APPROXIMATION OF FUNCTIONS ON THE INTEGERS

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How can algorithms be used to analyze nonrecursive functions? This question motivates the present work.

Let us suppose that a particular function, with natural numbers as arguments and values, is known to be completely defined but not recursive. Then by Church's thesis,[1] no algorithm gives the functional value for every argument. In some practical situation, however, where a particular sequence of arguments is of interest, it might suffice to have an "approximating algorithm" that performs as follows when applied to the successive arguments in the sequence: for each argument, the algorithm computes a number; for some arguments, this number may differ from the actual functional value, but after sufficiently many arguments have been processed, the proportion of such cases never exceeds a prescribed real number less than unity. If such an approximating algorithm exists whenever the given sequence of arguments is infinite, nonrepeating and effectively generable, then the given function is in some (conceivably useful) sense susceptible to analysis by mechanical means. Functions of this last kind are the object of our investigation; when the above notions are made precise in § 1, they are called "recursively approximable" functions.

In § 2 it is shown that uncountably many nonrecursive functions are recursively approximable; in § 3, that uncountably many functions are not recursively approximable.[2]

1. A number-theoretic notion of approximation. Given any function f, any partial function φ,[3] and any sequence $x_0, x_1, \cdots$ of natural numbers, let "err (n)" denote the number of natural numbers $i < n$ such that $f(x_i) \neq \varphi(x_i)$. If E is a real number and, for all sufficiently large n, err $(n)/n \leqq E$, then we say that φ *approximates*

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1 Cf [**3**].

2 An analogous notion of approximable function, involving finite sets of arguments rather than sequences, is considered in [**5**], where a function is called "m-in-n-computable" if there is an algorithm that produces at least m correct functional values for every set of n arguments. It was shown that uncountably many functions are not m-in-n computable for any $m > 0$. The existence of nonrecursive m-in-n-computable functions with $m > 0$ was left an open question; an affirmative answer, however, was soon provided by Dana Scott in an unpublished communication.

3 By "function" we mean, unless otherwise specified, "total singulary function" (in the sense of [**1**] p. xxi). A "partial function" is any singulary function whose domain is a subset of the natural numbers.

f with error E on the given sequence.

It may happen that, for every infinite nonrepeating recursive enumeration $x_0, x_1, \cdots$,[5] there is a partial recursive function φ that approximates f with error E on $x_0, x_1, \cdots$.[6] In this case we say that f is *recursively approximable with error E*. A function recursively approximable with some error <1 is called *recursively approximable.*

2. Recursively approximable functions. Are there recursively approximable functions other than the recursive functions? The Myhill-Friedberg notion of maximal set provides an affirmative answer through Theorem 2.1.[7] By Corollary 2.2 below, every recursion function f recursively approximates the uncountably many functions which agree with f on a maximal set.[8] In fact, we establish a stronger result as follows.

We consider an extension of the notion of maximal set. For convenience, a set C is called *cohesive* if it is infinite and, for every recursively enumerable set R, either $R \cap C$ or $\bar{R} \cap C$ is finite. A set is *quasi-maximal* if for some positive natural number m, its complement is the union of m cohesive sets. Thus the maximal sets are those quasi-maximal sets for which the number m can be taken as 1. Through Theorem 2.1, the notion of quasi-maximal set provides a sufficient condition that a function f be recursively approximable. This condition is that there exist a recursive function r that agrees on some quasi-maximal set with f.

THEOREM 2.1. *Let f be any function, r any recursive function, Q any quasi-maximal set such that f and r agree on Q. Then r recursively approximates f with arbitrary positive error on every infinite nonrepeating recursive enumeration.*

Proof. Assume that $\bar{Q} = C_1 \cup \cdots \cup C_m$ where the C's are cohesive sets. Let E be any positive real number and $x_0, x_1, \cdots$ any infinite nonrepeating recursive enumeration. Choose a natural number $p \geqq (m+1)/E$ and, for each natural number $j < p$, let $X_j = \{x_i \mid i \equiv j \bmod p\}$.

[4] In order to realize an approximating algorithm in the sense of the Introduction, it would be necessary to require that φ be defined for all x_i. It will be obvious, however, that the current results would be unaffected by this additional requirement.

[5] A "recursive enumeration" is any sequence $x(0), x(1), \cdots$ where x is a recursive function.

[6] Terminology regarding recursive functions and recursively enumerable sets is essentially that of [**3**]. However, "recursive" is used throughout for "general recursive," and the empty set is regarded as recursively enumerable.

[7] A set M is maximal if (i) $\bar{M}$ is infinite and (ii) for every recursively enumerable set R, either $R \cap \bar{M}$ or $\bar{R} \cap \bar{M}$ is finite. The existence of recursively enumerable maximal sets is established in [**2**].

[8] Functions r and f are said to agree on a set X if, for all $x \in X$, $r(x) = f(x)$.

For each $k\,(1 \leqq k \leqq m)$ we consider two cases.

Case 1. $\{x_0, x_1, \cdots\} \cap C_k$ is finite. Let q_k be the number of its members. Then the number of numbers $i < n$ such that $f(x_i) \neq r(x_i)$ and $x_i \in C_k$ is $\leqq q_k$.

Case 2. $\{x_0, x_1, \cdots\} \cap C_k$ is infinite. Then for some $j < p$, $X_j \cap C_k$ is infinite. Because C_k is cohesive and X_j is recursively enumerable, $\bar{X}_j \cap C_k$ is finite; let q_k be the number of its members. Now $\{x_i \mid f(x_i) \neq r(x_i)$ and $x_i \in C_k\} \subset C_k = (X_j \cap C_k) \cup (\bar{X}_j \cap C_k) \subset X_j \cup (\bar{X}_j \cap C_k)$. Therefore the number of numbers $i < n$ such that $f(x_i) \neq r(x_i)$ and $x_i \in C_k$ is $\leqq ((n-1)/p) + 1 + q_k$.

By hypothesis, if $f(x_i) \neq r(x_i)$ then $x_i \in C_1 \cup \cdots \cup C_m$. Hence $\mathrm{err}(n) \leqq m(n-1)/p + m + q_1 + \cdots + q_m$. Therefore

$$\mathrm{err}(n)/n \leqq m/p + (p(q_1 + \cdots + q_m) + mp - m)/(np)$$

and, for all $n \geqq p(q_1 + \cdots + q_m) + mp - m$, $\mathrm{err}(n)/n \leqq E$.

COROLLARY 2.2. *For every recursive function r, there are uncountably many functions f such that r recursively approximates f with arbitrary positive error on every infinite nonrepeating recursive enumeration.*

Proof. Given any recursive function r, choose any quasi-maximal set Q. For each subset S of $\bar{Q}$, let f_S be the function such that $f_S(x) = 1 \dot{-} r(x)$ if $x \in S$, $f_S(x) = r(x)$ otherwise. The functions f_S, being in one-to-one correspondence with the subsets of $\bar{Q}$, are uncountable.

For brevity, we will call a function "maximal" if it is not recursive and it agrees on some maximal set with some recursive function, "quasi-maximal" if it is not recursive and it agrees on some quasi-maximal set with some recursive function. In Theorem 2.1, the quasi-maximal functions were shown to be recursively approximable. By means of Theorems 2.5 and 2.6, we will show that there are uncountably many quasi-maximal functions, and consequently uncountably many recursively approximable functions, that are not maximal. For this purpose, let us define the *rank of a quasi-maximal set* Q to be the minimum number m such that $\bar{Q}$ is the union of m cohesive sets. Then define the *rank of a quasi-maximal function* f to be the minimum number m such that f agrees on some quasi-maximal set of rank m with some recursive function. Thus the maximal sets (functions) are the quasi-maximal sets (functions) of rank 1.

LEMMA 2.3. *If $C_1, \cdots C_m$ are cohesive sets, then every recursively enumerable subset of $C_1 \cup \cdots \cup C_m$ is finite.*

Proof. Assume that R is an infinite recursively enumerable subset of $C_1 \cup \cdots \cup C_m$. Then there is a recursive function r such that $r(0), r(1), \cdots$ enumerates R without repetition. Let $R_j = \{r(i) \mid i \equiv j \bmod m+1\}$ $(j = 0, \cdots, m)$. Then $R_0, \cdots, R_m$ are $m+1$ disjoint infinite recursively enumerable subsets of $C_1 \cup \cdots \cup C_m$. Hence at least two distinct R's, say R_j and R_k, have an infinite intersection with the same C_i. Since $R_k \subset \bar{R}_j$, it follows that $C_i \cap R_j$ and $C_i - R_j$ are infinite, contrary to the fact that C_i is cohesive.

LEMMA 2.4. *If Q and R are quasi-maximal sets and $R - Q$ is finite, then the rank of $Q \leqq$ rank of R.*

Proof. Let m be the rank of Q, n the rank of R. There are cohesive sets $D_1, \cdots, D_n$ such that

$$\bar{R} = D_1 \cup \cdots \cup D_n \,. \tag{2.1}$$

Then

$$\bar{Q} = (D_1 - Q) \cup \cdots \cup (D_n - Q) \cup (R - Q) \,. \tag{2.2}$$

Since $\bar{Q}$ is infinite and $R - Q$ is finite, at least one $D_i - Q$ is infinite. We may assume without loss of generality that the infinite sets $D_i - Q$ are $D_1 - Q, \cdots, D_h - Q$ where $1 \leqq h \leqq n$. Hence from (2.2)

$$\bar{Q} = ((D_1 - Q) \cup F) \cup \cdots \cup ((D_h - Q) \cup F) \tag{2.3}$$

where F is finite. For each i $(1 \leqq i \leqq h)$, $D_i - Q$, being an infinite subset of the cohesive set D_i, is obviously cohesive, hence $(D_i - Q) \cup F$ is cohesive. Since Q has rank m, it follows from (2.3) that $m \leqq h \leqq n$.

THEOREM 2.5. *For every natural number $m > 1$, there is a recursively enumerable quasi-maximal set of rank m. Hence there are infinitely many quasi-maximal sets that are not maximal.*

Proof. Choose a recursively enumerable maximal set Q_1 and let e be a recursive function such that $e(0), e(1), \cdots$ enumerates Q_1 without repetition. Define by induction on m the sets Q_m and C_m $(m = 1, 2, \cdots)$ thus.

$$C_1 = \bar{Q}_1; \text{ for all } m > 1,\ Q_m = e(Q_{m-1}) \text{ and } C_m = Q_{m-1} - Q_m \,.^9 \tag{2.4}$$

Clearly, each Q_m is recursively enumerable. By induction on m we establish the following properties of the Q's and C's. For all $m \geqq 1$,

$$Q_{m+1} \subset Q_m \,; \tag{2.5}$$

[9] For any function f and set X, we denote the image of X under the mapping f by "$f(X)$."

(2.6) $$C_{m+1} = e(C_m)\ ;$$

(2.7) $$C_m \text{ is cohesive;}$$

(2.8) $$\bar{Q}_m = C_1 \cup \cdots \cup C_m\ .$$

Basis. Let $m = 1$. Now $Q_2 = e(Q_1)$ and $Q_1 = e(N)$ where N is the set of all natural numbers. $Q_1 \subset N$; therefore (2.5) holds. Next, note that e is a one-to-one mapping. Hence $C_2 = e(N) - e(Q_1) = e(N - Q_1) = e(C_1)$; i.e. (2.6) holds. Because Q_1 is maximal, C_1 is cohesive; i.e. (2.7) holds. By (2.4), (2.8) holds.

Induction step. Let $m > 1$. By (2.4), $Q_{m+1} = e(Q_m)$ and $Q_m = e(Q_{m-1})$. By induction hypothesis, $Q_m \subset Q_{m-1}$. Therefore (2.5) holds. By (2.4), $C_{m+1} = e(Q_{m-1}) - e(Q_m) = e(Q_{m-1} - Q_m) = e(C_m)$; i.e. (2.6) holds. By induction hypothesis C_{m-1} is cohesive, hence infinite. Then by (2.6) C_m is infinite. Let R be any recursively enumerable set. The set $\{x \mid e(x) \in R\}$ (call it R') is recursively enumerable. In view of (2.6) and the fact that e is one-to-one, $C_m \cap R = e(C_{m-1} \cap R')$ and $C_m - R = e(C_{m-1} - R')$. Suppose that $C_m \cap R$ is infinite. Then $C_{m-1} \cap R'$ must be infinite. Then, because C_{m-1} is cohesive, $C_{m-1} - R'$ is finite, and consequently $C_m - R$ is finite. Thus (2.7) holds. Finally, in view of (2.5), $\bar{Q}_m = \bar{Q}_{m-1} \cup (Q_{m-1} - Q_m) = \bar{Q}_{m-1} \cup C_m$. Hence by induction hypothesis (2.8) holds.

Having established (2.5)–(2.8) we now show that, for all $m > 1$, Q_m has rank m. By (2.8) and (2.7), Q_m ash rank $\leqq m$. Let $D_1, \cdots, D_n$ be any cohesive sets such that $\bar{Q}_m = D_1 \cup \cdots \cup D_n$. By (2.8) each C_i has an infinite intersection with at least one D_k. Moreover, if $1 \leqq i < j \leqq m$, C_i and C_j cannot both have an infinite intersection with the same D_k. If they did, then by (2.8) $C_i \cap D_k \subset \bar{Q}_i \cap D_k$ and, by (2.4) and (2.5), $C_j \cap D_k \subset Q_{j-1} \cap D_k \subset Q_i \cap D_k$; then $Q_i \cap D_k$ and $\bar{Q}_i \cap D_k$ would both be infinite, contrary to the fact that Q_i is recursively enumerable and D_k is cohesive. Thus for each i between 1 and m there must be a distinct k between 1 and n. Therefore $n \geqq m$. We conclude that Q_m has rank m.

THEOREM 2.6. *For every natural number $m > 1$, there are uncountably many quasi-maximal functions of rank m. Hence there are uncountably many quasi-maximal functions that are not maximal.*

Proof. By Theorem 2.5 there is a recursively enumerable quasi-maximal set Q of rank m. For each of the uncountably many subsets S of $\bar{Q}$ let f_S be the function such that

$$f_S(x) = \begin{cases} 0 \text{ if } x \in Q, \\ 1 \text{ if } x \in S, \\ 2 \text{ otherwise.} \end{cases}$$

There are uncountably many functions f_S since they are in one-to-one correspondence with the sets S. If f_S were recursive, then $\bar{Q}$ would be the infinite recursively enumerable set $\{x \mid f_S(x) \neq 0\}$, contrary to Lemma 2.3. Hence each f_S is nonrecursive. Therefore, since f_S agrees on Q with the constant function 0, f_S is a quasi-maximal function of rank $\leqq m$.

Moreover, consider any quasi-maximal set R and any recursive function r such that f_S agrees on R with r. Now Q and $\{x \mid r(x) \neq 0\}$ are recursively enumerable and $\{x \mid r(x) \neq 0\} \cap Q \subset \bar{R}$. Hence $\{x \mid r(x) \neq 0\} \cap Q$, being a recursively enumerable subset of $\bar{R}$, is finite by Lemma 2.3. Hence $\{x \mid r(x) \neq 0\} - Q$, which $= \{x \mid r(x) \neq 0\} - (\{x \mid r(x) \neq 0\} \cap Q)$, is a recursively enumerable subset of $\bar{Q}$. Hence by Lemma 2.3 $\{x \mid r(x) \neq 0\} - Q$, is finite. Hence $R - Q$, which $\subset \{x \mid r(x) \neq 0\} - Q$, is finite. Hence by Lemma 2.4 R has rank $\geqq m$. Therefore f_S has rank m.

3. Functions that are not recursively approximable. It will now be shown that not every function is recursively approximable. That is to say, there are functions f with the following property: there is an infinite nonrepeating recursive enumeration $x_0, x_1, \cdots$ such that, for every real number $E < 1$ and every partial recursive function φ, φ does not approximate f with error E on $x_0, x_1, \cdots$.

Let us call a function f *constructively nonrecursive* if there is a recursive function g such that, for all natural numbers $e, f(g(e)) \neq \{e\}(g(e))$.[10] In view of Theorems 3.1 and 3.2, the constructively nonrecursive functions form an uncountably infinite subclass of the functions that are not recursively approximable.

THEOREM 3.1. *If a function is constructively nonrecursive, then it is not recursively approximable.*

Proof. Let f be any constructively nonrecursive function and g a recursive function such that, for all e,

$$f(g(e)) \neq \{e\}\,(g(e))\,. \tag{3.1}$$

First we will exhibit a recursive binary function c such that, for all i and e,

[10] For any $n \geqq 1$ and any $e, x_1, \cdots, x_n$, "$\{e\}\,(x_1, \cdots, x_n)$" denotes the ambiguous value $\varphi(x_1, \cdots, x_n)$ of the partial recursive n-ary function φ whose Gödel number is e. (Cf. [3], p. 340.)

$$c(i, e) > i \, , \tag{3.2}$$

$$f(c(i, e)) \neq \{e\}(c(i, e)) \, . \tag{3.3}$$

For this purpose, let ψ be the partial recursive quaternary function defined by

$$\psi(z, i, e, x) \simeq \begin{cases} (z)_x \text{ if } x \leqq i \, ,[11] \\ \{e\}(x) \text{ otherwise.} \end{cases} \tag{3.4}$$

Then there is a primitive recursive ternary function a such that

$$\{a(z, i, e)\}(x) \simeq \psi(z, i, e, x) \, .[12] \tag{3.5}$$

Now for any natural numbers i and e, let z be the number $\Pi_{j \leqq i} p_j^{f(j)}$.[13] Because a and g are completely defined, $g(a(z, i, e))$ is defined. Hence either $g(a(z, i, e)) \leqq i$ or $g(a(z, i, e)) > i$. But $g(a(z, i, e))$ cannot be $\leqq i$, for in that case

$$\begin{aligned} \{a(z, i, e)\}(g(a(z, i, e))) &\simeq [\text{by } (3.5)]\ \psi(z, i, e, g(a(z, i, e,))) \\ &\simeq [\text{by } (3.4)]\ (z)_{g(a(z,i,e))} \simeq f(g(a(z, i, e))) \, , \end{aligned}$$

contrary to (3.1). Hence $g(a(z, i, e)) > i$. Therefore $\mu z(g(a(z, i, e)) > i)$ is a recursive function of i and e. It now follows that (3.6) and (3.7) define b and c as recursive binary functions.

$$b(i, e) = \mu z(g(a(z, i, e)) > i) \, , \tag{3.6}$$

$$c(i, e) = g(a(b(i, e), i, e)) \, . \tag{3.7}$$

By (3.6) and (3,7), (3.2) holds. Now for any natural numbers i and e, assume that $f(c(i, e)) = \{e\}(c(i, e))$. Then $\{e\}(c(i, e))$ is defined and $f(g(a(b(i, e), i, e))) = [\text{by } (3.7)] f(c(i, e)) = \{e\}(c(i, e)) = [\text{by } (3.2) \text{ and } (3.4)]$ $\psi(b(i, e), i, e, c(i, e)) = [\text{by } (3.5)] \{a(b(i, e), i, e)\}(c(i, e)) = [\text{by } (3.7)] \{a(b(i, e), i, e)\}(g(a(b(i, e), i, e)))$, contrary to (3.1). Therefore (3.3) follows by contradiction.

Next, define the primitive recursive functions d and e and the recursive function x thus.

$$d(i) = \mu j((j + 1)! > i) \, ; \tag{3.8}$$

$$e(i) = d(i) \dot{-} (d(d(i)))! \, ; \tag{3.9}$$

$$x(i) = \begin{cases} 0 \text{ if } i = 0 \, , \\ c(x(i \dot{-} 1), e(i)) \text{ otherwise.} \end{cases} \tag{3.10}$$

By (3.2), $x(0) < x(1) < \cdots$, so that $x(0), x(1), \cdots$ is an infinite non-repeating recursive enumeration. We now show that, for any real

[11] For the notation $(z)_x$, cf. [**3**], p. 230.

[12] Cf. [**3**], § 65, Theorem XXIII.

[13] For the notation p_j, cf. [**3**], p. 230.

number $E < 1$ and any partial recursive function φ, φ does not approximate f with error E on $x(0), x(1), \cdots$, thereby proving that f is not recursively approximable. The proof is by contradiction. Thus, assume that φ approximates f with error E on $x(0), x(1), \cdots$. Then there is a natural number N such that

$$\text{for all } n > N, \ \text{err}\,(n)/n \leqq E\,. \tag{3.11}$$

Choose any Gödel number t of φ and let k be a natural number $>$ $\max(N, t, 1/(1-E))$. Then for the $(k! + t + 1)! - (k! + t)!$ natural numbers i such that $(k! + t)! \leqq i < (k! + t + 1)!$, $e(i) = t$; hence by (3.10) and (3.3) $f(x(i)) \neq \{t\}(x(i))$. Therefore $(k! + t + 1)! > N$ and $\text{err}\,((k!+t+1)!)/(k!+t+1)! \geqq 1-(k!+t)!/(k!+t+1)! = 1-1/(k!+t+1) > E$, contrary to (3.11).

For Theorem 3.2, we use the following notation from [4]. For any natural number e, "W_e" denotes the set of all numbers y such that, for some x, $\{e\}(x) = y$. A set P is *productive* if and only if there is a partial recursive function ψ such that, for all e, if $W_e \subset P$ then $\psi(e) \in P - W_e$.

THEOREM 3.2. *The representing function of any productive set is constructively nonrecursive. Hence uncountably many functions are not recursively approximable.*

Proof. Given any productive set P, let f be the function such that

$$f(x) = \begin{cases} 0 \text{ if } x \in P\,, \\ 1 \text{ otherwise.} \end{cases} \tag{3.12}$$

Myhill has shown that there is a recursive function g such that, for all natural numbers e,

$$g(e) \in (P - W_e) \cup (W_e - P)\,.\text{[14]} \tag{3.13}$$

Moreover there is a recursive function h such that, for all natural numbers e,

$$W_{h(e)} = \{y \mid \{e\}(y) = 0\}\,. \tag{3.14}$$

(For example, we can take for h the primitive recursive function $\Lambda x\, \mu y(y \geqq x \& \{e\}(y) = 0)$. For the Λ-notation, cf. [3], § 65.) Now let e be any natural number. By (3.12) $f(g(h(e))) = 0$ if and only if $g(h(e)) \in P$; hence by (3.13) if and only if $g(h(e)) \notin W_{h(e)}$; hence by (3.14) if and only if $\{e\}(g(h(e))) \neq 0$. Thus $g(h)$ is a recursive function such that, for all e, $f(g(h(e))) \neq \{e\}(g(h(e)))$. Therefore f is constructively

[14] Cf. [4], § 3.153.

nonrecursive.

It now follows from Theorem 3.1 that the representing functions of productive sets are not recursively approximable. Moreover, by [4], p. 47, there are uncountably many productive sets. Hence uncountably many functions are not recursively approximable.

REMARK. The proof of Theorem 3.2 can readily be generalized to show that a function f is constructively nonrecursive if there is a recursively enumerable set A and a productive set P such that $f(x) \in A$ if and only if $x \in P$.

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