G})$ for which $\mathfrak{L}(\mathfrak{M}) \neq (0)$.

LEMMA 3.1. *$\mathfrak{M}$ and $\mathfrak{L}(\mathfrak{M})$ are 2-sided ideals in $L^{**}(\mathfrak{G})$ and*

Received November 22, 1961. This research was supported by the National Science Foundation grant NSF-G-14111.

$\mathfrak{R}\mathfrak{L}(\mathfrak{M}) = \mathfrak{M}$.

Proof. Since $\mathfrak{M}$ is a left ideal, $\mathfrak{L}(\mathfrak{M})$ is a 2-sided ideal. Thus $\mathfrak{R}\mathfrak{L}(\mathfrak{M})$ is a 2-sided ideal containing $\mathfrak{M}$. However, the algebra $L^{**}(\mathfrak{G})$ contains [3, p. 855] a right identity E, so $\mathfrak{L}(\mathfrak{M}) \neq (0)$ implies $\mathfrak{R}\mathfrak{L}(\mathfrak{M})$ is proper, hence $\mathfrak{R}\mathfrak{L}(\mathfrak{M}) = \mathfrak{M}$, and $\mathfrak{M}$ is a 2-sided ideal.

In the next several lemmas we consider the consequences of the assumption $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$.

LEMMA 3.2. *If* $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$, *then* $\mathfrak{L}(\mathfrak{M}) = (L^{**}(\mathfrak{G}))A$, *with* $A = A^2$.

Proof. It follows from $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$ that $\mathfrak{L}^{**}(\mathfrak{G}) = \mathfrak{L}(\mathfrak{M}) + \mathfrak{M}$. Thus the right identity E satisfies $E = A + M$ with $A \in \mathfrak{L}(\mathfrak{M})$ and $M \in \mathfrak{M}$. Left multiplication by $F \in \mathfrak{L}(\mathfrak{M})$ yields $F = FE = FA$, so in particular $A = A^2$ and $\mathfrak{L}(\mathfrak{M}) \subset (L^{**}(\mathfrak{G}))A$. The reverse set inequality is immediate since $\mathfrak{L}(\mathfrak{M})$ is a left ideal.

We adopt as fixed notation $E = A + M$, with $A \in \mathfrak{L}(\mathfrak{M})$ and $M \in \mathfrak{M}$, throughout the section in which we are discussing $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$.

LEMMA 3.3. *For all* $F \in L^{**}(\mathfrak{G})$, $AF = AFA$.

Proof. As above $E = A + M$. Left multiplication by AF gives $AF = AFA$ since $A \in \mathfrak{L}(\mathfrak{M})$ and $FM \in \mathfrak{M}$.

LEMMA 3.4. *If* $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$, *then* $A\mathfrak{L}(\mathfrak{M})$ *is the set of complex multiples of* A.

Proof. Let $L \neq 0$ be an element of $A\mathfrak{L}(\mathfrak{M})$. Then by Lemma 3.3, $L = AL = ALA$. Since $L \neq 0$ and $A \in \mathfrak{L}(\mathfrak{M})$, it follows that $LA \notin \mathfrak{M}$ and $L \notin \mathfrak{M}$ Consequently $(L^{**}(\mathfrak{G}))LA$ is a left ideal not contained in $\mathfrak{M}$. Hence $L^{**}(\mathfrak{G}) = \mathfrak{M} + (L^{**}(\mathfrak{G}))LA$, and $E = N + CLA$, with $N \in \mathfrak{M}$. Left multiplication by A, and appropriate use of the right identity yields $A = ACLA = ACELA = AC(A + M)LA = ACALA = (ACA)(ALA)$. Thus the normed algebra $A\mathfrak{L}(\mathfrak{M})$ has A as an identity and each nonzero element has a left inverse. This implies that $A\mathfrak{L}(\mathfrak{M})$ is a complex normed division algebra and the lemma then follows from the Gelfand-Mazur theorem.

LEMMA 3.5. *If* $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$, *then there exist a multiplicative linear functional* φ *on* $L^{**}(\mathfrak{G})$ *such that* $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid \varphi(F) = 0\}$.

Proof. In view of Lemma 3.4 and the fact that $A\mathfrak{L}(\mathfrak{M})$ is a right ideal, we may define the complex number $\varphi(F)$ for $F \in L^{**}(\mathfrak{G})$ by $AF = \varphi(F)A$. Clearly φ is additive and by the use of Lemma 3.3 we see that

$\varphi(FG)A = AFG = AFAG = \varphi(F)\varphi(G)A$, so φ is multiplicative. It follows from Lemmas 3.1 and 3.2 that $\varphi(F) = 0$ if and only if $F \in \mathfrak{M}$.

THEOREM 3.6. *Let $\mathfrak{M}$ be a maximal modular left ideal in $L^{**}(\mathfrak{G})$ with $\mathfrak{L}(\mathfrak{M}) \neq (0)$. Then $\mathfrak{M}$ is a 2-sided ideal and either $(\mathfrak{L}(\mathfrak{M}))^2 = (\mathfrak{R}(\mathfrak{M}))^2 = (0)$ or there exist a multiplicative linear functional μ on $L(\mathfrak{G})$ such that $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$. In the latter case $\mathfrak{L}(\mathfrak{M})$ is a one-dimensional 2-sided ideal in $L^{**}(\mathfrak{G})$ and $\mathfrak{R}(\mathfrak{M}) = \mathfrak{R}(L^{**}(\mathfrak{G})) \oplus \mathfrak{L}(\mathfrak{M})$.*

Proof. If $(0) \neq \mathfrak{L}(\mathfrak{M}) \subset \mathfrak{M}$, then $\mathfrak{M} = \mathfrak{R}\mathfrak{L}(\mathfrak{M}) \supset \mathfrak{R}(\mathfrak{M})$, so $(\mathfrak{L}(\mathfrak{M}))^2 = (\mathfrak{R}(\mathfrak{M}))^2 = (0)$. If $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{M}$, let $\varphi \in L^{***}(\mathfrak{G})$ be the multiplicative linear function on $L^{**}(\mathfrak{G})$ whose existence is guaranteed by Lemma 3.5. Since π is a homomorphism, the functional $\mu = \varphi \circ \pi$ is a multiplicative linear functional on $L(\mathfrak{G})$. The null space of μ is then either $L(\mathfrak{G})$ or a modular ideal $\mathfrak{M}_*$ in $L(\mathfrak{G})$. If the first possibility prevails, $\pi L(\mathfrak{G}) \subset \mathfrak{M}$, and thus $0 = A(\pi x) = (\pi x)A$ for all $x \in L(\mathfrak{G})$. The w^*-density of $\pi L(\mathfrak{G})$ together with the w^*-continuity of left multiplication [2] in $L^{**}(\mathfrak{G})$ implies that $FA = 0$ for all $F \in L^{**}(\mathfrak{G})$. This contradicts $A = A^2 \neq 0$. We thus conclude that there is a maximal modular ideal $\mathfrak{M}_*$ in $L(\mathfrak{G})$ such that $\pi\mathfrak{M}_* \subset \mathfrak{M}$. Now [3, p. 865] the w^*-closure of $\pi\mathfrak{M}_*$ is a maximal modular left ideal $\mathfrak{M}_0$ in $L^{**}(\mathfrak{G})$. Let $F \in \mathfrak{M}_0$, then $F = w^* - \lim \pi x_\alpha$, $x_\alpha \in \mathfrak{M}_*$. Thus $0 = A(\pi x_\alpha) = (\pi x_\alpha)A$ for all α, so $FA = 0$, i.e. $A \in R(\mathfrak{M}_0)$. Therefore by Lemma 3.5, $\varphi(F)\varphi(A) = \varphi(FA) = 0$. However, since $A \notin \mathfrak{M}$, $\varphi(A) \neq 0$ and consequently $\varphi(F) = 0$, so $F \in \mathfrak{M}$. Therefore $\mathfrak{M}_0 \subset \mathfrak{M}$ and $\mathfrak{M} = \mathfrak{M}_0$. In particular $A \in \mathfrak{R}(\mathfrak{M})$. Also if $F \in \mathfrak{M}$, $F = w^* - \lim \pi x_\alpha$, $x_\alpha \in \mathfrak{M}_*$ and thus $F(\mu) = \lim \pi x_\alpha(\mu) = \lim \mu(x_\alpha) = \lim \varphi(\pi x_\alpha) = 0$. Since the set of $F \in L^{**}(\mathfrak{G})$ such that $F(\mu) = 0$ is a maximal modular ideal containing $\mathfrak{M}$, we see that $\mathfrak{M}$ has the appropriate form.

It now follows from Lemma 3.4 that $L^{**}(\mathfrak{G}) = \mathfrak{M} \oplus A\mathfrak{L}(\mathfrak{M})$ with the second summand one-dimensional. Since $A \in \mathfrak{R}(\mathfrak{M})$, it then follows from Lemmas 3.2 and 3.3 that $\mathfrak{L}(\mathfrak{M}) = A\mathfrak{L}(\mathfrak{M})$ and so is a one-dimensional 2-sided ideal. Since $A \in \mathfrak{R}(\mathfrak{M})$ we have $\mathfrak{R}(L^{**}(\mathfrak{G})) \oplus \mathfrak{L}(\mathfrak{M}) \subset \mathfrak{R}(\mathfrak{M})$. Also if $F \in \mathfrak{R}(\mathfrak{M})$, then $F = M_1 + \alpha A$, with $M_1 \in \mathfrak{M}$ and α complex. Since $A \in \mathfrak{R}(\mathfrak{M}) \cap \mathfrak{L}(\mathfrak{M})$, it is immediate that $M_1 \in \mathfrak{R}(L^{**}(\mathfrak{G}))$ which completes the proof.

4. Right annihilators. Again we let $\mathfrak{M}$ denote a maximal modular left ideal. If $\mathfrak{G}$ is not discrete [3, p. 856] then $(0) \neq \mathfrak{R}(L^{**}(\mathfrak{G})) \subset \mathfrak{R}(\mathfrak{M})$. On the other hand we saw in Theorem 3.6 that if $(0) \neq (\mathfrak{L}(\mathfrak{M}))^2$ then $\mathfrak{R}(\mathfrak{M}) = \mathfrak{R}(L^{**}(\mathfrak{G})) \oplus \mathfrak{L}(\mathfrak{M})$. Our object in this section is to investigate relationships between $\mathfrak{M}$ and $\mathfrak{R}(\mathfrak{M})$ with no hypothesis on $\mathfrak{L}(\mathfrak{M})$. As indicated in the introduction, we use $\mathfrak{R}^{**}$ for the radical of $L^{**}(\mathfrak{G})$.

4.1 LEMMA. *Either $\mathfrak{R}(\mathfrak{M}) \subset \mathfrak{R}^{**}$ or there exists an $F \in \mathfrak{R}(\mathfrak{M})$ which* *is not left quasi-regular.*

Proof. Suppose that the right ideal $\mathfrak{R}(\mathfrak{M})$ is left quasi-regular. Let $F \in \mathfrak{R}(\mathfrak{M})$. Since FD is left quasi-regular for all $D \in L^{**}(\mathfrak{S})$, we see [5, p. 17] that $(L^{**}(\mathfrak{S}))F$ is a left quasi-regular left ideal, and so is a quasi-regular left ideal and is included in $\mathfrak{R}^{**}$. Thus $EF \in \mathfrak{R}^{**}$. However, $EF - F \in \mathfrak{R}(L^{**}(\mathfrak{S})) \subset \mathfrak{R}^{**}$, so $F \in \mathfrak{R}^{**}$.

4.2 Lemma. *If $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$, there exists an $A \in \mathfrak{R}(\mathfrak{M})$ such that $0 \neq A = A^2$.*

Proof. By Lemma 4.1 there is an $F \in \mathfrak{R}(\mathfrak{M})$ which is not left quasi-regular. The left ideal $\{BF - B \mid B \in L^{**}(\mathfrak{S})\}$ is then a proper modular left ideal, so is contained in a maximal modular left ideal $\mathfrak{N}$. It follows from $BF = 0$ for $B \in \mathfrak{M}$ that $\mathfrak{N} = \mathfrak{M}$. Consequently $F^2 - F \in \mathfrak{M}$ and therefore $F^2 = F^3 = F^4$. Thus $A = F^2 \in \mathfrak{R}(\mathfrak{M})$, and $A \neq 0$ since otherwise F would be left quasi-regular.

We fix the notation in the remainder of this section so that A has the properties asserted in the lemma.

4.3 Lemma. *If $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$, then*
(i) $E = N + A$, $N \in \mathfrak{M}$,
(ii) $L^{**}(\mathfrak{S}) = \mathfrak{M} \oplus (L^{**}(\mathfrak{S}))A$, *and*
(iii) $(L^{**}(\mathfrak{S}))A$ *is a minimal left ideal of* $L^{**}(\mathfrak{S})$.

Proof. Since A has the properties asserted in Lemma 4.2, $A \notin M$ and therefore $L^{**}(\mathfrak{S}) = \mathfrak{M} \oplus (L^{**}(\mathfrak{S}))A$ with the sum clearly a direct sum. Let $E = N + BA$ with $N \in \mathfrak{M}$. Right multiplication by A yields $EA = BA$. Thus $BA - A = EA - A \in \mathfrak{R}(L^{**}(\mathfrak{S})) \subset \mathfrak{M}$. Another right multiplication by A yields $BA = A$, so $E = N + A$.

Suppose that $(0) \neq \mathfrak{J}$ is a left ideal in $(L^{**}(\mathfrak{S}))A$. Then $L^{**}(\mathfrak{S}) = M \oplus \mathfrak{J}$. Let $B \in L^{**}(\mathfrak{S})$. Then $BA = M_1 + I_1$ with $M_1 \in \mathfrak{M}$ and $I_1 \in \mathfrak{J}$. Right multiplication by A shows that $BA = I_1$ so $(L^{**}(\mathfrak{S}))A$ is a minimal left ideal.

Lemma 4.4. *If $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$, then there exists a $\varphi \in L^{***}(\mathfrak{S})$ such that for each $X \in L^{**}(\mathfrak{S})$, $(AX)^2 = \varphi(X)(AX)$.*

Proof. Since $(L^{**}(\mathfrak{S}))A$ is a minimal left ideal, $A(L^{**}(\mathfrak{S}))A$ is a division algebra and so by the Gelfand-Mazur theorem consists of the scalar multiples of A. For $X \in L^{**}(\mathfrak{S})$, define $\varphi(X)$ by $AXA = \varphi(X)A$. As defined φ is clearly linear. Moreover, $|\varphi(X)| \, ||A|| = ||\varphi(X)A|| = ||AXA|| \leqq ||A||^2||X||$, so $|\varphi(X)| \leqq ||A|| \, ||X||$ and $\varphi \in L^{***}(\mathfrak{S})$. The remaining assertion is now immediate.

4.5 LEMMA. *Let* $\mathfrak{M}_* = \{x \in L(\mathfrak{G}) \mid \pi x \in \mathfrak{M}\}$. *If* $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$, *then* $\mathfrak{M}_*$ *is a maximal modular ideal of* $L(\mathfrak{G})$.

Proof. Note first that $\mathfrak{L}(A)$ is a left ideal containing $\mathfrak{M}$, and $A \notin \mathfrak{L}(A)$. Therefore $\mathfrak{L}(A) = \mathfrak{M}$. It is an immediate consequence of the definition of φ given in the proof of Lemma 4.4 that φ is multiplicative on the center of $L^{**}(\mathfrak{G})$. Since $\pi L(\mathfrak{G})$ is central it follows that $\varphi\circ\pi$ is a multiplicative linear functional on $L(\mathfrak{G})$, and so has a null space which is either all of $L(\mathfrak{G})$ or is a maximal modular ideal of $L(\mathfrak{G})$. Now $(\pi x)A = A(\pi x)A = \varphi(\pi x)A$, so $\varphi(\pi x) = 0$ if and only if $\pi x \in \mathfrak{L}(A) = \mathfrak{M}$ or if and only if $x \in \mathfrak{M}_*$. If $\mathfrak{M}_*$ were all of $L(\mathfrak{G})$, then the w^*-continuity of left multiplication in $L^{**}(\mathfrak{G})$ together with the w^*-density of $\pi L(\mathfrak{G})$ would imply that $A^2 = 0$ which is not the case. Thus $\mathfrak{M}_*$ is a maximal modular ideal of $L(\mathfrak{G})$ as asserted.

We will use in the sequel two lemmas which are valid in the algebra B^{**} of the second conjugate space of a Banach algebra B. The notation is that of [3].

4.6 LEMMA. *A w^*-closed subspace $\mathfrak{J}$ of B^{**} is a left (right) ideal of B^{**} if and only if $\langle f, x\rangle \in \mathfrak{J}^\top$ for all $f \in \mathfrak{J}^\top$ and $x \in B$ ($[F, f] \in \mathfrak{J}^\top$ for all $f \in \mathfrak{J}^\top$ and $F \in B^{**}$).*

Proof. The argument will be given only for left ideals. Suppose $\mathfrak{J}$ is a left ideal and let $f \in \mathfrak{J}^\top$ and $x \in B$. Then for any $F \in \mathfrak{J}$, $(\pi x)F \in \mathfrak{J}$, so $0 = (\pi x)F(f) = F(\langle f, x\rangle)$. Consequently $\langle f, x\rangle \in \mathfrak{J}^\top$. Suppose next that $f \in \mathfrak{J}^\top$ and $x \in B$ implies $\langle f, x\rangle \in \mathfrak{J}^\top$. Then for $F \in \mathfrak{J}$ and $x \in B$, $0 = F(\langle f, x\rangle) = (\pi x)F(f)$, so $(\pi x)F \in \mathfrak{J}^{\top\perp} = \mathfrak{J}$. The w^*-density of πB in B^{**} together with the w^*-continuity of left multiplication and the w^*-closure of $\mathfrak{J}$ give $HF \in \mathfrak{J}$ for all $H \in \mathfrak{J}$ for all $H \in B^{**}$, so $\mathfrak{J}$ is a left ideal in B^{**}.

4.7 LEMMA. *If $\mathfrak{J}$ is a left ideal in B^{**}, then so is $\mathfrak{J}^{\top\perp}$.*

Proof. The subspace $\mathfrak{J}^{\perp\top}$ is w^*-closed. If $f \in \mathfrak{J}^\top$ and $x \in B$, then for any $F \in \mathfrak{J}$, $(\pi x)F \in \mathfrak{J}$ so $0 = (\pi x)F(f) = F(\langle f, x\rangle)$ and $\langle f, x\rangle \in \mathfrak{J}^\top$. Since $\mathfrak{J}^\top = \mathfrak{J}^{\top\perp\top}$, Lemma 4.6 yields the desired conclusion.

4.8 LEMMA. *If $\mathfrak{M}$ is a maximal modular left ideal of $L^{**}(\mathfrak{G})$ with $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$, then $\mathfrak{M}$ is w^*-closed.*

Proof. In view of Lemma 4.7, if $\mathfrak{M}$ were not w^*-closed, $L^{**}(\mathfrak{G}) = \mathfrak{M}^{\top\perp}$ and then $(0) = \mathfrak{M}^{\top\perp\top} = \mathfrak{M}^\top$. If A has the same meaning as in the earlier lemmas, $A^2 \neq 0$, so there is an $f_0 \in L^*(\mathfrak{G})$ such that $[A, f_0] \neq 0$. However, since $A \in \mathfrak{R}(\mathfrak{M})$, $[A, f_0] \in \mathfrak{M}^\top$. Thus $\mathfrak{M}$ is w^*-closed.

4.9 THEOREM. *Let $\mathfrak{M}$ be a maximal modular left ideal in $L^{**}(\mathfrak{G})$. Then either $(\mathfrak{R}(\mathfrak{M}))^2 = 0$ or there exists a multiplicative linear functional μ on $L(\mathfrak{G})$ such that $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$.*

Proof. If $\mathfrak{R}(\mathfrak{M}) \subset \mathfrak{R}^{**}$, then $\mathfrak{R}(\mathfrak{M}) \subset \mathfrak{M}$ and $(\mathfrak{R}(\mathfrak{M}))^2 = 0$. Suppose that $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{M}$. Let μ be the multiplicative linear functional on $L(\mathfrak{G})$ corresponding to the maximal modular ideal $\mathfrak{M}_*$ of Lemma 4.5. By Lemma 4.2 there is an $A \in \mathfrak{R}(\mathfrak{M})$, $0 \neq A = A^2$. Let $\mathfrak{K}$ be the closed span of $\{[A, f] \mid f \in L^*(\mathfrak{G})\}$. Since $A \in \mathfrak{R}(\mathfrak{M})$, $\mathfrak{K} \subset \mathfrak{M}^{\top}$, so $\mathfrak{K}^{\perp} \supset \mathfrak{M}$. Also if $g \in \mathfrak{K}$ and $x \in L(\mathfrak{G})$, then $g = \lim [A, g_n]$ and $\langle g, x\rangle = \lim \langle [A, g_n], x\rangle = \lim [A, \langle g_n, x\rangle]$, so $\langle g, x\rangle \in \mathfrak{K}$. Thus by Lemma 4.6, $\mathfrak{K}^{\perp}$ is a left ideal in $L^{**}(\mathfrak{G})$. Since $\mathfrak{K}^{\perp} \supset \mathfrak{M}$, either $\mathfrak{K}^{\perp} = \mathfrak{M}$ or $\mathfrak{K}^{\perp} = L^{**}(\mathfrak{G})$. The latter is impossible since $A^2 \neq 0$, and thus $\mathfrak{K}^{\perp} = \mathfrak{M}$. Now if $x \in \mathfrak{M}_*$ then $\pi x \in \mathfrak{M}$, so $x \in \mathfrak{K}^{\top}$. Thus $\mathfrak{K}^{\top} \subset \mathfrak{M}_*$ and $\mathfrak{K} \subset \mathfrak{K}^{\top\perp} \subset \mathfrak{M}_*^{\perp}$. However since the latter set consists of the scalar multiples of μ, so also must $\mathfrak{K}$. Thus $\mathfrak{M}$ has the indicated form.

5. **Existence.** The question of the existence of maximal modular left ideals in $L^{**}(\mathfrak{G})$ with $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$ or with $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$ is easily resolved if $\mathfrak{G}$ is compact. For $\mathfrak{G}$ not compact, necessary and sufficient conditions are given for the existence of ideals with the indicated properties, but no conclusion is reached as to whether or not the given condition is automatically satisfied.

5.1 THEOREM. *Let $\mathfrak{G}$ be an infinite compact abelian group, and let μ be a multiplicative linear functional on $L(\mathfrak{G})$. Let $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$. Then $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$ and $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$.*

Proof. Since $\mathfrak{G}$ is compact, its character group is discrete. The regularity of the Banach algebra $L(\mathfrak{G})$ then implies that there is an $e \in L(\mathfrak{G})$ such that $\mu(e) = 1$ and $\nu(e) = 0$ for every multiplicative linear functional ν on $L(\mathfrak{G})$ with $\nu \neq \mu$. The semi-simplicity of $L(\mathfrak{G})$ then implies $e = e^2 \neq 0$. Since πe is an idempotent in $L^{**}(\mathfrak{G})$ and thus $\pi e \notin \mathfrak{R}^{**}$, it suffices to show that $\pi e \in \mathfrak{L}(\mathfrak{M}) \cap \mathfrak{R}(\mathfrak{M})$. Also since πe is central it sufficies to show $\pi e \in \mathfrak{R}(\mathfrak{M})$. Now for $\pi x \in \mathfrak{M}$, $\nu(xe) = 0$ for all multiplicative linear functional ν on $L(\mathfrak{G})$ so $xe = 0$ and $(\pi x)(\pi e) = 0$. However, $\mathfrak{M}$ is [3, p. 865] the w^*-closure of $\{\pi x \mid \pi x \in \mathfrak{M}\}$, so the w^*-continuity of left multiplication shows that $\pi e \in \mathfrak{R}(\mathfrak{M})$ as desired.

5.2 LEMMA. *Let μ be a nonzero multiplicative linear functional on $L(\mathfrak{G})$. Then there exists $D \in L^{**}(\mathfrak{G})$ such that $D(\mu) = 1$, while if ν is a multiplicative linear functional on $L(\mathfrak{G})$ and $\nu \neq \mu$, then $D(\mu) = 0$.*

Proof. We use the notation for multiplicative linear functionals on $L(\mathfrak{G})$ corresponding to the interpretation of the functional as a member of the character group $\hat{\mathfrak{G}}$. Let M donote the almost periodic mean. Then for any multiplicative linear functionals μ_i on $L(\mathfrak{G})$, $\mu_i \neq \mu$ and for any complex numbers a_i, $i = 1, \cdots, n$,

$$\left|\left|\mu - \sum_1^n a_i \mu_i\right|\right| = \left|\left|1 - \sum_1^n a_i i \mu^{-1}\mu_i\right|\right| \geqq \left|M\left(1 - \sum_1^n a_i \mu^{-1}\mu_i\right)\right| = 1 \ ,$$

where the norm is that of $L^*(\mathfrak{G})$. Thus the distance from μ to the span of the other multiplicative functions is at least one. The desired functional $D \in L^{**}(\mathfrak{G})$ then exists as a consequence of the Hahn-Banach theorem. The author is indebted to a referee for the suggestion of the above proof for Lemma 5.2.

5.3 THEOREM. *Let $\mathfrak{Y}$ be the closed subspace of $L^*(\mathfrak{G})$ generated by the multiplicative linear functionals on $L(\mathfrak{G})$. Let $\mathfrak{Z}$ be the closed span of $\{[F, f] \mid F \in \mathfrak{Y}^\perp$ and $f \in L^*(\mathfrak{G})\}$.*

(i) *A necessary and sufficient condition that there exist a maximal modular left ideal $\mathfrak{M}$ in $L^{**}(\mathfrak{G})$ with $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$ is that $\mathfrak{Y} \not\subset \mathfrak{Z}$.*

(ii) *A necessary and sufficient condition that there exist a maximal modular left ideal $\mathfrak{M}$ in $L^{**}(\mathfrak{G})$ with $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$ is that there exist $B \notin \mathfrak{Y}^\perp$ such $[B, f] \in \mathfrak{Y}$ for all $f \in L^*(\mathfrak{G})$.*

Proof. Suppose first that there exists a maximal modular left ideal $\mathfrak{M}$ in $L^{**}(\mathfrak{G})$ with $\mathfrak{L}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$. Then by Theorem 3.6 there exists a multiplicative linear functional μ on $L(\mathfrak{G})$ such that $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) | F(\mu) = 0\}$. By Lemma 3.2, $\mathfrak{L}(\mathfrak{M}) = (L^{**}(\mathfrak{G}))A$ and $A^2 = A \neq 0$, so $A \notin \mathfrak{M}$. It follows that $A(\mu) = 1$. Suppose that $\mathfrak{Y} \subset \mathfrak{Z}$, so that $\mu \in \mathfrak{Z}$. Thus

$$\mu = \lim_n \sum_{i=1}^{m(n)} [G_{n,i}, f_{n,i}]$$

with $G_{n,i} \in \mathfrak{Y}^\perp$. Now $\mathfrak{Y}^\perp \subset \mathfrak{M}$ and $A \in \mathfrak{L}(\mathfrak{M})$ so $A \in \mathfrak{L}(\mathfrak{Y}^\perp)$. Thus

$$1 = A(\mu) = \lim_n \sum_{i=1}^{m(n)} AG_{n,i}(f_{n,i}) = 0 \ .$$

Consequently $\mathfrak{Y} \not\subset \mathfrak{Z}$.

Suppose that $\mathfrak{Y} \not\subset \mathfrak{Z}$. Then there exist some multiplicative linear functional μ on $L(\mathfrak{G})$ such that $\mu \notin \mathfrak{Z}$. Thus there exists $J \in L^{**}(\mathfrak{G})$ such that $J \in \mathfrak{Z}^\perp$ and $J(\mu) = 1$. Let $D \in L^{**}(\mathfrak{G})$ have the property asserted in Lemma 5.2. Let $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$. Clearly $\mathfrak{M}$ is a maximal modular left ideal of $L^{**}(\mathfrak{G})$. Let $H = JD$. Then $H(\mu) = J(\mu)D(\mu) = 1$, so $H \notin \mathfrak{Y}^\perp$ and therefore $H \notin \mathfrak{R}^{**}$. Let $P \in \mathfrak{M}$, and let $f \in L^*(\mathfrak{G})$. Then $HP(f) = JDP(f) = J([DP, f])$. Now if ν is any multiplicative linear functional on $L(\mathfrak{G})$, $(DP)(\nu) = D(\nu)P(\nu) = 0$, since Lemma 5.2 $D(\nu) = 0$ if $\nu \neq \mu$, while $P(\nu) = 0$ if $\nu = \mu$ since $P \in \mathfrak{M}$. Thus $DP \in \mathfrak{Y}^\perp$, and thus $[DP, f] \in \mathfrak{Z}$. However $J \in \mathfrak{Z}^\perp$, so $HP(f) = 0$. Since f was arbitrary in

$L^*(\mathfrak{G})$ and P arbitrary in $\mathfrak{M}$, we see that $H \in \mathfrak{L}(\mathfrak{M})$ and $H \notin R^{**}$ which completes the proof of the first half of the theorem.

Next, we suppose that there exist a maximal modular left ideal $\mathfrak{M}$ in $L^{**}(\mathfrak{G})$ with $\mathfrak{R}(\mathfrak{M}) \not\subset \mathfrak{R}^{**}$. By Theorem 4.9, there is a multiplicative linear functional μ on $L(\mathfrak{G})$ such that $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$. Also by Lemma 4.2, there exists $A \in \mathfrak{R}(\mathfrak{M})$ such that $A = A^2 \neq 0$. In particular. $A \notin \mathfrak{M}$, so $A \notin \mathfrak{Y}^\perp$ as $\mathfrak{Y}^\perp \subset \mathfrak{M}$. Let $f \in L^*(\mathfrak{G})$. Then $A \in \mathfrak{R}(\mathfrak{M})$ so $A \in \mathfrak{R}(\mathfrak{Y}^\perp)$. Thus for any $T \in \mathfrak{Y}^\perp$, $0 = TA(f) = T([A, f])$, and $[A, f] \in \mathfrak{Y}^{\top\perp} = \mathfrak{Y}$. Thus A has the required properties.

Finally, we suppose that there exist $B \notin \mathfrak{Y}^\perp$ such that $[B, f] \in \mathfrak{Y}$ for each $f \in L^*(G)$. Since $B \notin \mathfrak{Y}^\perp$, there exist a multiplicative linear functional μ such that $B(\mu) \neq 0$. Let $\mathfrak{M} = \{F \in L^{**}(\mathfrak{G}) \mid F(\mu) = 0\}$, so that $\mathfrak{M}$ is a maximal modular left ideal in $L^{**}(\mathfrak{G})$. By Lemma 5.2, there exist $A \in L^{**}(\mathfrak{G})$ such that $A(\mu) = 1$ and $A(\nu) = 0$ if ν is a multiplicative linear functional on $L(\mathfrak{G})$ different from μ. Now $AB(\mu) = A(\mu)B(\mu) \neq 0$, so $AB \notin \mathfrak{R}^{**}$. Let $P \in \mathfrak{M}$, then for $f \in L^*(\mathfrak{G})$, $[B, f] \in \mathfrak{Y}$, so

$$[B, f] = \lim_n \sum_{i=1}^{m(n)} c_{n,i} \mu_{n,i}$$

where each $\mu_{n,i}$ is a multiplicative linear functional on $\mathfrak{L}(\mathfrak{G})$ and each $c_{n,i}$ is a complex number. We choose the notation so that $\mu_{n,1} = \mu$. Hence by the stated properties of A and the fact that $[A, \nu] = A(\nu)\nu$ for any multiplicative linear functional μ on $L(\mathfrak{G})$ we see that

$$[AB, f] = [A, [B, f]] = \lim_n \sum_{i=1}^{m(n)} c_{n,i} A(\mu_{n,i}) \mu_{n,i} = \lim_n c_{n,1} \mu .$$

Thus $PAB(f) = P([AB, f]) = 0$, and since f was arbitrary in $L^*(\mathfrak{G})$ and P arbitrary in $\mathfrak{M}$ we have $AB \in \mathfrak{R}(\mathfrak{M})$. This completes the proof of Theorem 5.3.

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