THE SPACE OF REAL PARTS OF A FUNCTION ALGEBRA

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1. Introduction. Let X be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on X. We consider a closed subalgebra A of $C(X)$ which separates the points of X and contains the constants. We call A "a function algebra on X".

Let $Re\,A$ denote the class of functions u real and continuous on X such that for some f in A, $u = Re\,f$. Then $Re\,A$ is a real vector space of real continuous functions on X. What more can be said about $Re\,A$?

In [3] it was shown that $Re\,A$ cannot be closed under uniform convergence on X unless $A = C(X)$. Here we shall show that $Re\,A$ cannot be closed under multiplication unless $A = C(X)$. In other words:

Theorem 1: *If $Re\,A$ is a ring, then $A = C(X)$.*

This result was conjectured by K. Hoffman. As a corollary one gets the existence of a continuous function u on the unit circle having the following property: u has a continuous conjugate function (in the sense of Fourier theory) whereas u^2 does not. For we may take for A the algebra of continuous functions on the circle which extend analytically to the unit disk. Then $Re\,A$ is the class of all functions which are continuous and have continuous conjugates. But $A \neq C(X)$. Hence by Theorem 1, $Re\,A$ is not a ring, hence not closed under squaring, and so the desired u exists.

The existence of such a u had been shown in 1961 by J. P. Kahane (unpublished). It should be noted that if a function u is sufficiently smooth to have an absolutely convergent Fourier series, then u^2 does also, and hence u^2 does have a continuous conjugate.

2. The antisymmetric case. In this section we assume that A is anti-symmetric, i.e. contains no real functions except constants, and prove Theorem 1 under this hypothesis. This amounts to proving:

THEOREM 1′. *Let A be anti-symmetric and let $Re\,A$ form a ring. Then X consists of a single point.*

Assume X contains a point x_0 and another point x_1. We must deduce a contradiction. Fix u in $Re\,A$. Then (because of antisymmetry),

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there exists exactly one f in A with $u = Re\, f$ and $Im\, f(x_0) = 0$. The map: u into f is now a real-linear map of $Re\, A$ into A which is one-to-one. We can then norm $Re\, A$ by the norm N:

$$N(u) = \max_X |f| = ||f|| .$$

In this norm $Re\, A$ is then evidently a real Banach space. By standard application of the closed graph theorem, we have

LEMMA 1. *There exists a constant K such that for all u, u' in $Re\, A$*

$$N(u \cdot u') \leqq K \cdot N(u) \cdot N(u') .$$

LEMMA 2. *If p lies in $Re\, A$ and $p > 0$ on X, then* $\log p$ *is in $Re\, A$.*

Proof. Let S be the class of functions $u + iu'$ with u and u' in $Re\, A$. Then S is an algebra of complex-valued functions on X containing A as a subalgebra and closed under complex conjugation. Define $N(u + iu') = N(u) + N(u')$ and $||f||' = \sup_\theta N(e^{i\theta} f)$ for all f in S. Then S is a (complex) Banach space under $||\ \ ||'$ as norm and also $||f \cdot g||' \leqq K||f||' \cdot ||g||'$. Hence S is a Banach algebra under a norm equivalent to $||\ \ ||'$.

Let M_S denote the space of homomorphisms of S into the complex numbers and M_A be the corresponding space for A. Fix m in M_S. Restricted to A, m is an element σ of M_A. Also the map: f into $\overline{m(\bar{f})}$, restricted to A, is an element τ of M_A. Since p lies in $Re\, A$, we can find some r in A such that

$$p = \frac{1}{2}(r + \bar{r}) \quad \text{whence } m(p) = \frac{1}{2}(m(r) + m(\bar{r})) ,$$

or

$$m(p) = \frac{1}{2}(\sigma(r) + \overline{\tau(r)}) .$$

By hypothesis, $Re\, r = p > 0$ on X. Hence by a well-known property of function algebras, $Re\, \beta(r) > 0$ for all β in M_A. In particular $Re\, \sigma(r) > 0$ and $Re\, \tau(r) > 0$. Hence $Re\, m(p) > 0$.

Since this holds for all m in M_S, we can, by the general theory of Banach algebras, apply to p any function analytic in the right half-plane and still stay in the algebra S. Hence $\log p$ is an element of S, and, being real valued, of $Re\, A$.

Let now K^* be any positive number. Choose g in A with $g(x_0) = 0$

and $\|g\| = 1$. Let a be some point in X where $|g(a)| = 1$. Next choose φ analytic in $|z| < 1$, continuous in $|z| \leqq 1$, such that $0 < Re\,\varphi \leqq 1$ in $|z| \leqq 1$, $Im\,\varphi(0) = 0$ and $Im\,\varphi(g(a)) \geqq K^*$. Put $f = \varphi(g)$. Then f belongs to A and we have:

$$0 < Re\,f \leqq 1 \text{ on } X\,, \qquad Im\,f(x_0) = 0 \text{ and } \|f\| \geqq K^*\,.$$

Then $Re\,f$ is in $Re\,A$ and >0. By Lemma 2, then, $\log(Re\,f)$ also is in $Re\,A$, i.e. there is some F in A with $Re\,F = \log(Re\,f)$. Put now $V = \exp(\frac{1}{2}F)$. Then again V is in A. Also $|V|^2 = Re\,f$. Then $\max_x |V| = \|V\| \leqq 1$.

We now use the following identity, true for each complex z:

$$(Re\,z)^2 = \frac{1}{2}(Re\,z^2 + |z|^2)\,.$$

Applying this to V and using that $|V|^2 = Re\,f$, we get

$$(Re\,V)^2 = Re\Big(\frac{1}{2}(V^2 + f)\Big)\,.$$

Clearly for each h in A, we have $N(Re\,h) \geqq \|h\| - |Im\,h(x_0)|$. Hence $N((Re\,V)^2) \geqq \frac{1}{2}(\|V^2 + f\| - |Im\,V^2(x_0)|) \geqq \frac{1}{2}(K^* - 2)$, since $\|f\| \geqq K^*$ while $\|V^2\| \leqq 1$.

On the other hand, by Lemma 1,

$$N((Re\,V)^2) \leqq K\cdot(N(Re\,V))^2 \quad\text{and}\quad N((Re\,V)) \leqq 2\|V\| \leqq 2\,.$$

Since K^* is arbitrary while K is fixed, we have a contradiction. Thus Theorem 1′ is proved.

3. **The general case.** To deduce the result in the general case from Theorem 1′, we use the following theorem of Bishop [1]. (See also [2].):

THEOREM. *Let A be any function algebra on X. Then there exists a collection Φ of closed, pairwise disjoint sets covering X so that*

(a) *f in $C(X)$ and $f|K$ in $A|K$ for every K in Φ imply f in A;*

(b) *$A|K$ is closed in $C(K)$ for each K in Φ.*

(c) *$A|K$ is antisymmetric on K for each K in Φ.*

Because of Bishop's theorem, one has the following method of reasoning: let (P) be a property which has meaning for every function algebra A. Assume

(i) Whenever a given A has property (P), then so does each restriction algebra $A|K$ for K in Φ, and

(ii) Whenever A is antisymmetric on the space X and A has property (P), then X consists of a single point.

We then conclude, using the Theorem, that if A is a function algebra on a space X such that A has property (P), then $A = C(X)$. Thus, if (P) is the property "A is closed under complex conjugation", (i) and (ii) clearly hold, and one concludes the Stone-Weierstrass theorem.

If (P) is the property "$Re\, A$ is a ring", then (i) also clearly holds, and that (ii) holds was the content of Theorem 1′. Thus we may conclude Theorem 1.

References

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