DIVISORIAL VARIETIES

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Introduction. The purpose of the present work is to introduce a new type of algebraic varieties, called Divisorial varieties. The name comes from the fact that the topology of these varieties is determined by their positive divisors. See §3 for a more detailed discussion of the above statement.

In the first two sections we lay the groundwork for our study. The result obtained in Proposition 2.2 is new, and constitutes a natural generalization of a well known result of Serre. (See [3], page 235, and Lemma 2, page 98 of [5]).

Section 3 is devoted to the study of the categorical properties of divisorial varieties. We prove that locally closed subvarieties of divisorial varieties are divisorial, and that products and direct sums of divisorial varieties are divisorial. Furthermore we give a characterization of divisorial varieties which shows how such varieties are a natural generalization of the notion of projective varieties.

We show in §4 that all quasi-projective, and all nonsingular varieties are divisorial. A procedure is also given for constructing a large class of divisorial varieties which are neither quasi-projective nor nonsingular, both reducible and irreducible ones.

In §5 we study the additive group of equivalence classes (under linear equivalence) of locally linearly equivalent to zero divisors of a divisorial variety. We show that such group is generated by the semigroup of those classes which contain some positive members. As a matter of fact the statement of Corollary 5.1 is more general than the one above, but we omit the details here for brevity's sake. The results of §5 are a generalization of the operation of "adding hypersurface sections," well known to the Italian geometers for projective varieties.

Finally, in §6, we give one instance of a theorem which is known to be true for either quasi-projective or irreducible and nonsingular varieties, and show that it holds for divisorial varieties. The theorem considered, which we refer to as the polynomial theorem of Snapper, is Theorem 9.1 of [6], generalized by Cartier (See [1]) to either quasi-projective or irreducible and nonsingular varieties.

We believe that the notion of divisorial varieties represents a natural extension of the notion of quasi-projective varieties.

Our notation and terminology are essentially those of [3]. The word sheaf always means, unless other-wise specified, algebraic coherent

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sheaf. The symbol $\simeq$ is used to denote all sorts of isomorphisms, and the type has not been specified, unless there is danger of confusion. Whenever the expression $a \otimes a \otimes \cdots \otimes a$, m times, is meaningful, we shall denote it by $a^{(m)}$. When we refer to, say, Theorem 3.2, without any further designation, we mean Theorem 3.2 of the present work, to be found as the second theorem of the third section.

1. We wish to review briefly some of the ideas and theorems concerning the functorial properties of line classes; for a more detailed treatment see [6], §§1 to 5, and [7].

Let X denote an abstract algebraic variety, defined over an algebraically closed groundfield k. Let $\mathcal{O}_X$ denote the sheaf of local rings of X, and $\mathcal{O}_X^0$ the sheaf (not algebraic) of units of $\mathcal{O}_X$. The elements of the (multiplicative) first cohomology group $H^1(X, \mathcal{O}_X^0)$ are called the line classes of X.

Let $f \in H^1(X, \mathcal{O}_X^0)$ and let $\mathcal{U} = (U_i, I)$ be an indexed open covering of X which admits a 1-cocycle b with values in $\mathcal{O}_X^0$ which represents f. We shall briefly say that the system $(\mathcal{U}, b)$ represents f.

If F is an algebraic sheaf over X, there exists a uniquely defined (up to $\mathcal{O}_X$-isomorphisms) algebraic sheaf K, and local isomorphisms u_i: $K \mid U_i \to F \mid U_i$, such that, for every $x \in U_i \cap U_j$ and $a \in F_x$,

$$(u_i u_j^{-1})(a) = [b(i, j)(x)] \cdot a \ .$$

The sheaf K depends only upon F and f, while, of course, the local isomorphisms u_i depend upon the choice of the system $(\mathcal{U}, b)$. We denote the sheaf K by $f(F)$.

In this way f can be looked upon as a functor from the category of (classes of $\mathcal{O}_X$-isomorphic) algebraic sheaves and (classes of equivalent) $\mathcal{O}_X$-homomorphisms into the same category. Such functor is covariant and exact. Furthemore, if F and G are two algebraic sheaves over X, and f and g are two line classes of X, then

$$f(F) \otimes \sigma_X g(G) \simeq fg(F \otimes \sigma_X G) \ ,$$

where fg denotes the product in the group of line classes.

Since F and $f(F)$ are locally isomorphic sheaves, if F is of finite type or coherent so is $f(F)$, and conversely. Furthermore the stalk of $f(\mathcal{O}_X)$ over any point $x \in X$ has a unique maximal submodule, which we shall denote by n_x, corresponding to the unique maximal ideal m_x of $\mathcal{O}_{x,X}$.

2. **Sections of $f(\mathcal{O}_X)$.** We shall keep the same notation as in the previous section. Furthermore, for every sheaf F over X, and any subset U of X, we shall denote by $\Gamma(U, F)$ the set of sections of F over U.

PROPOSITION 2.1. Let X be an abstract algebraic variety, f a line class of X, and $s \in \Gamma[X, f(\mathcal{O}_X)]$. Then the set

$$X_s = \{x \in X \mid s(x) \notin n_x\}$$

is an open subset of X.

Proof. Let $(\mathcal{U}, b)$ be a system representing f, where $\mathcal{U} = (U_i, I)$. Let $u_i: f(\mathcal{O}_X) \mid U_i \to \mathcal{O}_X \mid U_i$ be the local isomorphisms as in §1. If $x \in U_i$ then, by the definition of n_x, $x \in X_s \cap U_i$ if, and only if,

$$(u_i \circ s)(x) \notin m_x$$

or, equivalently, if, and only if, $(u_i \circ s)(x) \in \mathcal{O}_X^\circ$, X. Since $\mathcal{O}_X^\circ$ is open in $\mathcal{O}_X$, and since the u_i is a local homeomorphism, $X_s \cap U_i$ is open in U_i. This proves the proposition.

The following proposition generalizes Proposition 5 of §43 of [3], as well as Lemma 2, page 98 of [5].

PROPOSITION 2.2. Let X be an abstract algebraic variety, f and g two line classes of X, U an open subset of X. Let $s \in \Gamma[X, g(\mathcal{O}_X)]$ and $t \in \Gamma[U, f(\mathcal{O}_X)]$ be given, such that $X_s \subset U$. Then, for a sufficiently high integer n, there exists a section $s^* \in \Gamma[X, fg^n(\mathcal{O}_X)]$, such that $s^* = t \otimes s^{(n)}$ on X_s.

Proof. Let $\mathcal{U} = (U_i, I)$, $\mathcal{W} = (V_\alpha, A)$ be open affine coverings of X which admit 1-cocycles with values in $\mathcal{O}_X^\circ$ representing f and g respectively. We may assume that $\mathcal{W}$ is a refinement of $\mathcal{U}$. Let u_i and v_α be the usual local isomorphisms. Since $X_s \subset U$ we have that $t \in \Gamma[X_s, f(\mathcal{O}_X)]$. Let:

$$t_\alpha = v_\alpha^{-1}(1) \quad ; \quad f_\alpha = v_\alpha \circ t$$
$$s_i = u_j^{-1}(1) \quad ; \quad g_i = u_i \circ s$$

Let $V_\alpha \subset U_i$. Observe that g_i is regular on U_i, and that

$$X_s \cap U_i = \{x \in U_i \mid g_i(x) \notin m_x\} .$$

Since V_α is affine and f_α is defined on $X_s \cap V_\alpha$, we see by Lemma 1 of §55 of [3] that there exists a sufficiently large integer m_α and a section

$$h_\alpha \in \Gamma(V_\alpha, \mathcal{O}_X)$$

such that

$$h_\alpha = f_\alpha \cdot g_i^{m_\alpha}$$

on $V_\alpha \cap X_s$. Since X is compact (we do not include T_2 in the definition of compactness) we may assume all m_α's to be equal, and denote their common value by m.

Let now $s'_\alpha \in \Gamma[V_\alpha, fg^m(\mathcal{O}_X)]$ be defined as follows:

$$s'_\alpha = h_\alpha \cdot (t_\alpha \otimes s_i^{(m)}) .$$

Clearly $s'_\alpha - s'_\beta = 0$ on $V_\alpha \cap V_\beta \cap X_s$. Hence, since g_i is regular on $V_\alpha \cap V_\beta$, we have that the section

$$(s'_\alpha - s'_\beta) \otimes s \in \Gamma[V_\alpha \cap V_\beta, fg^{m+1}(\mathcal{O}_X)]$$

is 0 on $V_\alpha \cap V_\beta$. Therefore the system of sections $s'_\alpha \otimes s$ defines a unique section s^* of $fg^{m+1}(\mathcal{O}_X)$ over X. On $V_\alpha \cap X_s$ we have:

$$\begin{aligned} s^* &= s'_\alpha \otimes s = h_\alpha \cdot t_\alpha \otimes s_i^{(m)} \otimes s \\ &= g_i^m \cdot f_\alpha \cdot t_\alpha \otimes s_i^{(m)} \otimes s = t \otimes s^{(m+1)} \end{aligned}$$

which finishes the proof.

COROLLARY 2.1. *Let X be an abstract algebraic variety, g a line class of X, $s \in \Gamma[X, g(\mathcal{O}_X)]$, h a regular function on X_s. Then, for a sufficiently high integer n, the section $h \cdot s^{(n)}$ can be extended to X.*

Proof. Let $f = 1$, $t = h$, $U = X_s$ in the above proposition.

REMARK. Let X be an irreducible, normal algebraic variety, with constant sheaf (not coherent) of rational functions denoted by E. There exists a group isomorphism between the multiplicative group $H^1(X, \mathcal{O}_X^\circ)$ and the additive group of equivalence classes (under linear equivalence) of locally linearly equivalent to zero divisors of X. If g is a line class of X shall denote by $|g|$ the equivalence class of divisors which corresponds to it. Then there exists an isomorphism between $\Gamma[X, g(E)]$ and $|g|$. Sections of $g(\mathcal{O}_X)$ over X correspond to the positive members of $|g|$. See [6], §5 for the proof of the above statements.

The geometrical meaning of Proposition 2.2 is then the following: if D is a locally linearly equivalent to zero divisor of X, such that the variety of its negative components is contained in the variety of some positive divisor (also locally linearly equivalent to zero), say P, then, for a sufficiently high integer n the divisor $D + nP$ is locally linearly equivalent to zero and positive.

PROPOSITION 2.3. Let X be an abstract algebraic variety, Y a locally closed subvariety of X. Then there exists a homomorphism

$$\varphi_n\colon H^n(X, \mathcal{O}_X^\circ) \longrightarrow H^n(Y, \mathcal{O}_Y^\circ) , \qquad n = 0, 1, \cdots$$

and, for every $f \in H^1(X, \mathscr{O}_X^\circ)$, there exists a homomorphism

$$\varphi_f\colon \Gamma[X, f(\mathscr{O}_X)] \longrightarrow \Gamma[Y, \varphi_1(f)(\mathscr{O}_Y)]$$

such that, for every $s \in \Gamma[X, f(\mathscr{O}_X]$,

$$Y \cap X_s = Y_{\varphi_f(s)} .$$

Proof. There exists a unitary ring epimorphism $\varphi\colon \mathscr{O}_X \mid Y \to \mathscr{O}_Y$, hence a sheaf homomorphism $\varphi'\colon \mathscr{O}_X^\circ \mid Y \to \mathscr{O}_Y^\circ$. This proves the existence of the homomorphisms φ_n.

Let now $f \in H^1(X, \mathscr{O}_X^\circ)$. Let $(\mathscr{U}, b)$ be a system which represents f, where $\mathscr{U} = (U_i, I)$. The system $(\mathscr{U}', b')$, where $\mathscr{U}' = (U_i \cap Y, I)$ and $b'(i, j) = \varphi' \circ b(i, j)$, represents $\varphi_1(f)$. Let

$$u_i\colon f(\mathscr{O}_X) \mid U_i \longrightarrow \mathscr{O}_X \mid U_i$$
$$u_i'\colon \varphi_1(f)(\mathscr{O}_Y) \mid Y \cap U_i \longrightarrow \mathscr{O}_Y \mid Y \cap U_i$$

be the usual local isomorphisms. Let $s \in \Gamma[X, f(\mathscr{O}_X)]$. We define $\varphi_f(s)$ by the formula:

$$\varphi_f(s)(x) = (u_i'^{-1} \circ \varphi \circ u_i \circ s)(x) \qquad x \in Y \cap U_i .$$

We easily verify that $\varphi_f(s)(x)$ does not depend on the index i. We now assert that $\varphi_f(s)$ does not depend on the particular system $(\mathscr{U}, b)$ chosen to represent f. Let therefore $(\mathscr{W}, c)$ be another such system, where $\mathscr{W} = (V_j, J)$. We proceed in steps.

Case 1. $\mathscr{W}$ is a refinement of $\mathscr{U}$, the mapping $t\colon J \to I$ is such that $c = t^*(b)$. From [6], Case 1 of Proposition 2.1 we know that the usual isomorphisms $v_j\colon f(\mathscr{O}_X) \mid V_j \to \mathscr{O}_X \mid V_j$ can be chosen in such a manner that $u_{t(j)} = v_j$ on V_j. The system $(\mathscr{W}', c')$, where $\mathscr{W}' = (Y \cap V_j, J)$ and $c'(j, j') = \varphi' \circ c(j, j')$, clearly represents $\varphi_1(f)$, hence we can furthermore choose the isomorphisms

$$v_j'\colon \varphi_1(f)(\mathscr{O}_Y) \mid Y \cap V_j \to \mathscr{O}_Y \mid Y \cap V_j$$

in such a manner that $u_t'(j) = v_j'$ on $Y \cap V_j$.

Hence, if $x \in Y \cap V_j$ we have

$$(u_{t(j)}'^{-1} \circ \varphi \circ u_{t(j)} \circ s)(x) = (v_j'^{-1} \circ \varphi \circ v_j \circ s)(x)$$

which finishes the proof of Case 1.

Case 2. $\mathscr{U} = \mathscr{W}$, b and c cohomologous. Hence there exists a 0-cochain e of $\mathscr{U}$, with values in $\mathscr{O}_X^\circ$, such that $b^{-1}c$ is the coboundary of e. We can hence choose the isomorphisms v_i in such a manner that, if $x \in U_i$, then $u_i = e(i)(x) \cdot v_i$, on the stalk of $f(\mathscr{O}_X)$ over x. Let

$e' = \varphi_0(e)$. Then it is easily seen that $b'^{-1}c'$ is the coboundary of e', hence, if $x \in Y \cap U_i$, $u'_i = e'(i)(x) \cdot v'_i$, on the stalk of $\varphi_1(f)(\mathcal{O}_Y)$ over x. Hence we have that $v'^{-1}_i = e'(i)(x) \cdot u'^{-1}_i$, and a trivial computation now finishes the proof of Case 2.

Case 3. The systems $(\mathcal{U}, b)$ and $(\mathcal{W}, c)$ are arbitrary. Let $\mathcal{W}'$ be a common refinement of $\mathcal{U}$ and $\mathcal{W}$. Hence there exist two cohomologous 1-cocycles of $\mathcal{W}'$ with values in $\mathcal{O}^\circ_X$, say g and h, such that the systems $(\mathcal{W}', g)$ and $(\mathcal{W}', h)$ represent f, and the pairs (b, g) and (c, h), with their respective coverings, fall under Case 1. Furthermore the pair (g, h) falls under Case 2, and this finishes the proof of Case 3.

The map φ_f is now easily seen to be a homomorphism.

It remains to prove that $Y \cap X_s = Y_{\varphi_f(s)}$. From the definition of $\varphi_f(s)$ we see immediately that, for $x \in Y$, $u_i[s(x)] \notin m_x$ implies $\varphi_f(s)(x)u'^{-1}_i(m'_x)$, where m'_x denotes the unique maximal ideal of $\mathcal{O}_{x,Y}$. Hence $Y \cap X_s$ is contained in $Y_{\varphi_f(s)}$. Conversely, if we have $[u'_i \circ \varphi_f(s)](x) \notin m'_x$, then $(\varphi \circ u_i \circ s)(x) \notin m'_x$, and since $\varphi^{-1}(m'_x) = m_x$, we have $(u_i \circ s)(x) \notin m_x$. Therefore $Y_{\varphi_f(s)} \subset Y \cap X_s$, which completes the proof of the proposition.

3. Divisorial varieties. Let X be an abstract algebraic variety, and let G_X denote the collection of open subsets of X. We define

$$B_X = \{U \in G_X \mid U = X_s, s \in \Gamma[X, g(\mathcal{O}_X)], g \in H^1(X, \mathcal{O}^\circ_X)\}$$

DEFINITION 3.1. An abstract algebraic variety X is called divisorial if B_X constitutes a base for the topology of X.

REMARK. Keeping in mind the remark of the previous section, the geometrical meaning of our definition becomes clear. If Y is irreducible and divisorial, then, for every point $x \in X$ and every closed subset Y of X, not containing x, there exists a positive divisor of X, which is locally linearly equivalent to zero and whose variety contains Y but not x. In other words the topology of X is entirely determined by the positive, locally linearly equivalent to zero divisors. This justifies our terminology.

We now begin the study of the categorical properties of divisorial varieties.

THEOREM 3.1. *Let X be a divisorial algebraic variety, and Y a locally closed subvariety of X. Then Y is a divisorial algebraic variety.*

Proof. Let U' be an open subset of Y and let $x \in U'$. Let U

be an open subset of X such that $U' = Y \cap U$. Since X is divisorial there exist a line class f of X and a section $s \in \Gamma[X, f(\mathcal{O}_X)]$ such that $x \in X_s \subset U$. By Proposition 2.3 the section $\varphi_f(s)$ of the sheaf $\varphi_1(f)(\mathcal{O}_Y)$ over Y is such that

$$Y_{\varphi_f}(s) = Y \cap X_s \, .$$

Hence $x \in Y_{\varphi_f}(s) \subset U'$, which proves the theorem.

THEOREM 3.2. *The direct sum of divisorial varieties is divisorial.*

Proof. Let X be the direct sum of $X_1, X_2, \cdots, X_n$. It is easily seen that

$$H^1(X, \mathcal{O}_X^\circ) \simeq \prod_{i=1}^{n} H^1(X_i, \mathcal{O}_{X_i}^\circ)$$

where the product on the right hand side is direct. Furthermore, if $f_r \in H^1(X_r, \mathcal{O}_{X_r}^\circ)$, and $s_r \in \Gamma[X_r, f_r(\mathcal{O}_{X_r})]$, then the rule

$$s(x) = \begin{cases} s_r(x) & \text{if } x \in X_r \\ 0 & \text{otherwise} \end{cases}$$

defines a section of $(1 \times 1 \times \cdots \times f_r \times \cdots \times 1)(\mathcal{O}_X)$ over X such that $X_s = X_{s_r}$. This proves the theorem.

Before proving that the category of divisorial varieties is a category with product, we need to prove the following very useful characterization of divisorial varieties.

THEOREM 3.3. *Let X be an abstract algebraic variety. A necessary and sufficient condition for X to be divisorial is the following: there exists an open affine covering $\mathcal{U} = (U_i, I)$ of X, line classes $g_1, g_2, \cdots, g_m$ of X, and sections $s_j \in \Gamma[X, g_j(\mathcal{O}_X)]$, $j = 1, 2, \cdots, m$, such that the collection of open sets $\{X_{s_j}, j = 1, 2, \cdots, m\}$ constitutes a covering of X which refines $\mathcal{U}$.*

Proof. The condition is obviously necessary, as it suffices to consider any open affine covering of X, and then use the fact that B_X is a base for the topology of X, and that X is compact.

To prove the sufficiency, let $x \in X$, and let Y be a closed subset of X, not containing x. Let $x \in X_{s_p}$ and $X_{s_p} \subset U_i$. Since U_i is affine, there exists a section h of $\mathcal{O}_X$ over X_{s_p} such that

$$h(x) \notin m_x \, ; \qquad h(y) \in m_y, \qquad y \in Y \cap X_{s_p} \, .$$

By Corollary 2.1 there exists a sufficiently high integer n such that the section $h \cdot s_p^{(n)}$ extends to a section $s*$ of $g_p^n(\mathcal{O}_X)$ over X.

Since $s_p(x) \notin n_x$, and $h(x) \notin m_x$, we have

$$s*(x) = h(x) \cdot s_p^{(n)}(x) \notin n'_x$$

where n'_x denotes the unique maximal submodule of $[g_p^n(\mathcal{O}_X]_x$. Furthermore, if $y \in Y \cap X_{s_p}$

$$s*(y) = h(y) \cdot s_p^{(n)}(y) \in n'_y .$$

Finally, n can be chosen high enough so that, if $y \notin X_{s_p}$, then $s*(y) \in n'_y$. Hence $x \in X_{s*} \subset X - Y$, and the proof is finished.

The above proof immediately yields the following corollary.

COROLLARY 3.1. *A necessary and sufficient condition for X to be divisorial is that there exists a finite number of line classes of X, say $g_1, g_2, \cdots, g_m$, such that the collection of open sets $\{X_s\}$, where s ranges among the sections over X of $g_j^n(\mathcal{O}_X)$, $j = 1, 2, \cdots, m$; $n = 1, 2, \cdots$, form a base for the topology of X.*

Proof. The condition is obviously sufficient. If X is divisorial, the proof of the above theorem shows that the line classes given by the criterion in the theorem satisfy the condition stated.

REMARK. Corollary 3.1 shows that the notion of divisorial variety is an extension of the notion of quasi-projective varieties in a natural way. In fact every quasi-projective variety satisfies the condition stated in the Corollary, *with only one line class*, namely the line class p of hyperplane sections, (sections of $p^n(\mathcal{O}_X)$ over X correspond to hypersurface sections) which was introduced by Serre in [3], §54, page 246.

We believe that a slight modification of the condition stated in Corollary 3.1, *with only one line class*, will yield a characterization of quasi-projective varieties.

The above reasoning already shows that every quasi-projective variety is divisorial. We shall give another proof of the same statement in the next section.

Let X, Y be abstract algebraic varieties. There exists a natural monomorphism

$$\mu\colon H^1(X, \mathcal{O}_X^\circ) \longrightarrow H^1(X \times Y, \mathcal{O}_{X \times Y}^\circ)$$

and, for every $g \in H^1(X, \mathcal{O}_X^\circ)$ a monomorphism

$$\mu_g\colon \Gamma[X, g(\mathcal{O}_X)] \longrightarrow \Gamma[X \times Y, \mu(g)(\mathcal{O}_{X \times Y})]$$

such that

$$(X \times Y)_{\mu_g(s)} = X_s \times Y .$$

The proof of the above statements is entirely straightforward, and we omit it here for brevity's sake. In what follows we will identify $H^1(X, \mathcal{O}_X^\circ)$ and $\Gamma[X, g(\mathcal{O}_X)]$ with their images in $H^1(X \times Y, \mathcal{O}_{X\times Y}^\circ)$ and $\Gamma[X \times Y, \mu(g)(\mathcal{O}_{X\times Y})]$ respectively. Similarly for Y.

THEOREM 3.4. *The product of divisorial varieties is a divisorial variety.*

Proof. Let X, Y be divisorial varieties. We shall use the criterion of Theorem 3.3. Accordingly, let $\mathscr{U} = (U_i, I)$, $g_1, g_2, \cdots, g_m, s_1, s_2, \cdots, s_m$ and $\mathscr{W} = (V_j, J)$, $h_1, h_2, \cdots, h_r, t_1, t_2, \cdots, t_r$ be the affine open coverings, line classes and sections satisfying the condition of Theorem 3.3 for X and Y respectively. Observe that:

$$\begin{aligned}(X \times Y)_{s_p \otimes t_q} &= (X \times Y)_{s_p} \cap (X \times Y)_{t_q} \\ &= (X_{s_p} \times Y) \cap (X \times Y_{t_q}) = X_{s_p} \times Y_{t_q}\end{aligned}$$

for all values of p from 1 to m and of q from 1 to r.

Hence the open affine covering

$$(U_i \times V_j, I \times J)$$

of $X \times Y$, the line classes $g_p h_q$ and the sections $s_p \otimes t_q$, $p = 1, \cdots, m$ and $q = 1, \cdots, r$, satisfy the condition of Theorem 3.3 applied to $X \times Y$. Hence $X \times Y$ is divisorial.

4. Existence of divisorial varieties. As we have already seen in the previous section, all quasi-projective varieties are divisorial. We shall show in the present section that the category of divisorial varieties also includes all nonsingular varieties and lots more.

We call an abstract algebraic variety factorial if the local ring of every one of its points is a unique factorization domain. As Zariski has shown in [9], all nonsingular varieties are factorial.

In what follows, if h is a rational function on an irreducible variety X, we shall denote by (h) the divisor of the function h on X.

THEOREM 4.1. *Every irreducible factorial variety is divisorial.*

Proof. Let X be an irreducible factorial variety, whose function field we shall denote by E. For every irreducible subvariety W of X, we denote by $\mathcal{O}_W$ the local ring of W in E.

Let U be an open subset of X, and let $x \in U$. We proceed in steps.

Case 1. $W = X - U$ is an irreducible subvariety of X. Since

$x \notin W$, it follows that $\mathcal{O}_x \not\subset \mathcal{O}_W$. Let hence $h \in E$ be such that $h \in \mathcal{O}_x$ and $h \notin \mathcal{O}_W$. Let $(h) = D_1 - D_2$, where D_1 and D_2 denote respectively the zeros and poles of the function h. Since $h \in \mathcal{O}_x$, we have $x \notin \mathrm{Var}(D_2)$, where $\mathrm{Var}(D)$ denotes the variety of the divisor D. Furthermore $y \in W$ implies $h \notin \mathcal{O}_y$, hence, since X is normal, $y \in \mathrm{Var}(D_2)$. Therefore $W \subset \mathrm{Var}(D_2)$. Since X is factorial, D_2 is locally linearly equivalent to zero, i.e. there exists an open covering $\mathcal{U} = (U_i, I)$ of X and rational functions $h_i \in E$, such that h_i is regular on U_i and $(h_i) = D_2$ on U_i. Hence, since X is normal, $(h_i/h_j) = 0$ on $U_i \cap U_j$ implies that the system h_i/h_j defines a 1-cocycle of $\mathcal{U}$ with values in $\mathcal{O}^\circ_X$. Let g be the line class of X represented by the system $(\mathcal{U}, h_i/h_j)$, and let u_i: $g(\mathcal{O}_X) \mid U_i \to \mathcal{O}_X \mid U_i$ be the usual isomorphisms. If we define $s(y) = (u_i^{-1} \circ h_i)(y)$, for $y \in U_i$, we clearly obtain a section s of $g(\mathcal{O}_X)$ over X such that

$$X_s \cap U_i = U_i - \mathrm{Var}(D_2) .$$

Hence $x \in X_s = X - \mathrm{Var}(D_2) \subset U$.

Case 2. $W = X - U$ is arbitrary. Let $W_1, W_2, \cdots, W_p$ be the irreducible components of W. From Case 1 we know that there exist line classes $g_1, \cdots, g_p$ of X and sections $s_i \in \Gamma[X, g_i(\mathcal{O}_X)]$ $i = 1, \cdots, p$, such that $x \in X_{s_i} \subset X - W_i$. We easily verify that the section

$$s = s_1 \otimes s_2 \otimes \cdots \otimes s_p \in \Gamma[X, g_1 g_2 \cdots g_p(\mathcal{O}_X)]$$

is such that $X_s = \bigcap_{i=1}^p X_{s_i}$, hence $x \in X_s \subset U$. This finishes the proof of the theorem.

THEOREM 4.2. *Every factorial variety is divisorial.*

Proof. By definition, every unique factorization domain is an integral domain. Hence every factorial variety is the direct sum of its irreducible components, which, by Theorem 4.1, are all divisorial. Then we apply Theorem 3.2.

THEOREM 4.3. *Every quasi-projective variety is divisorial.*

Proof. Projective space is nonsingular, hence divisorial. Then we apply Theorem 3.1.

THEOREM 4.4. *There exist divisorial varieties which are neither quasi-projective nor nonsingular, of any dimension* > 3.

Proof. There exist nonsingular, nonprojective varieties of any dimension > 2. (See [2]). We use any singular, quasi-projective variety, and apply Theorem 3.4.

REMARK. The above theorems provide us with a large class of divisorial varieties. It is not settled at the moment, though, whether there are divisorial surfaces which are not projective. Such surfaces must necessarily be singular, as it follows from the fact that every nonsingular surface is quasi-projective (See [8]).

For an example of a normal, nonprojective surface see [2], page 492.

5. The group of line classes of a divisorial variety. Let X be an abstract algebraic variety. As in [6], §4, we shall call *regular* any line class g of X such that, for some $s \in \Gamma[X, g(\mathcal{O}_X)]$, $X_s \neq \phi$. Let x be a fixed point of X. A regular line class g is called *free at* x if, for some $s \in \Gamma[X, g(\mathcal{O}_X)]$, $x \in X_s$. The set of line classes which are free at x is easily shown to form a subsemi-group of $H^1(X, \mathcal{O}_X^\circ)$, which we shall denote by L_x.

The following proposition generalizes the well known operation of "adding hypersurface sections." (See [6], Proposition 8.2.).

Let X be a divisorial algebraic variety, and let

$$\mathscr{U} = (U_i, I), g_1, \cdots, g_m, s_1, \cdots, s_m$$

be the open affine covering, line classes and sections satisfying the criterion of Theorem 3.3.

PROPOSITION 5.1. Let X be a divisorial algebraic variety, f a line class of X, x a fixed point of X. Then, for a sufficiently high integer n, and for some integer p between 1 and m, the line class fg_p^n is regular and free at x.

Proof. For a suitable open subset U of X, containing x, we can find a section $t \in \Gamma[U, f(\mathcal{O}_X)]$ such that $t(x) \notin n_x$. By Corollary 3.1 there exist an integer p, with $1 \leqq p \leqq m$, and a sufficiently high integer q such that the sheaf $g_p^q(\mathcal{O}_X)$ has a section s over X with $x \in X_s \subset U$.

Applying Proposition 2.2 to the line classes f and g_p^q, and their respective sections t and s, we see that, for a sufficiently high integer q' the section $t \otimes s^{(q')}$ extends to a section s^* of $fg_p^{qq'}(\mathcal{O}_X)$ over X. We have:

$$s^*(x) = t(x) \otimes s^{(q')}(x) \notin n'_x$$

where n'_x denotes the unique maximal submodule of $[fg_p^{qq'}(\mathcal{O}_X)]_x$. Hence $x \in X_{s^*}$, which finishes the proof of the proposition.

COROLLARY 5.1. *Let X be divisorial, and $x \in X$. The group generated by L_x in $H^1(X, \mathcal{O}_X^\circ)$ is $H^1(X, \mathcal{O}_X^\circ)$.*

Proof. By the above proposition, for any f in $H^1(X, \mathcal{O}_X^\circ)$, $fg_p^n \in L_x$. Clearly $g_p^n \in L_x$.

6. **The polynomial theorem of Snapper.** Let λ be an additive sheaf function, i.e. a function defined over the category of sheaves, with values in an arbitrary abelian group G, and such that the exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

implies $\lambda(F) = \lambda(F') + \lambda(F'')$. (See [5], §4, page 105, or [1], §3.).

The following theorem is an extension to divisorial varieties of the polynomial theorem proved by Snapper in [6], Theorem 9.1, as well as the more general form given by Cartier in [1], §4.

THEOREM 6.1. *Let λ be an additive sheaf function, and X a divisorial algebraic variety. Then, for every sheaf F over X and every finite set of line classes $f_1, \cdots, f_n$ of X, the expression*

$$\lambda[f_1^{m_1} \cdots f_n^{m_n}(F)]$$

is a polynomial in $m_1, \cdots, m_n$ of degree at most $\dim(\mathrm{Supp}\, F)$.

Proof. The theorem is an immediate consequence of the following lemma, which generalizes the theorem given in §3 of [1]. The formal algorithm used in §4 of the same paper, identically repeated, proves our theorem. Therefore we limit ourselves to the proof of the lemma.

LEMMA 6.1. *Let X be a divisorial algebraic variety, λ an additive sheaf function, g any line class of X. If $\lambda(F) = 0$ for every sheaf F such that* $\dim(\mathrm{Supp}\, F) < r$, *then* $\lambda(F) = \lambda[g(F)]$ *for every sheaf F with* $\dim(\mathrm{Supp}\, F) \leqq r$.

Proof. We proceed in steps.

Case 1. We assume $\dim(\mathrm{Supp}\, F) < r$. Since F and $g(F)$ are locally isomorphic we have $\dim(\mathrm{Supp}\, g(F)) < r$, hence

$$\lambda(F) = 0 = \lambda[g(F)] .$$

Case 2. We assume $\mathrm{Supp}\, F \subset S$, where S is an irreducible closed subset of X, and $\dim S \leqq r$. Let $x \in S$. Since X is divisorial, by

Corollary 5.1 we can write $g = f_1/f_2$, where $f_i \in L_x$, $i = 1, 2$. Let therefore $s_i \in \Gamma[X, f_i(\mathcal{O}_X)]$ be such that $x \in X_{s_i}$, $i = 1, 2$. We now define

$$\omega_i\colon F \longrightarrow f_i(F) \qquad i = 1, 2 .$$

as follows:

$$\omega(a) = a \otimes s_i(y) \qquad a \in F_y .$$

Since $x \in X_{s_i}$ we see that $s_i(x)$ generates the stalk of $f_i(\mathcal{O}_X)$ over x, hence ω_i induces an isomorphism on F_x. Therefore Supp ($\ker \omega_i$) and Supp ($\operatorname{coker} \omega_i$) are proper closed subsets of S, hence

$$\lambda(\ker \omega_i) = \lambda(\operatorname{coker} \omega_i) = 0 .$$

Since λ is additive, the exact sequence

$$0 \longrightarrow \ker \omega_i \longrightarrow F \longrightarrow f_i(F) \longrightarrow \operatorname{coker} \omega_i \longrightarrow 0$$

shows that $\lambda(F) = \lambda[f_i(F)]$. Let $F' = g(F)$. Then F and F' are locally isomorphic, hence $\operatorname{Supp} F = \operatorname{Supp} F'$. Hence, by the above proof applied to F', we obtain:

$$\lambda[g(F)] = \lambda(F') = \lambda[f_2(F')] = \lambda[f_2 g(F)] = \lambda[f_1(F)] = \lambda(F) .$$

Case 3. We only assume $\dim(\operatorname{Supp} F) \leqq r$. Let S_i be the irreducible components of $\operatorname{Supp} F$, and let T be the union of the closed sets $S_i \cap S_j$, for $i \neq j$. We have $\dim S_i \leqq r$, and $\dim T < r$. From [4], page 11, we know that there exist sheaves F_i, G, such that $\operatorname{Supp} F_i \subset S_i$, and $\operatorname{Supp} G \subset T$, and that there exists an exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow \sum F_i \longrightarrow 0_i$$

where the sum at right is direct. Applying Case 2 to each pair (F_i, S_i) we get $\lambda(F_i) = \lambda[g(F_i)]$, and from the exact sequence

$$0 \longrightarrow g(G) \longrightarrow g(F) \longrightarrow g(\sum F_i) \longrightarrow 0$$

we get $\lambda[g(F)] = \lambda[g(\sum F_i)]$. Hence:

$$\begin{aligned} \lambda(F) &= \lambda(\sum F_i) = \sum \lambda(F_i) = \sum \lambda[g(F_i)] \\ &= \lambda[\sum g(F_i)] = \lambda[g(\sum F_i)] = \lambda[g(F)] . \end{aligned}$$

This finishes the proof of the lemma.

Final Remark. We wish to point out the following question, which stems from the above study of divisorial varieties:

If a divisorial variety X has a line class g such that, for any finite set of points $P_1, \cdots, P_n$ of X, there exists an open affine subset

X_s, $s \in \Gamma[X, g(\mathscr{O}_X)]$, containing them, is then X quasi projective?

The above question is more restrictive, in a natural way, than the original one asked by Chevalley, (See [2], footnote to Introduction), and we believe the answer to be in the affirmative.

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