A THEOREM ON THE ACTION OF SO(3)

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1. Introduction. We shall use notions given in [1]. Let G be a compact Lie group acting on a locally compact Hausdorff space X. We denote by $F(G, X)$ the set of stationary points of G in X, that is, $F(G, X) = \{x \in X \mid Gx = x\}$. If G is a cyclic group generated by $g \in G$, $F(G, X)$ is also written $F(g, X)$.

Whenever $x \in X$, we call $Gx = \{gx \mid g \in G\}$ the *orbit* of x and $G_x = \{g \in G \mid gx = x\}$ the *isotropy group* at x. By a *principal orbit* we mean an orbit Gx such that G_x is minimal. By an *exceptional orbit* we mean an orbit of maximal dimension which is not a principal orbit. By a *singular orbit* we mean an orbit not of maximal dimension. Denote by U the union of all the principal orbits, by D the union of all the exceptional orbits and by B the union of all the singular orbits. Then U, D and B are all G-invariant and they are mutually disjoint. Moreover, $X = U \cup D \cup B$ and both B and $D \cup B$ are closed in X.

Denote by X^* the orbit space X/G and by π the natural projection of X onto X^*. Whenever $A \subset X$, A^* denotes the image πA. If X is a connected cohomology n-manifold over Z [1; p. 9], where Z denotes the ring of integers, then the following results are known.

(1.1) *U^* is connected* [1; p. 122] *so that whenever $x, y \in U$, G_x and G_y are conjugate.*

(1.2) $\dim_Z B^* \leqq \dim_Z U^* - 1$ *so that if r is the dimension of principal orbits and B_k is the union of all the k-dimensional singular orbits ($k < r$), then* $\dim_Z B_k \leqq n - r + k - 1$ [1; p. 118]. *Hence* $\dim_Z B \leqq n - 2$.

Denote by E^{n+1} the euclidean $(n + 1)$-space, by S^n the unit n-sphere in E^{n+1} and by SO(3) the rotation group of E^3. In this note G is to be SO(3) and X is to be a compact cohomology n-manifold over Z with $H^*(X; Z) = H^*(S^n; Z)$.

Let us first observe the following examples.

1. Let $G = \mathrm{SO}(3)$ act trivially on $X = S^1$. (Here we have $n = 1$.)
2. Let $G = \mathrm{SO}(3)$ act on $E^{n+1} = E^5 \times E^{n-4}$ $(n \geqq 4)$ by the definition

$$g(x, y) = (gx, y)\ ,$$

where the action of G on E^5 is an irreducible orthogonal action. Then G acts on $X = S^n$ and in this action, the 2-dimensional orbits are all

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projective planes, $F(G, X)$ is an $(n-5)$-sphere and for every $x \in U$, G_x is a dihedral group of order 4.

3. Let $G = \mathrm{SO}(3)$ act on $E^{n+1} = E^3 \times E^3 \times E^{n-5} (n \geqq 5)$ by the definition

$$g(x, y, z) = (gx, gy, z) ,$$

where the action on E^3 is the familiar one. Then G acts on $X = S^n$ and in this action, the 2-dimensional orbits are all 2-spheres, $F(G, X)$ is an $(n-6)$-sphere and for every $x \in U$, G_x is the identity group.

In all three examples, $D = \phi$ and $\dim B = n - 2$. The orbit space X^* is X itself in the first example and it is a closed $(n-3)$-cell with boundary B^* in the other two examples.

The purpose of this note is to prove that if X is a compact cohomology n-manifold over Z with $H^*(X; Z) = H^*(S^n; Z)$, then every action of $G = \mathrm{SO}(3)$ on X with $\dim_Z B = n - 2$ strongly resembles one of these examples. In fact, we shall prove the following:

THEOREM. *Let X be a compact cohomology n-manifold over Z with $H^*(X; Z) = H^*(S^n; Z)$ and let $G = \mathrm{SO}(3)$ act on X with $\dim_Z B = n - 2$. Then $D = \phi$ and one of the following occurs.*

1. *$n = 1$ and G acts trivially on X.*

2. *$n \geqq 4$ and for every $x \in U$, G_x is a dihedral group of order 4. Moreover, the 2-dimensional ordits are all projective planes and $F(G, X)$ is a compact cohomology $(n-5)$-manifold over Z_2 with $H^*(F(G, X); Z_2) = H^*(S^{n-5}; Z_2)$, where Z_2 denotes the prime field of characteristic 2.*

3. *$n \geqq 5$ and for every $x \in U$, G_x is the identity group. Moreover, the 2-dimensional orbits are all 2-spheres and $F(G, X)$ is a compact cohomology $(n-6)$-manifold over Z_2 with $H^*(F(G, X); Z_2) = H^*(S^{n-6}; Z_2)$.*

In the last two cases, B^* is a compact cohomology $(n-4)$-manifold over Z with $H^*(B^*; Z) = H^*(S^{n-4}; Z)$ and X^* is a compact Hausdorff space which is cohomologically trivial over Z and such that $X^* - B^*$ is a cohomology $(n-3)$-manifold over Z.

The proof of this theorem is given in the next three sections.

2. The set D. Let X be a connected cohomology n-manifold over Z and let $G = \mathrm{SO}(3)$ act on X with $\dim_Z B = n - 2$. If G acts trivially on X, it is clear that $n = 1$ and that $D = \phi$. Hence we shall assume that the action of G on X is nontrivial.

Since G is a 3-dimensional simple group which has no 2-dimensional subgroup, it follows that

(2.1) *G acts effectively on X and no orbit is 1-dimensional.*

(2.2) *Principal orbits are 3-dimensional so that for every $x \in U \cup D$, G_x is finite.*

By (2.1), principal orbits are either 2-dimensional or 3-dimensional. If principal orbits are 2-dimensional, then $B = F(G, X)$ so that, by (1.2), $\dim_Z B < n - 2$, contrary to our assumption.

(2.3) *Denote by B_2 the union of all the 2-dimensional orbits. Then $\dim_Z B_2 = n - 2$ so that $B_2 \neq \phi$ and $n \geqq 4$. Moreover, whenever Gz is a 2-dimensional orbit, G_z is either a circle group or the normalizer of a circle group and accordingly Gz is either a 2-sphere or a projective plane.*

By (2.2), $n = \dim_Z X \geqq \dim_Z U \geqq 3$. We infer that $B_2 \neq \phi$ so that $n - 2 = \dim_Z B_2 \geqq 2$. Hence $n \geqq 4$.

(2.4) *Let $x \in U$. Whenever $y \in D$, there is a $g \in G$ such that G_x is a normal subgroup of G_{gy}.*

Let S be a connected slice at y [1; p. 105]. Then S is a connected cohomology $(n - 3)$-manifold over Z and G_y acts on S. As seen in [7], S is also a connected cohomology $(n - 3)$-manifold over Z_p for every prime p, where Z_p denotes the prime field of characteristic p.

Let $x' \in S \cap U$. We claim that $G_{x'}$ is a normal subgroup of G_y. Since G_y is a finite group (see (2.2)) and $G_{x'}$ is a subgroup of G_y, there exists a neighborhood N of the identity in G such that $N^{-1}G_{x'}N \cap G_y = G_{x'}$. Let V be a neighborhood of x' such that whenever $x'' \in V$, $hG_{x''}h^{-1} \subset G_{x'}$ for some $h \in N$. (For the existence of V, see [4; p. 216].) Then for every $x'' \in V \cap S$, $G_{x''} \subset N^{-1}G_{x'}N \cap G_y = G_{x'}$ so that $G_{x''} = G_{x'}$. Therefore $G_{x'}$ leaves every point of $V \cap S$ fixed. Since S is a connected cohomology $(n - 3)$-manifold over Z_p for every prime p, it follows from Newman's theorem [6] that $G_{x'}$ leaves every point of S fixed. Hence $G_{x'} = \{g \in G_y \mid gx'' = x'' \text{ for all } x'' \in S\}$, which is clearly a normal subgroup of G_y. By (1.1), G_x and $G_{x'}$ are conjugate so that our assertion follows.

(2.5) *Let $x \in U$. Whenever Gz is 2-dimensional, there is a $g \in G$ such that $G_x \subset G_{gz}$. Hence G_x is either cyclic or dihedral and it is cyclic if there is a 2-dimensional orbit which is a 2-sphere.*

For the rest of this section, we assume that

$$H_c^*(X; Z) = H^*(S^n; Z) .$$

Under this assumption, $H_c^0(X; Z) = H^0(S^n; Z) = Z$. Hence X is compact.

(2.6) *Let T be a circle group in G. Then $F(T, X)$ is a compact cohomology $(n - 4)$-manifold over Z with $H^*(F(T, X); Z) = H^*(S^{n-4}; Z)$.*

Since $F(T, X)$ intersects every singular orbit at one or two points, $\dim_Z F(T, X) = \dim_Z B^* = n - 4$. Hence our assertion follows [1; Chapters IV and V].

(2.7) *Let $g \in G$ be of order p^α, where p is a prime and α is a positive integer. If $g \in G_x$ for some $x \in U \cup D$, then $F(g, X)$ is a compact cohomology $(n-2)$-manifold over Z_p with $H^*(F(g, X); Z_p) = H^*(S^{n-2}; Z_p)$. Hence $F(g, X)$ intersects every principal orbit.*

It is known that X is also a compact cohomology n-manifold over Z_p with $H^*(X; Z_p) = H^*(S^n; Z_p)$. Since G is connected, g preserves the orientation of X. It follows that for some $r < n$ of the same parity, $F(g, X)$ is a compact cohomology r-manifold over Z_p with $H^*(F(g, X); Z_p) = H^*(S^r; Z_p)$ [1; Chapters IV and V].

Let T be the circle group in G containing g. By (2.6), $F(g, X) \cap B = F(T, X)$ is a compact cohomology $(n-4)$-manifold over Z_p. Since, by hypothesis, there exists a point of $U \cup D$ contained in $F(g, X)$, $F(g, X) \cap B$ is properly contained in $F(g, X)$ so that $r = n - 2$. Hence $F(g, X)$ is a compact cohomology $(n-2)$-manifold over Z_p with $H^*(F(g, X); Z_p) = H^*(S^{n-2}; Z_p)$.

Since $\dim_Z D^* < n - 3$ [1; p. 121] and since $F(g, X)$ intersects every exceptional orbit at a set of dimension $\leqq 1$, it follows that $\dim_{Z_p}(F(g, X) \cap D) \leqq \dim_Z(F(g, X) \cap D) < n - 2$. But we have $\dim_{Z_p} F(g, X) = n - 2$ and $\dim_{Z_p}(F(g, X) \cap B) = n - 4$. Therefore $F(g, X) \cap U \neq \phi$. Hence, by (1.1), $F(g, X)$ intersects every principal orbit.

(2.8) *Let $x \in U$ and $y \in D$. Let p be a prime and let α be a positive integer. If G_y has an element of order p^α, so does G_x.*

Let $g \in G_y$ be of order p^α. By (2.7), $F(g, X) \cap Gx \neq \phi$ so that for some $h \in G$, $hx \in F(g, X)$. Hence $h^{-1}gh$ is an element of G_x of order p^α.

(2.9) $D = \phi$.

Suppose that $D \neq \phi$. Let $x \in U$ and $y \in D$ be such that G_x is a proper normal subgroup of G_y (see (2.4)). We first claim that G_y is dihedral.

It is well known that a finite subgroup of SO(3) is either cyclic or dihedral or tetrahedral or octahedral or icosahedral. If G_y is cyclic, so is G_x. Let the order of G_y be $p_1^{s_1} \cdots p_k^{s_k}$, where $p_1, \cdots p_k$ are distinct primes and $s_1, \cdots, s_k$ are positive integers. Then for every $i = 1, \cdots, k$, G_y contains an element of order $p_i^{s_i}$ so that, by (2.8), G_x also contains an element of order $p_i^{s_i}$. Hence G_x is of order $\geqq p_1^{s_1} \cdots p_k^{s_k}$ and consequently $G_x = G_y$, contrary to the fact that G_x is a proper subgroup of G_y. If G_y is either tetrahedral or octahedral or icosahedral, then by (2.8), G_x contains a subgroup of order 2 and a subgroup of order 3. In case G_x is octahedral, it also contains a subgroup of order 4. Hence G_x, as a normal subgroup of G_y, is equal to G_y, contrary to our hypothesis. This proves that G_y is dihedral.

Now the order of G_y is even. It follows from (2.7) that whenever $g \in G$ is of order 2, $F(g, X)$ is a compact cohomology $(n-2)$-manifold over Z_2 with $H^*(F(g, X); Z_2) = H^*(S^{n-2}; Z_2)$. Let H be a dihedral subgroup of G of order 4. By Borel's theorem [1; p. 175], $F(H, X)$ is a compact cohomology $(n-3)$-manifold over Z_2 with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$. Since $\dim_{Z_2}(F(H, X) \cap (D \cup B)) \leqq \dim_Z (F(H, X) \cap (D \cup B)) < n-3$, it follows that $F(H, X) \cap U$ is not null. Hence we may assume that $H \subset G_x \subset G_y$.

Let T be the circle group in G such that its normalizer contains G_y. Then $H \cap T \subset G_x \cap T \subset G_y \cap T$ so that $G_y \cap T$ is a cyclic group and $G_x \cap T$ is a proper subgroup of $G_y \cap T$ of even order. Let the order of $G_y \cap T$ be $2^{s_0} p_1^{s_1} \cdots p_k^{s_k}$, where $p_1, \cdots, p_k$ are distinct odd primes and $s_0, s_1, \cdots, s_k$ are positive integers. By (2.8), there are $k+1$ elements $g_0, g_1, \cdots, g_k$ of G_x of order $2^{s_0}, p_1^{s_1}, \cdots, p_k^{s_k}$ respectively. Since $p_1, \cdots, p_k$ are odd, $g_1 \cdots, g_k$ are in $G_x \cap T$. Therefore no element of $G_x \cap T$ is of order 2^{s_0}. But this implies that $s_0 > 1$ so that $g_0 \in G_x \cap T$. Hence we have arrived at a contradiction.

3. Case that the 2-dimensional orbits are all projective planes.

Let X be a compact cohomology n-manifold over Z with $H^*(X; Z) = H^*(S^n; Z)$ and let $G = \mathrm{SO}(3)$ act *nontrivially* on X with $\dim_Z B = n-2$. Throughout this section, we assume that for some $x \in U$, G_x is of even order.

(3.1) *Let H be a dihedral subgroup of G of order* 4 *and let M be the normalizer of H that is the octahedral group containing H. Then $F(H, X)$ is a compact cohomology $(n-3)$-manifold over Z_2 with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$ and $K = M/H$ is isomorphic to the symmetric group of three elements and acts on $F(H, X)$. Moreover, the natural map of $F(H, X)/K$ into X^* is onto.*

By (2.7), for every $g \in G$ of order 2, $F(g, X)$ is a compact cohomology $(n-2)$-manifold over Z_2 with $H^*(F(g, X); Z_2) = H^*(S^{n-2}; Z_2)$. It follows from Borel's theorem [1; p. 175] that $F(H, X)$ is a compact cohomology $(n-3)$-manifold over Z_2 with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$.

Clearly $K = M/H$ is isomorphic to the symmetric group of three elements and the action of M on $F(H, X)$ induces an action of K on $F(H, X)$. Moreover, there is a natural map $f: F(H, X)/K \to X^*$.

Let $z \in F(H, X) \cap B$. If $Gz = z$, then $F(H, X) \cap Gz = z$. If Gz is 2-dimensional, then G_z contains H so that by (2.3) it is the normalizer of a circle group. Therefore any two isomorphic dihedral subgroups of

G_z are conjugate in G_z. Let g be an element of G with $gz \in F(H, X)$. It is clear that $g^{-1}Hg \subset g^{-1}G_{gz}g = G_z$ so that for some $h \in G_z$, $h^{-1}g^{-1}Hgh = H$ or $gh \in M$. Hence $gz = ghz \in Mz$. This proves that $F(H, X) \cap Gz \subset Mz$.

From these results it follows that $F(H, X)$ intersects every singular orbit at a finite set. [This and one or two facts mentioned below can be seen by examining the standard action of SO(3) on S^2 or on P^2 (viewed as the acts of lines through the region in E^3).] Therefore, by (1.2), $\dim_Z (F(H, X) \cap B) \leqq \dim_Z B^* < n - 3$. As a consequence of this result and that $D = \phi$ (see (2.9)), we have $F(H, X) \cap U \neq \phi$. Hence $F(H, X)$ intersects every principal orbit and consequently it intersects every orbit. This proves that the natural map $f: F(H, X)/K \to X^*$ is onto.

(3.2) *Every 2-dimensional orbit is a projective plane and intersects $F(H, X)$ at exactly three points.*

Let Gz be a 2-dimensional orbit. By (3.1), $F(H, X)$ intersects Gz so that we may assume that $z \in F(H, X)$. Since G_z contains H, it follows from (2.3) that G_z is the normalizer of a circle group. Hence Gz is a projective plane.

In the proof of (3.1) we have shown that $F(H, X) \cap Gz \subset Mz$. But it is clear that $Mz \subset F(H, X) \cap Gz$. Hence

$$F(H, X) \cap Gz = Mz = M/(M \cap G_z) .$$

Since M is of order 24 and $M \cap G_z$ is of order 8, it follows that $F(H, X) \cap Gz$ contains exactly three points.

(3.3) *B^* is a compact cohomology $(n - 4)$-manifold over Z with $H^*(B^*; Z) = H^*(S^{n-4}; Z)$.*

Let T be a circle group in G. It is clear that $F(T, X) \subset B$. Since, by (2.1) and (3.2), every singular orbit is either a point or a projective plane, it follows that $F(T, X)$ intersects every singular orbit at exactly one point. Therefore the natural projection π maps $F(T, X)$ homeomorphically onto B^* and hence our assertion follows from (2.6).

(3.4) *Let $Y = F(H, X) - F(G, X)$. Then $\bar{Y} = F(H, X)$ and every point of Y has a neighborhood V in Y which is a cohomology $(n - 3)$-manifold over Z and such that the isotropy group is constant on $V - B$.*

Let T be a circle group whose normalizer N contains H. Then $F(H, X) \supset F(N, X) = F(T, X) \supset F(G, X)$. Since $F(H, X)$ is a compact $(n - 3)$-manifold over Z_2 (see (3.1)) and since $F(T, X)$ is a compact $(n-4)$-manifold over Z_2 (see (2.6)), it follows that the closure of $F(H, X)- F(T, X)$ is $F(H, X)$. Hence $\bar{Y} = F(H, X)$.

Let $x \in Y \cap U$ and let S be a slice at x. Then S is a cohomology $(n-3)$-manifold over Z. Moreover, $G_y = G_x$ for all $y \in S$ so that $S \subset Y$. Since both S and Y are cohomology $(n-3)$-manifolds over Z_2, it follows that S is open in Y. Hence our assertion follows by taking S as V.

Let $z \in Y \cap B$ and let S be a slice at z. Then S is a cohomology $(n-2)$-manifold over Z and G_z is the normalizer of a circle group T acting on S. Whenever $x \in S \cap U$, $G_x \cap T$ is a finite cyclic group in T and the index of $G_x \cap T$ in G_x is 2 because G_x in a dihedral subgroup of G_z. Since the order of G_x is independent of $x \in S \cap U$, so is the order of $G_x \cap T$. Hence $G_x \cap T$ is independent of $x \in S \cap U$ so that for $x \in F(H, S) \cap U$.

$$G_x S = H(G_x \cap T)S = HS = S$$

and

$$F(G_x, S) = F(G_x/(G_x \cap T), S) = F(H/(H \cap T), S) = F(H, S) .$$

Let Q be a neighborhood of the identity of G such that $Q^{-1}TQ \cap G_z = T$. If $gy \in F(H, X)$ with $g \in Q$ and $y \in S$, then $g^{-1}Hg \subset g^{-1}G_{gy}g = G_y \subset G_z$ so that $g^{-1}(H \cap T)g \subset Q^{-1}TQ \cap G_z = T$. Therefore $g^{-1}Tg = T$ or $g \in G_z$. Hence $gy \in G_z y \subset S$. This proves that $F(H, S) = F(H, X) \cap S = F(H, X) \cap QS$ is open in $F(H, X)$ so that it is a cohomology $(n-3)$-manifold over Z_2.

Since S is a cohomology $(n-2)$-manifold over Z with

$$F(H/(H \cap T), S) = F(H, S) ,$$

it follows that $F(H, S)$ is also a cohomology $(n-3)$-manifold over Z. (If Z_2 acts on a cohomology m manifold over Z with $F(Z_2)$ being a cohomology $(m-1)$-manifold over Z_2, then $F(Z_2)$ is also a cohomology $(m-1)$-manifold over Z.) That G_x is constant on $F(H, S) \cap U$ is a direct consequence of the fact that $F(G_x, S) = F(H, S)$ for all $x \in F(H, S) \cap U$.

(3.5) *Y is a connected cohomology $(n-3)$-manifold over Z and the isotropy group is constant on $Y - B$.*

By (3.4), Y is a cohomology $(n-3)$-manifold over Z. Let T be a circle group in G whose normalizer N contains H. Then $F(H, X) \supset F(N, X) = F(T, X) \supset F(G, X)$. From (2.6) and (3.1), it is easily seen that $F(H, X) - F(T, X)$ has exactly two components with $F(T, X)$ as their common boundary. By (2.3), there exists a point z of $F(T, X)$ such that Gz is a projective plane so that $z \in F(T, X) - F(G, X)$. Hence Y is connected.

Let $x \in Y \cap U$. Then $F(G_x, X) \cap Y$ is clearly closed in Y. But, by (3.4), it is also open in Y. Hence, by the connectedness of Y, $F(G_x, X) \cap Y = Y$.

(3.6) *Whenever* $x \in F(H, X) \cap U$, $G_x = H$. *Hence for every* $x \in U$, G_x *is a dihedral group of order* 4.

Let x be a point of $F(H, X) \cap U$. Since $H \subset G_x$, $F(H, X) \supset F(G_x, X)$. But, by (3.4) and (3.5), $F(H, X) \subset F(G_x, X)$. Hence $F(H, X) = F(G_x, X)$.

It is clear that $G' = \{g \in G \mid gF(H, X) = F(H, X)\}$ is a closed subgroup of G containing M. Since $F(H, X) = F(G_x, X)$, G_x is a normal subgroup of G' so that G' is contained in the normalizer of G_x. But, by (2.5), G_x is dihedral and H is the only dihedral group whose normalizer contains M. It follows that $G_x = H$. Hence, by (1.1), the isotropy group at any point of U is a dihedral group of order 4.

(3.7) *Whenever* $x \in F(H, X)$, $F(H, X) \cap Gx = Kx$ *which contains one point or three points or six points according as* Gx *is* 0-*dimensional or* 2-*dimensional or* 3-*dimensional.*

If Gx is 0-dimensional, it is clear that $F(H, X) \cap Gx = x = Kx$. If Gx is 2-dimensional, we have shown in the proof of (3.2) that $F(H, X) \cap Gx = Mx = Kx$ which contains exactly three points.

Now let Gx be 3-dimensional. If g is an element of G with $gx \in F(H, X)$, then, by (3.6), $gHg^{-1} = gG_xg^{-1} = G_{gx} = H$ so that $g \in M$. Therefore $F(H, X) \cap Gx \subset Mx$. But it is obvious that $Mx \subset F(H, X) \cap Gx$. Hence

$$F(H, X) \cap Gx = Mx = Kx$$

which clearly contains six points.

From this result, it is easily seen that the natural map f: $F(H, X)/K \to X^*$ is a homeomorphism onto.

(3.8) *Whenever* $a \in K$ *is of order* 2, *we abbreviate* $F(a, F(H, X))$ *by* $F(a)$. *Then* $F(a) \subset B$ *and* $F(a)$ *is a compact cohomology* $(n-4)$-*manifold over* Z *with* $H^*(F(a); Z) = H^*(S^{n-4}; Z)$. *Moreover*, $F(H, X) - F(a)$ *contains exactly two components* V *and* V' *with* $aV = V'$.

Whenever $x \in F(H, X) \cap U$, $G_x = H$ (see (3.6)) so that $x \notin F(a)$. Hence $F(a) \subset B$. Let $a = a'H$ with a' being of order 4 and let T be the circle group containing a'. Then $F(a) = F(T, X)$ and hence the first part follows from (2.6). Now $F(H, X)$ is a compact cohomology $(n-3)$-manifold over Z_2 with $H^*(F(H, X); Z_2) = H^*(S^{n-3}; Z_2)$ and $F(a) = F(a, F(H, X))$ is a compact cohomology $(n-4)$-manifold over Z_2. The second part follows.

(3.9) $F(H, X) - B$ *contains exactly six components and whenever* P *is a component of* $F(H, X) - B$, $KP = F(H, X) - B$ *and the natural* *projection π maps P homeomorphically onto U^*.*

Let P be a component of $F(H, X) - B$. Since the isotropy group is constant on P (see (3.5)), the natural projection π defines a local homeomorphism $\pi'\colon P \to U^*$. By (3.7), for every $x^* \in U^*$, $\pi'^{-1}x^*$ contains no more than six points. We infer that π' is closed so that $\pi'P$ is both open and closed in U^*. Hence, by the connectedness of U^*, $\pi'P = U^*$.

Let Q be a second component of $F(H, X) - B$ and let $y \in Q$. Then there is a point $x \in P$ such that $\pi x = \pi y$. Therefore, by (3.7), for some $k \in K$, $y = kx$ so that $Q = kP$. Hence $KP = F(H, X) - B$.

Let $x \in P$. By (3.8), x and ax belong to different components of $F(H, X) - F(a) \supset F(H, X) - B$. Therefore aP is a component of $F(H, X) - B$ different from P. Similarly, bP and cP are components of $F(H, X) - B$ different from P.

If aP, bP and cP are not distinct, say $bP = cP$, then $\{k \in K \mid kP = P\}$ is of order 3 so that P and $aP = bP = cP$ are the only two components of $F(H, X) - B$. Now $F(H, Z) - B = F(H, Z) - (F(a) \cup F(b) \cup F(c))$ and $F(a)$, $F(b)$, $F(c)$ are manifold over Z of dimension one less than the dimension of $F(H)$. Hence $F(H, X) \cap B = F(a) \cap F(b) \cap F(c) = F(G, X)$. This is impossible, because the intersection of $F(H, X)$ and a 2-dimensional orbit is contained in B but not contained in $F(G, X)$. From this result it follows that P, aP, bP, cP are distinct components of $F(H, X) - B$. Hence P, aP, bP, cP, bcP, cbP are all the distinct components of $F(H, X) - B$.

Now it is clear that for every $x^* \in U^*$, $\pi'^{-1}x^*$ contains exactly one point. Hence π' is a homeomorphism.

(3.10) *Let P be a component of $F(H, X) - B$. Then the map of $G/H \times P$ onto U defined by $(gH, x) \to gx$ is a homeomorphsim onto. Hence U is homeomorphic to the topological product of a principal orbit and U^*.*

This is an immediate consequence of (3.5) and (3.9).

(3.11) *The closure of $F(a) - F(G, X)$ is equal to $F(a)$. Hence* $\dim_{Z_2} F(G, X) \leqq \dim_Z F(G, X) \leqq n - 5$.

Suppose that the closure of $F(a) - F(G, X)$ is not equal to $F(a)$. Then there is a point z of $F(G, X)$ and a neighborhood A of z such that $A \cap F(a) = A \cap F(G, X)$. Since $A \cap F(G, X) \subset F(b)$ and since, by (3.8), both $A \cap F(G, X)$ and $F(b)$ are cohomology $(n-4)$-manifolds over Z, $A \cap F(G, X)$ is open in $F(b)$ so that we may assume that $A \cap F(G, X) = A \cap F(b)$. Similarly, we may assume that $A \cap F(G, X) = A \cap F(c)$. Hence $A \cap F(G, X) = A \cap F(H, X) \cap B$. By (3.1) and (3.8), we may also assume that $KA = A$ and $A \cap (F(H, X) - F(a))$ contains exactly two components Q and Q'. Now both Q and Q' are contained in $F(H, X) - B$ and $aQ = bQ = Q'$ Therefore $abQ = Q$ so that ab maps the component of $F(H, X) - B$ containing Q into itself, contrary to (3.9).

Since, by (3.8), $F(a)$ is a cohomology $(n-4)$-manifold over Z and since $F(G, X)$ is nowhere dense in $F(a)$, it follows that $\dim_{Z_2} F(G, X) \leqq \dim_Z F(G, X) \leqq n - 5$.

(3.12) *If $n = 4$, then $F(G, X)$ is null.*

This is a direct consequence of (3.11).

(3.13) *Let T be a circle group in G, let N be the normalizer of T and let A be an orbit. If A is a projective plane, then A/T is an arc and N/T acts trivially on A/T so that $F(N/T, A/T) = A/T = A/N$. If A is 3-dimensional, then A/T is a 2-sphere and A/N is a closed 2-cell so that $F(N/T, A/T)$ is a circle.*

If A is a projective plane, it is clear that A/T is an arc and N/T acts trivially on A/T. Therefore $A/N = A/T = F(N/T, A/T)$.

Now let A be 3-dimensional. By (3.6), we may let $A = G/H = \{gH | g \in G\}$. Therefore A/T is the double coset space $(G/H)/T$ and $(G/T)/H$ are homeomorphic. Since G/T is a 2-sphere and since every element of H preserves the orientation of G/T, it follows that $(G/T)/H$ is a 2-sphere. Hence A/T is a 2-sphere.

As seen in [3], the double coset space $(G/N)/H$ is a closed 2-cell. Since A/N may be regarded as the double coset space $(G/H)/N$ which is homeomorphic to $(G/N)/H$, we infer that A/N is a closed 2-cell.

From these results, it follows that $f(N/T, A/T)$ is a circle.

(3.14) *X^* is cohomological trivial over Z.*

Let N be the normalizer of a circle group T in G. Then N/T is a cyclic group of order 2 which acts on X/T with $(X/T)/(N/T) = X^*$. Since, by (2.6), $H^*(F(T, X); Z) = H^*(S^{n-4}; Z)$, it follows that $H(X/T; Z) = H^*(S^{n-1}; Z)$ [1; p. 65].

By (3.13), $F(N/T, B/T) = B/T$ and for every singular orbit A, A/T is either a single point or an arc. It follows from the Vietoris map theorem that $H^*(B/T; Z) = H^*(B^*; Z) = H^*(S^{n-4}; Z)$ (see (3.3)). By (3.10) and (3.13), $F(N/T, U/T)$ is homeomorphic to the topological product of a circle and U^* so that $H^{n-2}(F(N/T, U/T); Z) \neq 0$. Therefore $H^*(F(N/T, X/T); Z) = H^*(S^{n-2}; Z)$. Hence $H^*(X/N; Z) = 0$. By (3.13), for every orbit A, A/N is either a single point or an arc or a closed 2-cell. It follows from the Vietoris map theorem that $H^*(X^*; Z) = H^*(X/N; Z) = 0$.

(3.15)
$$H_c^k(U^*; Z_2) = \begin{cases} Z_2 & \textit{for } k = n-3\,; \\ 0 & \textit{otherwise}\,. \end{cases}$$

This follows from (3.3), (3.14) and the cohomology sequence of (X^*, B^*).

(3.16)
$$H_c^k(U; Z_2) = \begin{cases} Z_2 & \textit{for } k = n-3,\ n\,; \\ Z_2 \oplus Z_2 & \textit{for } k = n-2,\ n-1\,; \\ 0 & \textit{otherwise}\,. \end{cases}$$

Since for a principal orbit A, we have

$$H^k(A; Z_2) = \begin{cases} Z_2 & \text{for } k = 0, 3\,; \\ Z_2 \oplus Z_2 & \text{for } k = 1, 2\,; \\ 0 & \text{otherwise}\,, \end{cases}$$

our assertion follows from (3.10) and (3.15).

As a consequence of (3.16) and the cohomology sequence of (X, B), we have

(3.17)
$$H^k(B; Z_2) = \begin{cases} Z_2 & \textit{for } k = 0,\ n-4\,; \\ Z_2 \oplus Z_2 & \textit{for } k = n-3,\ n-2\,; \\ 0 & \textit{otherwise}\,. \end{cases}$$

(3.18) *Let T be a circle group in G and let $n \geqq 5$. Then*

$$H_c^k(F(T, X) - F(G, X); Z_2) = \begin{cases} \tilde{H}^{k-1}(F(G, X); Z_2) \ (\textit{the reduced group}) & \textit{for } k = 1\,; \\ H^{k-1}(F(G, X); Z_2) \oplus Z_2 & \textit{for } k = n-4\,; \\ H^{k-1}(F(G, X); Z_2) & \textit{otherwise}\,. \end{cases}$$

This follows from (2.6) and the cohomology sequence of $(F(T, X), F(G, X))$.

(3.19) *Let $n > 5$. Then*

$$H_c^k(B - F(G, X); Z_2) \begin{cases} H^k(B; Z_2) & \textit{for } k > n-4\,; \\ H^k(B; Z_2) \oplus H^{k-1}(F(G, X); Z_2) & \textit{for } k = n-4\,; \\ H^{k-1}(F(G, X); Z_2) & \textit{for } k = 2, \cdots, n-5\,; \\ \tilde{H}^{k-1}(F(G, X); Z_2) & \textit{for } k = 1\,. \end{cases}$$

This follows from the cohomology sequence of $(B, F(G, X))$.

(3.20) *$B - F(G, X)$ is homeomorphic to the topological product of a projective plane and $F(T, X) - F(G, X)$. Hence*

$$\begin{aligned} &H_c^k(B - F(G, X); Z_2) \\ &\quad = H_c^k(F(T, X) - F(G, X); Z_2) \oplus H_c^{k-1}(F(T, X) - F(G, X); Z_2) \\ &\qquad\qquad \oplus H_c^{k-2}(F(T, X) - F(G, X); Z_2) . \end{aligned}$$

The first part follows from the that $F(T, X) - F(G, X)$ is a cross-section of the transformation group $(G, B - F(G, X))$ on which the isotropy group is constant. The second part follows from the first part and the fact that if A is a projective plane, then

$$H^k(A; Z_2) = \begin{cases} Z_2 & \text{for } k = 0, 1, 2; \\ 0 & \text{otherwise.} \end{cases}$$

(3.21) $\dim_{Z_2} F(G, X) = n - 5$. *If $n = 4$, then B contains exactly two projective planes. If $n = 5$, then $F(G, X)$ contains exactly two points. If $n > 5$, then $H^{n-5}(F(G, X); Z_2) = Z_2$ so that $F(G, X)$ is not null.*

Setting $k = n - 2$ in (3.20), we have, by (2.6) and (3.17),

$$Z_2 \oplus Z_2 = H_c^{n-4}(F(T, X) - F(G, X); Z_2) .$$

If $n = 4$, then, by (3.12), $H^\circ(F(T, X); Z_2) = Z_2 \oplus Z_2$ so that $F(T, X)$ contains exactly two points. Hence B contains exactly two projective planes.

If $n = 5$, then $H_c^1(F(T, X) - F(G, X); Z_2) = \tilde{H}^\circ(F(G, X); Z_2) \oplus H^1(F(T, X); Z_2)$ so that $\tilde{H}^\circ(F(G, X); Z_2) = Z_2$. Hence $F(G, X)$ contains exactly two points.

If $n > 5$, it follows from (3.18) that $H^{n-5}(F(G, X); Z_2) = Z_2$. Hence $F(G, X)$ is not null.

(3.22) $H^*(F(G, X); Z_2) = H^*(S^{n-5}; Z_2)$.

For $n = 4$ and 5, the result has been shown in (3.12) and (3.21). For $n > 5$, our assertion follows from (3.18), (3.19), (3.20) and (3.21).

(3.23) *$F(G, X)$ is a compact cohomology $(n - 5)$-manifold over Z_2.*

To prove (3.23), we have only to localize the preceding computations. Details are omitted.

REMARK. There is no difficulty to use Z in place of Z_2 in these computations. However, the computations over Z will not strengthen our final results (3.22) and (3.23).

4. Case that the 2-dimensional orbits are all 2-spheres.

Let X be a compact cohomology n-manifold over Z with $H^*(X; Z) = H^*(S^n; Z)$ and let $G = \mathrm{SO}(3)$ act nontrivially on X with $\dim_Z B = n - 2$.

Throughout this section, we assume that for some $x \in U$, G_x is of odd order.

(4.1) *Let H be a dihedral subgroup of G of order 4. Then $F(H, X)$ is a compact cohomology $(n-6)$-manifold over Z_2 with $H^*(F(H, X); Z_2) = H^*(S^{n-6}; Z_2)$. Hence $n \geqq 5$.*

Let $g \in G$ be of order 2 and let T be the circle group in G containing g. Since for some $x \in U$, G_x is of odd order, $F(g, X) \subset B$ so that $F(g, X) = F(T, X)$ is a compact cohomology $(n-4)$-manifold over Z_2 with $H^*(F(g, X); Z_2) = H^*(S^{n-4}; Z_2)$. By Borel's theorem [1; p. 175], $F(H, X)$ is a compact cohomology $(n-6)$-manifold over Z_2 with $H^*(F(H, X); Z_2) = H^*(S^{n-6}; Z_2)$. From this result it follows that $n - 6 \geqq -1$. Hence $n \geqq 5$.

(4.2) *The 2-dimensional orbit are all 2-spheres.*

Suppose that this assertion is false. Then there is, by (2.3), a projective plane Gz. Denote by T the identity component of G_z and by H a dihedral subgroup of G_z of order 4. Let S be a connected slice at z. Then S is a cohomology $(n-2)$-manifold over Z and G_z acts on S. Moreover, $F(T, S) = F(T, X) \cap S$ is open in $F(T, X)$ so that it is a cohomology $(n-4)$-manifold over Z. Hence we may let S be so chosen that $F(T, S)$ is connected and that both S and $F(T, S)$ are orientable.

Since T is a circle group and since $\dim_Z S - \dim_Z F(T, S) = 2$, it follows that S/T is a connected cohomology $(n-3)$-manifold over Z with boundary $F(T, S)$ [1; p. 196]. Hence we have a connected cohomology $(n-3)$-manifold Y over Z obtained by doubling S/T on $F(T, S)$ [1; p. 196]. Since S is orientable, so is $S/T - F(T, S)$. It follows from the connectedness of $F(T, S)$ that Y is orientable.

It is clear that $K = G_z/T$ is a cyclic group of order 2 which acts on S/T with $KF(T, S) = F(T, S)$. Since $F(K, F(T, S)) = F(H, S)$ is a cohomology $(n-6)$-manifold over Z_2, we infer from the dimensional parity that K preserves the orientation of $F(T, S)$ [1; p. 79].

The action of K on S/T defines a natural action of K on Y which also preserves the orientation of Y. Hence $\dim_{Z_2} F(K, Y) > n - 6$ so that for some $y^* = Ty \in S/T - F(T, S)$, $Ky^* = y^*$. But this implies that $G_z y = Ty$ so that y is a point of D, contrary to (2.9). Hence (4.2) is proved.

(4.3) *$F(G, X)$ is a compact cohomology $(n-6)$-manifold over Z_2 with $H^*(F(G, X); Z_2) = H^*(S^{n-6}; Z_2)$.*

By (4.2), $F(G, X) = F(H, X)$. Hence our assertion follows from (4.1).

(4.4) *Whenever $x \in U$, G_x is the identity group.*

If X is strongly paracompact, the result can be found in [5]. But an unpublished result of Yang shows that it is true in general.

(4.5) *B^* is a compact cohomology $(n-4)$-manifold over Z with $H^*(B^*; Z) = H^*(S^{n-4}; Z)$.*

Proof. Let T be a circle group in G and N its normalizer. Then $F(T, X)$ is a compact cohomology $(n-4)$-manifold over Z with $H^*(F(T, X); Z) = H^*(S^{n-4}; Z)$ and N/T is a cyclic group of order 2 acting on $F(T, X)$ with $F(T, X)/(N/T) = B^*$. Therefore $H^*(B^*; Z)$ is finitely generated [1; p. 44]. If H is a dihedral subgroup of N of order 4, it is easily seen that $F(N/T, F(T, X)) = F(H, X)$ so that $F(N/T, F(T, X))$ is a compact cohomology $(n-6)$-manifold over Z_2 with $H^*(F(N/T, F(T, X)); Z_2) = H^*(S^{n-6}; Z_2)$. Hence, by the dimensional parity theorem, N/T preserves the orientation of $F(T, X)$.

By [1; pp. 63–64],

$$H^*(B^*; Z_2) = H^*(F(T, X)/(N/T); Z_2) = H^*(S^{n-4}; Z_2) .$$

We now use the following diagram from [1; p. 45]

$$\begin{tikzcd} \cdots \arrow[r] & H^k(B^*; Z) \arrow[r, "2"] \arrow[dr, "\pi^*"'] & H^k(B^*; Z) \arrow[r, "q"] & H^k(B^*; Z_2) \arrow[r] & \cdots \\ & & H^k(F(T, X); Z) \arrow[u, "\mu"'] & & \end{tikzcd}$$

in which the horizontal sequence is exact and the triangle is commutative. For $k \neq 0,\ n-4$, we have $H^k(B^*; Z_2) = 0$ and $H^k(F(T, X); Z) = 0$; hence $H^k(B^*; Z) = 0$. For $k = 0$, we have $H^0(B^*; Z) = Z$, because B^* is clearly connected. For $k = n-4$, $H^{n-4}(B^*; Z)$ is a finitely generated group with $H^{n-4}(B^*; Z) \otimes Z_2 = H^{n-4}(B^*; Z_2) = Z_2$. It follows from the universal coefficient theorem that there is a finite subgroup K of $H^{n-4}(B^*; Z)$ of odd order such that $H^{n-4}(B^*; Z)/K$ is Z or Z_2. Since $K = 2K = \mu\pi^*K = 0$, $H^{n-4}(B^*; Z) = Z$ or Z_2. But $H^{n-4}(B^*; Z) \neq Z_2$, because N/T preserves the orientation of $F(T, X)$. Hence $H^{n-4}(B^*; Z) = Z$.

By localizing this result, we can show that B^* is a cohomology $(n-4)$-manifold over Z near every point of $F(G, X)$. (This result is also shown in [2].) Since the projection of $F(T, X) - F(G, X)$ onto $B^* - F(G, X)$ is a local homeomorphism, B^* is a cohomology $(n-4)$-manifold over Z near every point of $B^* - F(G, X)$. Hence B^* is a compact cohomology $(n-4)$-manifold over Z.

(4.6) *Let T be a circle group in G and let N be the normalizer of T. Then $H^*(B/N; Z) = H^*(S^{n-4}; Z)$.*

Let A be a singular orbit. If A is a single point, so is A/N. If A

is a 2-sphere, we may let $A = G/T$. Therefore $A/N = (G/T)/N$ is homeomorphic to $(G/N)/T$ which is known to be a closed 2-cell [3]. Hence A/N is a closed 2-cell.

Since, by (2.1) and (4.2), every singular orbit is either a single point or a 2-sphere, it follows from Vietoris map theorem that $H^*(B/N; Z) = H^*(B^*; Z)$. Hence our assertion follows from (4.5).

(4.7) $$H^k(X/N; Z) = \begin{cases} Z & \textit{for } k = 0 ; \\ Z_2 & \textit{for } k = n-1 ; \\ 0 & \textit{otherwise.} \end{cases}$$

Since $H^*(F(T, X); Z) = H^*(S^{n-4}; Z)$, it follows that $H^*(X/T; Z) = H^*(S^{n-1}; Z)$. Now N/T is a cyclic group of order 2 acting on X/T with $(X/T)/(N/T) = X/N$.

Let A be an orbit. If A is 3-dimensional, then, by (4.4), A/T is a 2-sphere and N/T acts freely on A/T. If A is a 2-sphere, then A/T is an arc and $F(N/T, A/T)$ is a single point. If A is a point, then $F(N/T, A/T) = A/T = A$. Hence $F(N/T, X/T)$ is homeomorphic to B^* so that, by (4.5), $H^*(F(N/T, X/T); Z_2)$.

As in the proof of (4.5), we can show that

(4.8) $$H_c^k(U/N; Z) = \begin{cases} Z & \textit{for } k = n-3 , \\ Z_2 & \textit{for } k = n-1 , \\ 0 & \textit{otherwise.} \end{cases}$$

(4.9) *There is an exact sequence*

$$\cdots \to H_c^{k-3}(U^*; Z_2) \to H_c^k(U^*; Z) \to H_c^k(U/N; Z) \to H_c^{k-2}(U^*; Z_2) \to \cdots .$$

By (4.4), G acts freely on U. Hence we have the desired exact sequence as seen in [3].

(4.10) $$H_c^k(U^*; Z) = \begin{cases} Z & \textit{for } k = n-3 , \\ 0 & \textit{otherwise.} \end{cases}$$

Since $\dim_Z U^* = n-3$, we have

$$H_c^k(U^*; Z) = 0 \quad \text{for } k > n-3 .$$

It follows from (4.9) and (4.8) that $H_c^{n-3}(U^*; Z_2) = H_c^{n-1}(U/N; Z) = Z_2$. From (4.9), it is easily seen that $H_c^{n-3}(U^*; Z) = Z \oplus I$, where $I = im(H_c^{n-6}(U^*; Z_2) \to H_c^{n-3}(U^*; Z))$ so that every element of I different from 0 is of order 2. By the universal coefficient theorem,

$$\begin{aligned} Z_2 = H_c^{n-3}(U^*; Z_2) &= H_c^{n-3}(U^*; Z) \otimes Z_2 \oplus \mathrm{Tor}(H^{n-2}(U^*; Z), Z_2) \\ &= Z_2 \oplus I . \end{aligned}$$

Hence $I = 0$, proving that

$$H_c^{n-3}(U^*; Z) = Z .$$

If $k < n-3$, then by (4.8) and (4.9), $H_c^k(U^*; Z) = H_c^{k-3}(U^*; Z_2)$. Hence for $k < n-3$,

$$H_c^k(U^*; Z) = 0 .$$

(4.11) *X^* is cohomologically trivial over Z.*

This is an easy consequence of (4.5), (4.10) and the cohomology sequence of (X^*, B^*).

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