ON DENSITIES OF SETS OF LATTICE POINTS

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1. Introduction. Let $A$ be a set of positive integers, and for any positive integer $x$ denote by $A(x)$ the number of integers of $A$ which are not greater than $x$. Then the Schnirelmann density of $A$ is defined [4] to be the quantity

$$\alpha = \operatorname{glb} \frac{A(x)}{x}.$$  

For any $k$ sets $A_1, \ldots, A_k$ of positive integers, $k \geq 2$, let the sum set $A_1 + \cdots + A_k$ be the set of all nonzero sums $a_1 + \cdots + a_k$ for which each $a_i, i = 1, \ldots, k$, is either contained in $A_i$ or is 0. Let $kA$ be the set $A_1 + \cdots + A_k$ with $k$ summands.

Schnirelmann [4] and Landau [2] have shown that if $A$ and $B$ are two sets of positive integers with $C = A + B$, and if $\alpha, \beta, \gamma$ are the Schnirelmann densities of $A, B, C$, respectively, then $\gamma \geq \alpha + \beta - \alpha \beta$, and if $\alpha + \beta \geq 1$ then $\gamma = 1$. They have also shown that if $A$ is a set of positive integers whose Schnirelmann density is positive then $A$ is a basic sequence for the set of positive integers, or, in other words, there exists a positive integer $k$ such that every positive integer can be written as the sum of at most $k$ elements of $A$.

We will show that by using extensions of the methods employed by Schnirelmann and Landau the above results can be generalized to certain sets of vectors in a discrete lattice (for definition and discussion see [3, pp. 28–31] or [5, pp. 141–145]). Without loss of generality it may be assumed that the components of the vectors in such a lattice are rational integers. The usual identification of algebraic integers with lattice points then gives an immediate extension of these results to algebraic integers.

2. Notation and definitions. Let $Q_n$ be the set of all $n$-dimensional lattice points $(x_1, \ldots, x_n)$, $n \geq 1$, for which each $x_i, i = 1, \ldots, n$, is a nonnegative integer and at least one $x_i$ is positive. Define the sum of subsets of $Q_n$ in the same manner as was done for sets of positive integers, and for any subsets $A$ and $B$ of $Q_n$ let $A - B$ denote the set of all elements of $A$ which are not in $B$. If $A$ and $S$ are subsets of $Q_n$ and $S$ is finite let $A(S)$ be the number of elements in $A \cap S$.

**Definition 1.** A finite nonempty subset $R$ of $Q_n$ will be called a
**fundamental subset** of $Q_n$ or, briefly, a **fundamental set**, if whenever an element $(r_1, \cdots, r_n)$ is in $R$ then all elements $(x_1, \cdots, x_n)$ of $Q_n$ such that $x_i \leq r_i$, $i = 1, \cdots, n$, are also in $R$.

**DEFINITION 2.** Let $A$ be any subset of $Q_n$. The **density** of $A$ is defined to be the quantity

$$
\alpha = \frac{\text{glb } A(R)}{Q_n(R)}
$$

taken over all fundamental sets $R$.

3. **Extension of the Landau-Schnirelmann results.** Throughout this section we let $A$ and $B$ be subsets of $Q_n$ with $C = A + B$, and let $\alpha, \beta, \gamma$ be the densities of $A$, $B$, $C$, respectively.

**THEOREM 1.** If $\alpha + \beta \geq 1$ then $\gamma = 1$.

**Proof.** Assume $\gamma < 1$. Then there exists a fundamental set $R$ for which $C(R) < Q_n(R)$, which in turn implies that there exists an element $(x_{i1}^0, \cdots, x_{in}^0)$ in $Q_n - C$. Let $R_\alpha$ be the set of all elements $(x_1, \cdots, x_n)$ in $Q_n$ for which $x_i \leq x_{i0}^0$, $i = 1, \cdots, n$. Then for any $(x_1, \cdots, x_n)$ in $R_\alpha$ either $(x_1, \cdots, x_n)$ is in $A$, or $(x_1, \cdots, x_n) = (x_{i1}^0, \cdots, x_{in}^0) - (b_1, \cdots, b_n)$ for some $(b_1, \cdots, b_n)$ in $B \cap R_\alpha$, or neither, but not both. In particular, $(x_{i1}^0, \cdots, x_{in}^0)$ is neither. Hence,

$$
A(R_\alpha) + B(R_\alpha) \leq Q_n(R_\alpha) - 1
$$

and

$$
\alpha + \beta \leq \frac{A(R_\alpha) + B(R_\alpha)}{Q_n(R_\alpha)} < 1
$$

which is a contradiction. Therefore $\gamma = 1$.

**THEOREM 2.** $\gamma \geq \alpha + \beta - \alpha \beta$.

**Proof.** Let $\omega_1, 1 \leq i \leq n$, be that vector in $Q_n$ for which the $i$th component is 1 and the other components, if any, are 0. If any one of the vectors $\omega_1, \cdots, \omega_n$ is missing from $A$ then $\alpha = 0$ and the theorem is trivial. Hence we assume all the vectors $\omega_1, \cdots, \omega_n$ are in $A$. We must show

$$
(1) \quad \frac{C(R)}{Q_n(R)} \geq \alpha + \beta - \alpha \beta
$$

for all fundamental sets $R$. If $C(R) = Q_n(R)$ then (1) holds, since
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\((1 - \alpha)(1 - \beta) \geq 0\) implies \(1 \geq \alpha + \beta - \alpha\beta\). Therefore we assume \(C(R) < Q_n(R)\) and, consequently, \(A(R) < Q_n(R)\).

Let \(H = R - A\). We will show that there exist vectors \(a^{(1)}, \ldots, a^{(s)}\) in \(A\) and sets \(L_1, \ldots, L_s\) with the following properties.

(i) \(L_i \subseteq H\) and \(L_i\) is not empty, \(i = 1, \ldots, s\).

(ii) The sets \(L'_i = \{x - \alpha^{(i)} | x \in L_i\}\) are fundamental sets.

(iii) \(L_i \cap L_j = \emptyset\) for \(i \neq j\).

(iv) \(H = L_1 \cup \cdots \cup L_s\).

Let the elements of \(R\) be ordered so that \((x_1, \ldots, x_n) > (x'_1, \ldots, x'_n)\) if \(x_i > x'_i\) or if \(x_i = x'_i, \ldots, x_p = x'_p, x_{p+1} > x'_{p+1}\). For every \(h = (h_1, \ldots, h_n)\) in \(H\), let \(A_h\) be the set of all \((a_1, \ldots, a_n)\) in \(A\) such that each \(a_i \leq h_i\). The sets \(A_h\) are finite sets, hence they contain (in our ordering) a largest vector. Let \(a^{(1)}, \ldots, a^{(s)}\) be all the distinct vectors that are largest vectors in any \(A_h\). Let \(L_i\) be the set of all vectors \(x\) in \(H\) such that \(a^{(i)}\) is the largest vector in \(A_x\).

That (i), (iii), and (iv) are satisfied follows immediately from this definition of the \(L_i\). To prove (ii) consider a vector \(y = (y_1, \ldots, y_n)\) such that

\[(2) \quad x_j \geq y_j \geq a^{(i)}_j,\]

where \(x = (x_1, \ldots, x_n)\) is in \(L_i\) and \(y \neq a^{(i)}\). Suppose \(y \in L_k, k \neq i\). Then

\[(3) \quad x_j \geq y_j \geq a^{(k)}_j\]

and \(a^{(k)} \geq a^{(i)}\). But (2) and (3) and \(x \in L_i\) imply \(a^{(k)} \leq a^{(i)}\), hence \(a^{(k)} = a^{(i)}\). Similarly, \(y \in A\) implies \(y = a^{(i)}\). This proves (ii).

If \(b \in B \cap L_i\) then \(a^{(i)} + b\) is in \(C \cap L_i\), hence in \(C - A\). Therefore,

\[
C(R) \geq A(R) + B(L'_i) + \cdots + B(L'_s)
\geq A(R) + \beta[Q_n(L'_i) + \cdots + Q_n(L'_s)]
= A(R) + \beta[Q_n(L_i) + \cdots + Q_n(L_s)]
= A(R) + \beta[Q_n(H)]
= A(R) + \beta[Q_n(R) - A(R)]
= (1 - \beta)A(R) + \beta[Q_n(R)]
\geq (1 - \beta)\alpha[Q_n(R)] + \beta[Q_n(R)],
\]

and

\[
\frac{C(R)}{Q_n(R)} \geq \alpha + \beta - \alpha\beta,
\]

which completes the proof.
Corollary 1. Let $A_1, \ldots, A_k$ be any $k$ subsets of $\mathbb{Q}_n$, $k \geq 2$, let $\alpha_i$ be the density of $A_i$ for $i = 1, \ldots, k$, and let $d(A_1 + \cdots + A_k)$ be the density of $A_1 + \cdots + A_k$. Then

$$1 - d(A_1 + \cdots + A_k) \leq (1 - \alpha_1) \cdots (1 - \alpha_k).$$

Proof. If $k = 2$ then Theorem 2 implies that $1 - d(A_1 + A_2) \leq 1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2 (1 - \alpha_1) (1 - \alpha_2)$. Hence assume $1 - d(A_1 + \cdots + A_{k-1}) \leq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})$. Then

$$1 - d(A_1 + \cdots + A_{k-1} + A_k) \leq [1 - d(A_1 + \cdots + A_{k-1})] (1 - \alpha_k) \leq (1 - \alpha_1) \cdots (1 - \alpha_{k-1}) (1 - \alpha_k).$$

Corollary 2. If $A$ is any subset of $\mathbb{Q}_n$ with density $\alpha > 0$ then there exists an integer $k > 0$ such that $kA = \mathbb{Q}_n$.

Proof. There exists an integer $m > 0$ such that $(1 - \alpha)^m \leq 1/2$. Let $d(mA)$ be the density of $mA$. Then Corollary 1 implies that $1 - d(mA) \leq (1 - \alpha)^m \leq 1/2$, or $d(mA) \geq 1/2$. From Theorem 1, $d(mA) + d(mA) \geq 1$ implies $d(2mA) = 1$, or $2mA = \mathbb{Q}_n$.

4. Remark. We may identify $\mathbb{Q}_2$ with the set of nonzero Gaussian integers $x + yi$ for which $x$ and $y$ are both nonnegative rational integers. Luther Cheo [1] defined density for subsets of this $\mathbb{Q}_2$ as follows, using our notation.

Definition 3. Let $x_0 + y_0 i$ be any element of $\mathbb{Q}_2$ and $S$ the set of all $x + yi$ in $\mathbb{Q}_2$ such that $x \leq x_0$ and $y \leq y_0$. Then for any subset $A$ of $\mathbb{Q}_2$ the density of $A$ is the quantity

$$\alpha_e = \frac{\text{glb} A(S)}{Q_0(S)}.$$

Cheo proved Theorem 1 for his density and also a theorem which implies that if $ji$ is in $A$ for all $j = 1, 2, \ldots$, and if $\alpha_e, \beta_e, \gamma_e$ are the Cheo densities of $A, B, C = A + B$, respectively, then

$$\gamma_e \geq \alpha_e + \beta_e - \alpha_e \beta_e.$$

We cannot remove the requirement that all $ji$ be in $A$ by means of an argument like that used to establish Theorem 2 since it would be necessary to partition $H$ in such a way that the sets $L_i'$ are of the type $S$ used in defining the Cheo density, and this is not always possible. Consider, for example, the set $R = \{x + yi: x + yi \text{ is in } \mathbb{Q}_2, x \leq 4, y \leq 3\}$, and let $A \cap R = \{1, i, 3 + 3i\}$. Then $H = R - A$ cannot be so partitioned, as the reader can easily verify.
REFERENCES


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