INTEGRAL EQUATIONS IN NORMED ABELIAN GROUPS

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1. Introduction. Suppose Z is an additive abelian group with additive identity element N and a "norm" $\|\cdot\|$ such that $\|N\| = 0$, and if $z, w \in Z$, then $\|z + w\| \leqq \|z\| + \|w\|$, $\|-z\| = \|z\|$, and $\|z\| > 0$ unless $z = N$. Suppose furthermore that Z is complete with respect to the metric induced by this norm. Let B denote the set of all transformations from Z into Z. Suppose $[a, b]$ is a closed number interval, $A \in Z$, and each of F and G is a function from $[a, b]$ into B.

Under suitable restrictions on F and G, we wish to find a function Y from $[a, b]$ into Z satisfying the integral equation

$$Y(x) = A + \int_a^x dG \cdot FY\,, \tag{1.1}$$

where FY denotes the function from $[a, b]$ into Z defined by $[FY](x) = F(x)Y(x)$. Notice that parentheses are used in denoting the image of a number, but not in denoting the image of an element of B. We wish to express a solution of (1.1) as a product integral

$$Y(x) = \pi_a^x(1 + dG \cdot F)A\,. \tag{1.2}$$

The terms "integral" and "product integral" will be defined in the next section, but the notation is quite suggestive, taking $1z = z$ for $z \in Z$.

A related problem has been treated by J. W. Neuberger [1]. Let us perform a "change of variable." That is, let R denote the function from $[a, b]$ into B defined by $R(x)z = \int_a^x dG \cdot Fz$, where Fz denotes the function from $[a, b]$ into Z defined by $[Fz](x) = F(x)z$. Then (1.1) becomes, at least formally

$$Y(x) = A + \int_a^x dR \cdot Y\,. \tag{1.3}$$

Under suitable restrictions, Neuberger expresses solutions of (1.3) as the product integral

$$Y(x) = \pi_a^x(1 + dR)A\,, \tag{1.4}$$

or, in Neuberger's notation

$$Y(x) = {}_a\pi^x(T, A)\,, \qquad T(p, q) = 1 + R(p) - R(q)\,.$$

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With sufficient hypothesis, (1.1) and (1.3) are equivalent, but it can happen that (1.3) has a solution when (1.1) does not, and that the product integral (1.4) exists when (1.2) does not. Part of the difference between this paper and [1] lies in the attacking of the problem (1.1) directly instead of its reduction (1.3). This difference is not trivial even when the two problems are equivalent. For instance, error estimates for approximates of $\pi_a^x(1 + dR)A$ are likely to assume that the approximate was obtained with an exact knowledge of R. Certainly, this knowledge is unattainable for a great many (F, G) combinations. One can obtain error estimates for approximates of $\pi_a^x(1 + dG \cdot F)A$ which involve no such assumption. Also, this paper employs a weaker substitute for the standard Lipschitz condition.

2. Definitions and notation. If $[u, v]$ is a subinterval of $[a, b]$, then a partition of $[u, v]$ means a finite increasing number sequence with first term u and last term v. If Δ is a partition of $[u, v]$, then the statement that Δ' is a refinement of Δ means that Δ' is a partition of $[u, v]$ which has Δ as a subsequence. A partition shall mean a partition of some subinterval of $[a, b]$. If Δ is a partition, then $|\Delta|$ means the integer which is two less than the number of terms of Δ, and we write $\Delta = \{\Delta_j\}_{j=0}^{|\Delta|+1}$. If x and y are terms of a partition Δ, and $x < y$, then the section of Δ from x to y means the maximal subsequence of Δ which is a partition of $[x, y]$; that is, if $x = \Delta_p$, $y = \Delta_q$, and $p < q$, then $\{\Delta_j\}_{j=p}^{q}$ is the section of Δ from x to y. If Δ is a partition, then the statement that X is an interpolating sequence for Δ means that X is a finite number sequence $\{X_j\}_{j=0}^{|\Delta|}$ such that $X_j \in [\Delta_j, \Delta_{j+1}]$ for $j = 0, 1, \cdots, |\Delta|$. If $\Delta' = \{\Delta_j\}_{j=p}^{q}$ is a section of the partition Δ, and X is an interpolating sequence for Δ, then $\{X_j\}_{j=p}^{q-1}$ is called the Δ'-section of X. If H is a function from $[a, b]$ into B (or a number set), and Δ is a partition, then ΔH_j means the transformation (or number) $[H(\Delta_{j+1}) - H(\Delta_j)]$ for $j = 0, 1, \cdots, |\Delta|$.

If H is a function from $[a, b]$ into B, Q if a function from $[a, b]$ into Z, Δ is a partition, and X is an interpolating sequence for Δ, then

$$\Sigma(\Delta, X, H, Q) \text{ means } \sum_{j=0}^{|\Delta|} \Delta H_j Q(X_j) .$$

If $[u, v]$ is a subinterval of $[a, b]$, then the statement that J is the integral $\int_u^v dH \cdot Q$ means $J \in Z$, and for each $\varepsilon > 0$, there is a partition Δ of $[u, v]$ such that

$$\| J - \Sigma(\Delta', X', H, Q) \| < \varepsilon$$

if Δ' is a refinement of Δ, and X' is an interpolating sequence for Δ'.

We define $\int_u^u dH \cdot Q = N$, and notice that the existence of $\int_u^v dH \cdot Q$ implies that

$$\int_u^v dH \cdot Q = \int_u^x dH \cdot Q + \int_x^v dH \cdot Q$$

for $u < x < v$.

If each of H and Q is a function from $[a, b]$ into B, $P \in Z$, Δ is a partition, X is an interpolating sequence for Δ, and we write $P_0 = P$ and

$$P_{k+1} = [1 + \Delta H_k Q(X_k)]P_k$$

for $k = 0, 1, \cdots, |\Delta|$, then we get

$$\begin{aligned} P_{k+1} &= P + \sum_{j=0}^{k} \Delta H_j Q(X_j) P_j \\ &= [1 + \Delta H_k Q(X_k)] \cdots [1 + \Delta H_1 Q(X_1)] \cdot [1 + \Delta H_0 Q(X_0)]P , \end{aligned}$$

and in particular, we denote $P_{|\Delta|+1}$ by $\pi(\Delta, X, H, Q)P$. If $[u, v]$ is a subinterval of $[a, b]$, then the statement that J is the product integral $\pi_u^v(1 + dH \cdot Q)P$ means that $J \in Z$, and for each $\varepsilon > 0$, there is a partition Δ of $[u, v]$ such that

$$\| J - \pi(\Delta', X', H, Q)P \| < \varepsilon$$

if Δ' is a refinement of Δ and X' is an interpolating sequence for Δ'. $\pi_u^u(1 + dH \cdot Q)P$ means P.

3. **Integrals.** Suppose $M \subset Z$, $K \subset Z$, $A \in M$, $F(x)z \in K$ for $x \in [a, b]$ and $z \in M$, and $F(x)z = F(a)A$ for $x \in [a, b]$ and $z \notin M$. Suppose that the collection $\{F(x)z\}(x \in [a, b])$ is equi-uniformly continuous on M. That is, there is a nondecreasing function E from $[0, \infty)$ into $[0, \infty)$ with $E(0) = E(0+) = 0$ such that

$$\| F(x)z - F(x)w \| \leqq E(\| z - w \|) \tag{3.1}$$

for $x \in [a, b]$ and $z, w \in M$. Let E denote one such function. Suppose U is a nondecreasing function from $[0, \infty)$ into $[0, \infty)$ with $U(0) = U(0+) = 0$, h is a continuous real valued function which is nondecreasing on $[a, b]$, and

$$\| Dz - Dw \| \leqq | h(v) - h(u) | \, U(\| z - w \|) \tag{3.2}$$

for $u, v \in [a, b]$, $D = [G(v) - G(u)]$, and $z, w \in K$. Let W denote the composite function $U[E]$. Notice that W is nondecreasing and $W(0) = W(0+) = 0$. Suppose that $\int_a^b dG \cdot Fz$ exists for all $z \in M$, and, as in the introduction, let R denote the function from $[a, b]$ into B defined by

$$R(x)z = \int_a^x dG \cdot Fz \, .$$

THEOREM 3.1. *If Y is a continuous function from $[a, b]$ into M, then $\int_a^b dR \cdot Y$ exists. Moreover, if C is an equicontinuous collection of functions from $[a, b]$ into M, then the approximating sums for $\int_a^b dR \cdot Y$ converge uniformly for all $Y \in C$.*

Proof. Let us first show that if $[u, v]$ is a subinterval of $[a, b]$, $D = R(v) - R(u)$, and $z, w \in M$, then

$$\| Dz - Dw \| \leqq [h(v) - h(u)] W(\| z - w \|) \, . \tag{3.1.1}$$

It follows from the definition of R that

$$Dz = \int_u^v dG \cdot Fz \quad \text{and} \quad Dw = \int_u^v dG \cdot Fw \, .$$

It $\varDelta$ is a partition of $[u, v]$, X is an interpolating sequence for $\varDelta$, Σz denotes $\Sigma(\varDelta, X, G, Fz)$, and Σw denotes $\Sigma(\varDelta, X, G, Fw)$, then

$$\| Dz - Dw \| \leqq \| Dz - \Sigma z \| + \| \Sigma w - Dw \| + \| \Sigma z - \Sigma w \| \, ,$$

$$\| \Sigma z - \Sigma w \| \leqq \sum_{j=0}^{|\varDelta|} \| \varDelta G_j F(X_j) z - \varDelta G_j F(X_j) w \| \, ,$$

and, applying the inequalities (3.2) and (3.1), in that order, we get

$$\begin{aligned} \| \Sigma z - \Sigma w \| &\leqq \sum_{j=0}^{|\varDelta|} (\varDelta h_j) U(\| F(X_j) z - F(X_j) w \|) \\ &\leqq \sum_{j=0}^{|\varDelta|} (\varDelta h_j) U[E(\| z - w \|)] = [h(v) - h(u)] W(\| z - w \|) \, . \end{aligned}$$

This establishes the inequality (3.1.1).

Suppose C is an equicontinuous collection of functions from $[a, b]$ into M, $\varepsilon > 0$, $\delta > 0$, $[h(b) - h(a)] W(\delta) < \varepsilon$, $\delta' > 0$, $\| Y(v) - Y(u) \| < \delta$ for $| u - v | < \delta'$ and $Y \in C$, $\varDelta$ is a partition of $[a, b]$ with mesh less than δ', and X is an interpolating sequence for $\varDelta$.

Suppose $\varDelta'$ is a refinement of $\varDelta$, X' is an interpolating sequence for $\varDelta'$, and $Y \in C$. For each $p = 0, 1, \cdots, |\varDelta|$, let $\varDelta^p$ denote the section of $\varDelta'$ from $\varDelta_p$ to $\varDelta_{p+1}$ and let X^p denote the $\varDelta^p$-section of X'. Then

$$\begin{aligned} \| \Sigma(\varDelta, X, R, Y) - \Sigma(\varDelta', X', R, Y) \| &\leqq \sum_{p=0}^{|\varDelta|} \| \varDelta R_p Y(X_p) - \Sigma(\varDelta^p, X^p, R, Y) \| \\ &\leqq \sum_{p=0}^{|\varDelta|} \sum_{j=0}^{|\varDelta^p|} \| \varDelta^p R_j Y(X_p) - \varDelta^p R_j Y(X_j^p) \| \\ &\leqq \sum_{p=0}^{|\varDelta|} \sum_{j=0}^{|\varDelta^p|} (\varDelta^p h_j) W(\| Y(X_p) - Y(X_j^p) \| < \varepsilon \, . \end{aligned}$$

THEOREM 3.2. *If Y is a continuous function from $[a, b]$ into M, then $\int_a^x dG \cdot FY = \int_a^x dR \cdot Y$ for all $x \in [a, b]$. Moreover, if the approximating sums for $\int_a^b dG \cdot Fz$ converge uniformly for all $z \in M$, and C is an equicontinuous collection of functions from $[a, b]$ into M, then the approximating sums for $\int_a^b dG \cdot FY$ converge uniformly for all $Y \in C$.*

Proof. Let us prove the second statement. Suppose the approximating sums for $\int_a^b dG \cdot Fz$ converge uniformly for all $z \in M$, C is an equicontinuous collection of functions from $[a, b]$ into M, and $\varepsilon > 0$. Suppose $\delta > 0$, $[h(b) - h(a)]W(\delta) < \varepsilon$, $\delta' > 0$, $\| Y(v) - Y(u) \| < \delta$ for $| v - u | < \delta'$ and $Y \in C$, Δ is a partition of $[a, b]$ with mesh less than δ', and X is an interpolating sequence for Δ. We see from the argument for Theorem 3.1 that

$$\left\| \int_a^b dR \cdot Y - \Sigma(\Delta, X, R, Y) \right\| \leqq \varepsilon$$

for all $Y \in C$.

For each $p = 0, 1, \cdots, |\Delta|$, let Δ^p denote a partition of $[\Delta_p, \Delta_{p+1}]$ such that, if Δ' is a refinement of Δ^p, X' is an interpolating sequence for Δ', and $z \in M$, then

$$\left\| \int_{\Delta_p}^{\Delta_{p+1}} dG \cdot Fz - \Sigma(\Delta', X', G, Fz) \right\| < \varepsilon/(|\Delta| + 1) .$$

Notice that

$$\int_{\Delta_p}^{\Delta_{p+1}} dG \cdot Fz = \Delta R_p z$$

for $z \in M$ and $p = 0, 1, \cdots, |\Delta|$.

Let Δ' denote the refinement of Δ which has Δ^p as its section from Δ_p to Δ_{p+1} for $p = 0, 1, \cdots, |\Delta|$. We wish to show that

$$\left\| \int_a^b dR \cdot Y - \Sigma(\Delta'', X'', G, FY) \right\| < 3\varepsilon$$

if Δ'' is a refinement of Δ', X'' is an interpolating sequence for Δ'', and $Y \in C$.

Suppose Δ'' is a refinement of Δ', X'' is an interpolating sequence for Δ'', and $Y \in C$. For each $p = 0, 1, \cdots, |\Delta|$, let α^p denote the section of Δ'' from Δ_p to Δ_{p+1}, let β^p denote the α^p-section of X'', and let $z_p = Y(X_p)$. Notice that α^p is a refinement of Δ^p for $p = 0, 1, \cdots, |\Delta|$.

$$\begin{aligned}
&\left\|\int_a^b dR\cdot Y-\Sigma(\Delta'',X'',G,FY)\right\| \\
&\quad \leqq \left\|\int_a^b dR\cdot Y-\Sigma(\Delta,X,R,Y)\right\| \\
&\qquad +\sum_{p=0}^{|\Delta|}\|\Delta R_p z_p-\Sigma(\alpha^p,\beta^p,G,Fz_p)\| \\
&\qquad +\sum_{p=0}^{|\Delta|}\|\Sigma(\alpha^p,\beta^p,G,Fz_p)-\Sigma(\alpha^p,\beta^p,G,FY)\| \\
&\quad < 2\varepsilon+\sum_{p=0}^{|\Delta|}\sum_{j=0}^{|\alpha^p|}\|\alpha^p G_j F(\beta_j^p)z_p-\alpha^p G_j F(\beta_j^p)Y(\beta_j^p)\| \\
&\quad \leqq 2\varepsilon+[h(b)-h(a)]W(\delta)<3\varepsilon\,.
\end{aligned}$$

Only a slight modification of this argument is required to establish the first statement of the theorem.

REMARK. The first statement of Theorem 3.2 establishes the equivalence of the integral equations (1.1) and (1.3). The following example shows how markedly the problems may differ under a slightly altered hypothesis. In particular, the inequality (3.1) cannot be replaced by the weaker statement that, for each $x\in[a,b]$, the transformation $F(x)$ is continuous on M. In this example, the hypothesis of Theorem 3.2 is satisfied except for the above mentioned replacement. Moreover, $\|F(x)z\|$ is bounded, Fz is a step function for all $z\in M$, $\|[G(v)-G(u)]z\|\leqq 2\,|v-u|$ for $z\in K$ and $u, v\in[a,b]$, and $R(x)z=x$ for $x\in[a,b]$ and $z\in M$ (Z is the set of all real numbers in this example so that $x\in Z$ if $x\in[a,b]$).

Suppose C is a Cantor set lying in the closed number interval $[0,1]$, containing 0 and 1, and having the property that $C\cap[0,x]$ has positive length for all $x>0$. Let the complementary segments of C be arranged in a sequence $\{S_n\}_{n=1}^{\infty}$. For each n, let a_n denote the left end of S_n, b_n the right end of S_n, and m_n the midpoint $(a_n+b_n)/2$. Let h_n denote the function from $[a_n,b_n]$ onto $[0,1]$ defined by

$$h_n(x)=(x-a_n)/(m_n-a_n)\quad\text{for } a_n\leqq x\leqq m_n\,,$$

and

$$h_n(x)=(b_n-x)/(b_n-m_n)\quad\text{for } m_n\leqq x\leqq b_n\,.$$

Let π denote the Euclidean plane, and let I_n denote the closed vertical interval in π with ends (m_n,a_n) and (m_n,b_n). Let f denote the function from π onto $[1,2]$ defined by

$$f(x,y)=1\quad\text{if } (x,y)\text{ is in no } I_n\,,$$

and

$$f(x,y)=1+h_n(y)\quad\text{if } (x,y)\in I_n\,.$$

f is bounded, $f(x, y)$ is continuous in y for each x, and $f(x, y)$ is a step function in x for each y. If y is a number, and $x \in [0, 1]$, then

$$\int_0^x f(t, y)dt = x ,$$

because $f(t, y) = 1$ except for at most one number t. If y is a real valued function defined on $[0, 1]$, then $f(t, y(t)) = 1$ except for at most countably many numbers t, so that

$$\int_0^x f(t, y(t))dt = x$$

for all $x \in [0, 1]$, provided the integral exists. Therefore, if

$$y(x) = \int_0^x f(t, y(t))dt$$

for all $x \in [0, 1]$, it follows that $y(x) = x$ for all $x \in [0, 1]$. But $\int_0^x f(t, t)dt$ does not exist if $x > 0$, because $f(t, t)$ has oscillation 1 at all $t \in C$.

Take Z to be the set of all real numbers, $M = [0, 1]$, $K = [1, 2]$, $[a, b] = [0, 1]$, $G(x)z = xz$, $F(x)z = f(x, z)$, and $A = 0$. Then $R(x)z = x$ for all $x \in [a, b]$. Take $Y(x) = x$ for all $x \in [a, b]$. Then

$$Y(x) = A + \int_a^x dR \cdot Y = \int_0^x 1dt = x$$

for all $x \in [a, b]$, but

$$\int_a^x dG \cdot FY = \int_0^x f(t, t)dt$$

does not exist if $x > a$.

THEOREM 3.3. *If M is compact, then the approximating sums for $\int_a^b dG \cdot Fz$ converge uniformly for all $z \in M$.*

Proof. Suppose M is compact, $\varepsilon > 0$, $\delta > 0$, and $[h(b)-h(a)]\, W(\delta) < \varepsilon$. Let M' denote a finite subset of M such that, if $z \in M$, then there is a $w \in M'$ such that $\| z - w \| < \delta$. Let Δ denote a partition of $[a, b]$ such that, if Δ' is a refinement of Δ, X' is an interpolating sequence for Δ', and $w \in M'$, then

$$\left\| \int_a^b dG \cdot Fw - \Sigma(\Delta', X', G, Fw) \right\| < \varepsilon .$$

An observation of the inequality (3.1.1) and an observation of the argument used in obtaining this enequality reveals the fact that

$$\left\| \int_a^b dG \cdot Fz - \Sigma(\Delta', X', G, Fz) \right\| < 3\varepsilon$$

if Δ' is a refinement of Δ, X' is an interpolating sequence for Δ', and $z \in M$.

THEOREM 3.4. *If we remove the condition that* $\int_a^b dG \cdot Fz$ *exists for all* $z \in M$, *and suppose that the collection* $\{Fz\}(z \in M)$ *is an equicontinuous collection of functions from* $[a, b]$ *into* K, *then not only does it follow that the integral* $\int_a^b dG \cdot Fz$ *exists for all* $z \in M$, *but also that the approximating sums for this integral converge uniformly for all* $z \in M$.

Proof. Suppose that the collection $\{Fz\}(z \in M)$ is equicontinuous. Suppose $\varepsilon > 0$, $\delta > 0$, $[h(b) - h(a)]U(\delta) < \varepsilon$, $\delta' > 0$, $\| F(u)z - F(v)z \| < \delta$ for $|u - v| < \delta'$ and $z \in M$, Δ is a partition of $[a, b]$ with mesh less than δ', and X is an interpolating sequence for Δ. Then

$$\| \Sigma(\Delta, X, G, Fz) - \Sigma(\Delta', X', G, Fz) \| < \varepsilon$$

if Δ' is a refinement of Δ, X' is an interpolating sequence for Δ', and $z \in M$.

THEOREM. 3.5. *Suppose* $\varepsilon > 0$, Y_1 *and* Y_2 *are two continuous functions from* $[a, b]$ *into* M, *and*

$$\left\| Y_j - A - \int_a^x dG \cdot FY_j \right\| < \varepsilon/2$$

for all $x \in [a, b]$, $j = 1, 2$. *Then*

$$\int_\varepsilon^y [1/W(s)]ds \leqq h(x) - h(a)$$

if $x \in [a, b]$ *and* $y = \| Y_2(x) - Y_1(x) \| > 0$.

Proof. If $a < x \leqq b$, Δ is a partition of $[a, x]$, X is an interpolating sequence for Δ, Σ_1 denotes $\Sigma(\Delta, X, G, FY_1)$, and Σ_2 denotes $\Sigma(\Delta, X, G, FY_2)$, then

$$\begin{aligned} \| Y_2(x) - Y_1(x) \| &< \varepsilon + \left\| \int_a^x dG \cdot FY_1 - \int_a^x dG \cdot FY_2 \right\| \\ &\leqq \varepsilon + \left\| \int_a^x dG \cdot FY_1 - \Sigma_1 \right\| + \left\| \Sigma_2 - \int_a^x dG \cdot FY_2 \right\| + \| \Sigma_1 - \Sigma_2 \| , \end{aligned}$$

and

$$\| \Sigma_1 - \Sigma_2 \| \leqq \sum_{j=0}^{|\Delta|} (\Delta h_j) W(\| Y_2(X_j) - Y_1(X_j) \|) .$$

Therefore,

$$\| Y_2(x) - Y_1(x) \| \leqq \varepsilon + \int_a^x W(\| Y_2 - Y_1 \|) dh$$

for all $x \in [a, b]$. Let

$$D(x) = \varepsilon + \int_a^x W(\| Y_2 - Y_1 \|) dh$$

for all $x \in [a, b]$. Then, if $a \leqq u < v \leqq b$, it follows that

$$0 \leqq D(v) - D(u) = \int_u^v W(\| Y_2 - Y_1 \|) dh \leqq \int_u^v W(D) dh$$

so that D is continuous and nondecreasing.

Suppose $x \in [a, b]$, and $\| Y_2(x) - Y_1(x) \| > \varepsilon$. Then $D(x) > \varepsilon$, and $x > a$. Let c denote a number in the open interval (a, x) such that $D(c) > \varepsilon$. Then $D(t) > \varepsilon$ and $W[D(t)] > 0$ for all $t \in [c, x]$. If $[u, v]$ is a subinterval of $[c, x]$, it follows that

$$D(v) - D(u) \leqq [h(v) - h(u)] W(D(v)) ,$$

and

$$[D(v) - D(u)] / W(D(v)) \leqq h(v) - h(u) .$$

If Δ is a partition of $[c, x]$ then

$$\sum_{j=0}^{|\Delta|} (\Delta D_j) / W(D(\Delta_{j+1})) \leqq h(x) - h(c) ,$$

so that

$$h(x) - h(c) \geqq \int_c^x (1 / W[D]) dD = \int_{D(c)}^{D(x)} [1 / W(s)] ds .$$

The conclusion follows readily.

COROLLARY. *Suppose the improper integral* $\int_0^1 [1/W(s)] ds$ *diverges. Then there are not two continuous functions* Y *from* $[a, b]$ *into* M *such that* $Y(x) = A + \int_a^x dG \cdot FY$ *for all* $x \in [a, b]$.

Proof. If Y_1 and Y_2 are two such functions, $x \in [a, b]$, and $y = \| Y_2(x) - Y_1(x) \| > 0$, then

$$\int_{\varepsilon}^{y}[1/W(s)]ds \leqq h(x) - h(a)$$

for all $\varepsilon > 0$, a contradiction. For an earlier theorem of this type see W. F. Osgood [2], page 344.

4. Product integrals. Suppose that, for some $z_0 \in K$, the function Gz_0 from $[a, b]$ into Z is continuous and of bounded variation. Then, if K is bounded, it follows from inequality (3.2) that G is of bounded variation with continuous total variation in the following sense. There is a continuous function V (called a variation function, see [1], page 530) from $[a, b] \times [a, b]$ into $[0, \infty)$ such that $V(p, q) = V(q, p)$, $V(p, p) = 0$, $V(p, r) = V(p, q) + V(q, r)$ for $a \leqq p \leqq q \leqq r \leqq b$, and

$$\|[G(v) - G(u)]z\| \leqq V(u, v) \tag{4.1}$$

for $u, v \in [a, b]$ and $z \in K$. Let us now require that K be bounded and denote by V one such variation function. It is of interest in connection with the corollary to Theorem 3.5 to notice that now, if Y is a function from $[a, b]$ into M such that $Y(x) = A + \int_a^x dG \cdot FY$ for all $x \in [a, b]$, then $\|Y(v) - Y(u)\| \leqq V(u, v)$ for all $u, v \in [a, b]$. Suppose $r > 0$, $z \in M$ if $\|z - A\| < r$, and $V(a, b) < r$.

THEOREM 4.1. *Suppose $P \in M$, $a \leqq u < v \leqq b$, $z \in M$ if $\|z - P\| \leqq V(u, v)$, Δ is a partition of $[u, v]$, X is an interpolating sequence for Δ, $P_0 = P$, and $P_{k+1} = [1 + \Delta G_k F(X_k)]P_k$ for $k = 0, 1, \cdots, |\Delta|$. Then*
(i) *$P_k \in M$ for $k = 0, 1, \cdots, |\Delta| + 1$,*
(ii) *$\|P_m - P_n\| < V(\Delta_m, \Delta_n)$ for $m, n = 0, 1, \cdots, |\Delta| + 1$, and*
(iii) *if $J = \pi_u^v(1 + dG \cdot F)P$, then $\|J - P\| \leqq V(u, v)$.*

Proof. $\|P_1 - P_0\| = \|\Delta G_0 F(X_0)P_0\| \leqq V(\Delta_0, \Delta_1)$. If $k < |\Delta| + 1$, and $\|P_k - P_0\| \leqq V(\Delta_0, \Delta_k)$, then

$$\begin{aligned}\|P_{k+1} - P_0\| &\leqq \|P_{k+1} - P_k\| + \|P_k - P_0\| \\ &= \|\Delta G_k F(X_k)P_k\| + \|P_k - P_0\| \\ &\leqq V(\Delta_k, \Delta_{k+1}) + V(\Delta_0, \Delta_k) = V(\Delta_0, \Delta_{k+1}) .\end{aligned}$$

Therefore, $\|P_k - P_0\| \leqq V(\Delta_0, \Delta_k) \leqq V(u, v)$ so that $P_k \in M$ for $k = 0, 1, \cdots, |\Delta| + 1$. This establishes (i), and (ii) and (iii) follow quite readily.

THEOREM 4.2. *Suppose $Y(x) = \pi_a^x(1 + dG \cdot F)A$ for all $x \in [a, b]$. Then $Y(v) = \pi_u^v(1 + dG \cdot F)Y(u)$ for $a \leqq u < v \leqq b$, so that $\|Y(v) - Y(u)\| \leqq V(u, v)$, and Y is a continuous function from $[a, b]$ into M.*

Proof. It follows from (iii) of Theorem 4.1 that $\| Y(x) - A \| \leqq V(a, x) < r$, so that $Y(x) \in M$ for $x \in [a, b]$. Suppose $a \leqq u < v \leqq b$. If $\| z - Y(u) \| \leqq V(u, v)$, then $\| z - A \| \leqq V(a, v) < r$, so that $z \in M$. If $\| P - Y(u) \| < r - V(a, v)$, and $\| z - P \| \leqq V(u, v)$, then $\| z - A \| < r$, and $z \in M$.

If Δ is a partition of $[u, v]$, then let H_Δ denote the function from $[0, \infty)$ into $[0, \infty)$ which is obtained in the following manner. Let

$$H_1(\delta) = \delta + (\Delta h_0) W(\delta) ,$$

let

$$H_{k+1}(\delta) = H_k(\delta) + (\Delta h_k) W[H_k(\delta)]$$

for $k = 1, \cdots, |\Delta|$, and let $H_\Delta = H_{|\Delta|+1}$. Notice that H_Δ is nondecreasing and $H_\Delta(0) = H_\Delta(0+) = 0$, since H_Δ is the composition of $|\Delta|$ functions having these properties.

If Δ is a partition of $[u, v]$, X is an interpolating sequence for Δ, $\| P - Y(u) \| < r - V(a, v)$, $Y_0 = Y(u)$, $P_0 = P$, and

$$Y_{k+1} = [1 + \Delta G_k F(X_k)] Y_k ,$$
$$P_{k+1} = [1 + \Delta G_k F(X_k)] P_k$$

for $k = 0, 1, \cdots, |\Delta|$, then (i) of Theorem 4.1 assures us that $P_k, Y_k \in M$ for $k = 0, 1, \cdots, |\Delta| + 1$. Moreover, if $\delta_k = \| P_k - Y_k \|$, then we get

$$\delta_{k+1} \leqq \delta_k + (\Delta h_k) W(\delta_k) ,$$

so that

$$\delta_{|\Delta|+1} = \| \pi(\Delta, X, G, F)P - \pi(\Delta, X, G, F) Y(u) \| \leqq H_\Delta(\delta_0) .$$

Suppose $\varepsilon > 0$, and let Δ denote a partition of $[a, v]$ such that u is a term of Δ, and

$$\| Y(v) - \pi(\Delta', X', G, F)A \| < \varepsilon/2$$

if Δ' is a refinement of Δ, and X' is an interpolating sequence for Δ'. Let α denote the section of Δ from a to u, and let β denote the section of Δ from u to v. We wish to show that

$$\| Y(v) - \pi(\beta', X', G, F) Y(u) \| < \varepsilon$$

if β' is a refinement of β and X' is an interpolating sequence for β'.

Suppose β' is a refinement of $\beta, \delta > 0, \delta < r - V(a, v)$, and $H_{\beta'}(\delta) < \varepsilon/2$. Let α' denote a refinement of α such that

$$\| Y(u) - \pi(\alpha', X', G, F)A \| < \delta$$

if X' is an interpolating sequence for α'. Let Δ' denote the refinement of Δ which has α' as its section from a to u and β' as its section from u to v. Let X' denote an interpolating sequence for Δ', let X^α denote the α'-section of X', and let X^β denote the β'-section of X'. Then

$$\begin{aligned}
&\| Y(v) - \pi(\beta', X^\beta, G, F) Y(u) \| \\
&\quad \leqq \| Y(v) - \pi(\beta', X^\beta, G, F)\pi(\alpha', X^\alpha, G, F)A \| \\
&\qquad + \| \pi(\beta', X^\beta, G, F)\pi(\alpha', X^\alpha, G, F)A - \pi(\beta', X^\beta, G, F) Y(u) \| < \varepsilon ,
\end{aligned}$$

since

$$\pi(\beta', X^\beta, G, F)\pi(\alpha', X^\alpha, G, F)A = \pi(\Delta', X', G, F)A .$$

Thus $Y(v) = \pi_u^v(1 + dG \cdot F) Y(u)$, and by (iii) of Theorem 4.1 we have $\| Y(v) - Y(u) \| \leqq V(u, v)$.

THEOREM 4.3. *Suppose* $Y(x) = \pi_a^x(1 + dG \cdot F)A$ *for all* $x \in [a, b]$. *Then* $Y(x) = A + \int_a^x dG \cdot FY$ *for all* $x \in [a, b]$.

Proof. Suppose $a < x \leqq b$, $\varepsilon > 0$, $\delta > 0$, $[h(x) - h(a)] W(\delta) < \varepsilon$, $\delta' > 0$, and $V(u, v) < \delta/3$ if $|u - v| < \delta'$.

Let Δ denote a partition of $[a, x]$ with mesh less than δ' such that

$$\left\| \int_a^x dG \cdot FY - \Sigma(\Delta', X', G, FY) \right\| < \varepsilon$$

if Δ' is a refinement of Δ and X' is an interpolating sequence for Δ'.

For each $k = 1, \cdots, |\Delta| + 1$, let Δ^k denote a partition of $[a, \Delta_k]$ such that, if Δ' is a refinement of Δ^k, and X' is an interpolating sequence for Δ', then

$$\| Y(\Delta_k) - \pi(\Delta', X', G, F)A \| < \min [\varepsilon, \delta/3] .$$

Let Δ' denote a refinement of Δ which has as a term every number which is a term of any Δ^k for $k = 1, \cdots, |\Delta| + 1$, and let X' denote an interpolating sequence for Δ'.

Let $A_0 = A$, and let

$$A_{k+1} = [1 + \Delta' G_k F(X'_k)] A_k$$

for $k = 0, 1, \cdots, |\Delta'|$. For each $k = 0, 1, \cdots, |\Delta'|$, let $m(k)$ denote the greatest integer m such that $m \leqq k$ and Δ'_m is a term of Δ. Then for each $k = 0, 1, \cdots, |\Delta'|$, we have

$$\begin{aligned}
&\| A_k - Y(X'_k) \| \leqq \| A_k - A_{m(k)} \| + \| A_{m(k)} - Y(\Delta'_{m(k)}) \| \\
&\quad + \| Y(\Delta'_{m(k)}) - Y(X'_k) \| < V(\Delta'_{m(k)}, \Delta'_k) + (\delta/3) + V(\Delta'_{m(k)}, X'_k) < \delta .
\end{aligned}$$

Therefore

$$\left\| Y(x) - A - \int_a^x dG \cdot FY \right\| \leqq \| Y(x) - A_{|\Delta'|+1} \|$$
$$+ \sum_{k=0}^{|\Delta'|} \| \Delta' G_k F(X'_k) A_k - \Delta' G_k F(X'_k) Y(X'_k) \|$$
$$+ \left\| \Sigma(\Delta', X', G, FY) - \int_a^x dG \cdot FY \right\|$$
$$< \varepsilon + [h(x) - h(a)] W(\delta) + \varepsilon < 3\varepsilon ,$$

since

$$A_{|\Delta'|+1} = A + \sum_{k=0}^{|\Delta'|} \Delta' G_k F(X'_k) A_k .$$

DEFINITION. If Δ is a partition of $[a, b]$, X is an interpolating sequence for Δ, and Y is the function from $[a, b]$ into M defined by $Y(a) = A$, and

$$Y(x) = \{1 + [G(x) - G(\Delta_k)] F(X_k)\} Y(\Delta_k)$$

for $x \in [\Delta_k, \Delta_{k+1}]$, $k = 0, 1, \cdots, |\Delta|$, then Y is called the approximate solution constructed from (Δ, X, G, F, A). Such a function Y is a continuous function from $[a, b]$ into M and satisfies $\| Y(v) - Y(u) \| \leqq V(u, v)$ for $u, v \in [a, b]$.

If $\varepsilon > 0$, then the statement that Δ is an ε-approximate partition of $[a, b]$ for (G, F, A) means Δ is a partition of $[a, b]$, and, if Δ' is a refinement of Δ, X' is an interpolating sequence for Δ', and Y is the approximate solution constructed from (Δ', X', G, F, A), then

$$\left\| Y(x) - A - \int_a^x dG \cdot FY \right\| < \varepsilon \quad \text{for all } x \in [a, b] .$$

THEOREM 4.4. *Suppose $\varepsilon > 0$, and the approximating sums for $\int_a^b dG \cdot Fz$ converge uniformly for all $z \in M$. Then there is an ε-approximate partition of $[a, b]$ for (G, F, A).*

Proof. Let C denote the collection of all functions Y from $[a, b]$ into M such that $\| Y(v) - Y(u) \| \leqq V(u, v)$ for all $u, v \in M$. Then C is an equicontinuous collection. Suppose $\delta > 0$, $[h(b) - h(a)]\, W(\delta) < \varepsilon/4$, $\delta' > 0$, and $V(u, v) < \min [\delta, \varepsilon/4]$ if $|u - v| < \delta'$. Let Δ denote a partition of $[a, b]$ with mesh less than δ' such that, if Δ' is a refinement of Δ, X' is an interpolating sequence for Δ', and $Y \in C$, then

$$\left\| \int_a^b dG \cdot FY - \Sigma(\Delta', X', G, FY) \right\| < \varepsilon/4 .$$

We shall show that Δ is an ε-approximate partition of $[a, b]$ for (G, F, A). Suppose Δ' is a refinement of Δ, X' is an interpolating sequence for Δ', and Y is the approximate solution constructed from (Δ', X', G, F, A). Then

$$Y(\Delta'_{k+1}) = A + \sum_{j=0}^{k} \Delta' G_j F(X'_j) Y(\Delta'_j)$$

for $k = 0, 1, \cdots, |\Delta'|$. Also, $Y \in C$.

For $k = 1, \cdots, |\Delta'| + 1$, we have

$$\begin{aligned}
&\left\| A + \int_a^{\Delta'_k} dG \cdot FY - Y(\Delta'_k) \right\| \\
&\quad = \left\| \int_a^{\Delta'_k} dG \cdot FY - \sum_{j=0}^{k-1} \Delta' G_j F(X'_j) Y(\Delta'_j) \right\| \\
&\quad \leqq \left\| \int_a^{\Delta'_k} dG \cdot FY - \sum_{j=0}^{k-1} \Delta' G_j F(X'_j) Y(X'_j) \right\| \\
&\qquad + \sum_{j=0}^{k-1} \| \Delta' G_j F(X'_j) Y(X'_j) - \Delta' G_j F(X'_j) Y(\Delta'_j) \| \\
&\quad \leqq (\varepsilon/4) + [h(\Delta'_j) - h(a)] W(\delta) < \varepsilon/2 .
\end{aligned}$$

Suppose $a < x \leqq b$, and k is an integer such that $x \in [\Delta'_k, \Delta'_{k+1}]$. Then

$$\| Y(\Delta'_k) - Y(x) \| \leqq V(\Delta'_k, x) < \varepsilon/4 ,$$

and

$$\left\| \int_a^x dG \cdot FY - \int_a^{\Delta'_k} dG \cdot FY \right\| \leqq V(\Delta'_k, x) < \varepsilon/4 ,$$

so that

$$\left\| A + \int_a^x dG \cdot FY - Y(x) \right\| < \varepsilon .$$

THEOREM 4.5. *If M is compact, then*

(i) *there is a continuous function Y from $[a, b]$ into M such that $Y(x) = A + \int_a^x dG \cdot FY$ for all $x \in [a, b]$, and*

(ii) *if there is only one such function Y from $[a, b]$ into M, then $Y(x) = \pi_a^x(1 + dG \cdot F)A$ for all $x \in [a, b]$.*

Proof. Suppose M is compact. For each $n = 1, 2, \cdots$, let Δ^n denote a $(1/n)$-approximate partition of $[a, b]$ for (G, F, A), let X^n denote an interpolating sequence for Δ^n, and let Y_n denote the approximate solution constructed from (Δ^n, X^n, G, F, A). Since the Y_n form an equicontinuous collection, some subsequence of $\{Y_n\}_{n=1}^\infty$ converges uniformly to a continuous function Y from $[a, b]$ into M. Let Y denote one such function. Then $Y(x) = A + \int_a^x dG \cdot FY$ for all $x \in [a, b]$.

Suppose $x \in [a, b]$, and $Y(x)$ is not the product integral $\pi_a^x(1 + dG \cdot F)A$. Then $x > a$. For each n, require that the above defined Δ^n have x as a term, and let α^n denote its section from a to x. There is a positive number ε such that, if n is a positive integer, then there is a refinement α' of α^n and an interpolating sequence β' for α' such that

$$|| Y(x) - \pi(\alpha', \beta', G, F)A || \geqq \varepsilon .$$

Let ε denote one such positive number, and for each n, let α'^n denote such a refinement of α^n, β'^n such an interpolating sequence for α'^n, Δ'^n a refinement of Δ^n which has α'^n as its section from a to x, X'^n an interpolating sequence for Δ'^n which has β'^n as its α'^n-section, and H_n the approximate solution constructed from $(\Delta'^n, X'^n, G, F, A)$.

Some subsequence of $\{H_n\}_{n=1}^\infty$ converges uniformly to a continuous function H from $[a, b]$ into M. Let H denote one such function. Then $H(t) = A + \int_a^t dG \cdot FH$ for all $t \in [a, b]$. Since

$$H_n(x) = \pi(\alpha'^n, \beta'^n, G, F)A$$

for all n, it follows that $|| H(x) - Y(x) || \geqq \varepsilon$.

THEOREM 4.6. *Suppose that the approximating sums for* $\int_a^b dG \cdot Fz$ *converge uniformly for all* $z \in M$, *and that the improper integral* $\int_0^1 [1/W(s)]ds$ *diverges. Then* $\pi_a^x(1 + dG \cdot F)A$ *exists for all* $x \in [a, b]$.

Proof. Suppose $a < x \leqq b$, and $\varepsilon > 0$. Let δ denote a positive number such that $y < \varepsilon$ if

$$\int_\delta^y [1/W(s)]ds \leqq [h(x) - h(a)] .$$

Let Δ denote a $(\delta/2)$-approximate partition of $[a, b]$ for (G, F, A) which has x as a term, and let α denote the section of Δ from a to x. Suppose β is an interpolating sequence for α, α' is a refinement of α, and β' is an interpolating sequence for α'. Let

$$y = || \pi(\alpha, \beta, G, F)A - \pi(\alpha', \beta', G, F)A || .$$

It follows from Theorem 3.5 and the definitions of α, β, α', and β' that either $y = 0$, or

$$\int_\delta^y [1/W(s)]ds \leqq h(x) - h(a) ,$$

so that $y < \varepsilon$.

REMARKS. Limits on the difference

$$\| \pi_a^x(1 + dG \cdot F)A - \pi(\Delta, X, G, F)A \|$$

may be obtained by observing the arguments for Theorems 4.4 and 4.6, together with whatever theorem or theorems from § 3 might be appropriate to the problem at hand. In case the approximating sums for $\int_a^b dG \cdot Fz$ do not converge uniformly, then the theorems requiring this condition can still be applied to the reduced problem (1.3). Let I denote the function from $[a, b]$ into B defined by $I(x)z = z$ for $z \in M$ and $I(x)z = A$ for $z \notin M$, replace F by I, take $K = M$, replace G by R, replace U by W, and take $E(s) = s$ for all $s \geqq 0$. This still covers more problems than [1] because of the weaker substitute for the Lipschitz condition.

BIBLIOGRAPHY

1. J. W. Neuberger, *Continuous products and nonlinear integral equations*, Pacific J. Math., **8** (1958), 529-549.

2. W. F. Osgood, *Beweis der Existenz einer Lösung der Differentialgleichung* $dy/dx = f(x, y)$ *ohne hinzunahme der Cauchy Lipschitz' chen Bedingung*, Monatsh. f. Math. u. Phys., **9** (1898), 331-345.

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