A SUBFUNCTION APPROACH TO A BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

J. W. BEBERNES

1. Introduction. Consider the ordinary, second-order differential equation

$$y'' = f(x, y, y') \tag{1.1}$$

where $f(x, y, y')$ is a real-valued function defined on the region

$$T = \{(x, y, y') \mid a \leqq x \leqq b, |y| < \infty, |y'| < \infty\},$$

a and b finite.

The purpose of this paper is to determine sufficient conditions which when placed on $f(x, y, y')$ guarantee the existence of a unique solution of the two-point boundary value problem (BVP):

$$y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta. \tag{1.2}$$

A solution of the BVP: $y'' = f(x, y, y')$, $y(x_1) = y_1$, $y(x_2) = y_2$, where $a \leqq x_1 \leqq x_2 \leqq b$ will be defined to be a function $y(x)$ which is of class C^2 and satisfies (1.1) on (x_1, x_2), which is continuous on $[x_1, x_2]$, and which assumes the given boundary values at x_1 and x_2.

The following assumptions will be placed on $f(x, y, y')$ as needed.

(A_0) $f(x, y, y')$ is continuous on T.

(A_1) $f(x, y, y')$ is a non-decreasing function of y for each fixed x and y' in T.

(A_2) $f(x, y, y')$ satisfies a Lipschitz condition with respect to y' on each fixed compact subset to T.

The primary results of this paper are the following two theorems.

THEOREM 6.2. *If*

(1) $f(x, y, y')$ *satisfies* A_0, A_1, *and* A_2,

(2) *there exists a positive continuous function* $\phi(u)$ *defined for* $u \geqq 0$ *such that*

$$|f(x, y, y') - f(x, y, 0)| \leqq K_s\phi(|y'|)$$

where K_s *is a constant depending on compact subsets* S *of*

$$\{(x, y) \mid a \leqq y \leqq b, |y| < \infty\}, \quad (x, y) \in S, \quad |y'| < \infty,$$

Received Feburary 13, 1963. Research was in part supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command. This paper is a portion of a doctoral thesis written under the supervision of Professor Lloyd K. Jackson at the University of Nebraska.

and

$$\int_0^\infty \frac{udu}{\phi(u)+1} = \infty ,$$

(3) $|f(x, 0, y') - f(x, 0, 0)| \leqq K|y'|$ *for* $a \leqq x \leqq b$ *and* $|y'| < \infty$, *then there exists a unique solution of the* BVP (1.2) *of class* $C^2[a, b]$.

THEOREM 6.3. *If given the* BVP: $y'' = f(x, y, y')$, $y(a) = 0 = y(b)$, *with*

(1) $f(x, y, y')$ *satisfying* A_0, A_1, A_2, *and*

(2) $|f(x, 0, y') - f(x, 0, 0)| \leqq K|y'|$ *for all* $a \leqq x \leqq b$ *and* $|y'| < +\infty$, *then there exists a unique solution of the* BVP *of class* $C^2[a, b]$.

Lees [4] proved that if $f(x, y, y')$ satisfies A_0, A_1, and, in place of A_2, a uniform Lipschitz condition with respect to y', then there exists a unique solution of the BVP (1.2) of class $C^2[a, b]$. Note that Lees' result is immediate from Theorem 6–2. Whereas Lees used the method of finite differences, we shall attack the BVP (1.2) employing the theory of subfunctions developed by Fountain and Jackson [3]. In [3], Fountain and Jackson utilized the theory of subfunctions to construct a so-called generalized solution of the BVP (1.2) in the sense that if a solution exists it will be this generalized solution by uniqueness. To construct this generalized solution, they assumed $f(x, y, y')$ satisfies A_0, A_1, and, in place of A_2, that:

(A_2') $f(x, y, y')$ satisfies a Lipschitz condition with respect to y and y' on each fixed compact subset of T.

In this paper, we shall construct the generalized solution of the BVP (1.2) as in [3] under assumptions A_0, A_1 and A_2. Most of the proofs carry over with only slight modifications. By knowing properties of this generalized solution, additional conditions can then be imposed on $f(x, y, y')$ to assure a solution of the BVP (1.2).

2. A "local" existence theorem and a maximum principle.

The two theorems in this section form the basis for the construction of the generalized solution of the BVP (1.2). The "local" existence theorem is known and was first proven by E. Picard [6, pp. 9–36] in a slightly weaker form.

THEOREM 2.1. *Let* $f(x, y, y')$ *satisfy* A_0, *then given* $M, N > 0$ *there is a* $\delta(M, N) > 0$ *such that the* BVP:

$$y'' = f(x, y, y') , \quad y(x_1) = y_1 , \quad y(x_2) = y_2 \tag{2.1}$$

has a solution of class C^2 *on* $[x_1, x_2]$ *for any points* (x_1, y_1) *and* (x_2, y_2) *with* $x_1, x_2 \in [a, b]$, $|x_1 - x_2| \leqq \delta$, $|y_1| \leqq M$, $|y_2| \leqq M$, *and* $|(y_1 - y_2)/(x_1 - x_2)| \leqq N$.

The details of the proof of this result will not be given. Note that the BVP (2.1) has a solution if and only if the BVP:

$$\begin{aligned} y'' &= f(x, z(x) + px + q, z'(x) + p) \\ z(x_1) &= 0 \qquad z(x_2) = 0 \end{aligned} \tag{2.2}$$

has a solution, and that the BVP (2.2) has a solution if there is a function $z(x) \in C'[x_1, x_2]$ which is a solution of

$$z(x) = \int_{x_1}^{x_2} G(x, s) f(s, z(s) + ps + q, z'(s) + p) ds \tag{2.3}$$

where

$$G(x, s) = \begin{cases} (x_1 - s)(x_2 - x)(x_2 - x_1)^{-1}, & x_1 \leqq s \leqq x \\ (x_1 - x)(x_2 - s)(x_2 - x_1)^{-1}, & x \leqq s \leqq x_2 . \end{cases}$$

The Schauder-Tychonoff fixed-point theorem [2, p. 456] can be employed to show the existence of a solution of (2.3) of class C' on $[x_1, x_2]$.

The following corollary follows as an immediate consequence of the proof of Theorem 2.1.

COROLLARY 2.1. *Let* $M > 0$, $N > 0$ *be fixed and let* $\delta(M, N) > 0$ *be as in Theorem* 2.1. *Then given any* $\varepsilon > 0$ *and any* δ_1, $0 < \delta_1 \leqq \delta(M, N)$, *there is an* η, $0 < \eta \leqq \delta_1$, *such that for any points* (x_1, y_1) *and* (x_2, y_2) *with* $x_1, x_2 \in [a, b]$, $|x_1 - x_2| \leqq \eta$, $|y_1| \leqq M$, $|y_2| \leqq M$, *and* $|(y_1 - y_2)(x_1 - x_2)^{-1}| \leqq N$, *there is a solution* $y(x)$ *of the* BVP (2.1) *on* $[x_1, x_2]$ *such that* $|y(x) - w(x)| \leqq \varepsilon$ *and* $|y'(x) - w'(x)| \leqq \varepsilon$ *where* $w(x)$ *is the linear function with* $w(x_1) = y_1$ *and* $w(x_2) = y_2$.

For any function g defined on $[x_1, x_2]$ and $x_0 \in (x_1, x_2)$, define $\bar{D}g(x_0) = \limsup [g(x_0 + \delta) - g(x_0 - \delta)]/2\delta$ as $\delta \to 0$ and $\underline{D}g(x_0) = \liminf [g(x_0 + \delta) - g(x_0 - \delta)]/2\delta$ as $\delta \to 0$.

LEMMA 2.1. *If* $f(x, y, y')$ *satisfies* A_0 *and* A_1 *and if the functions* $\phi(x)$ *and* $\psi(x)$ *satisfy*:

(i) $\phi, \psi \in C[c, d] \cap C'(c, d)$ *for* $[c, d] \subset [a, b]$,

(ii) $\underline{D}\phi'(x) \geqq f(x, \phi(x), \phi'(x))$ *and* $\bar{D}\psi'(x) \leqq f(x, \psi(x), \psi'(x))$ *on* (c, d) *with at least one of these being strict inequality on* (c, d), *and*

(iii) $\phi(c) - \psi(c) \leqq M$ *and* $\phi(d) - \psi(d) \leqq M$ *where* $M \geqq 0$, *then*

$$\varphi(x) - \psi(x) < M \quad \text{on } (c, d) .$$

Observe that it suffices to consider only the case $M = 0$. If we assume the conclusion is false, we have an almost immediate contradiction.

LEMMA 2.2. *If* $f(x, y, y')$ *satisfies* A_0, A_1, *and* A_2, *and if there is a function* ϕ *such that*:

(i) $\phi \in C'[x_1, x_2]$ *where* $[x_1, x_2] \subset [a, b]$,

(ii) $\underline{D}\phi'(x) \geqq f(x, \phi(x), \phi'(x))$ *on* (x_1, x_2),

then given $\varepsilon > 0$ *there is a function* $\phi_1 \in C'[x_1, x_2]$ *such that* $\phi(x) - \varepsilon \leqq \phi_1(x) \leqq \phi(x)$ *on* $[x_1, x_2]$ *and*

$$\underline{D}\phi_1'(x) > f(x, \phi_1(x), \phi_1'(x)) \quad on \ (x_1, x_2) .$$

Proof. Let $|\phi(x)| + |\phi'(x)| \leqq R$ on $[x_1, x_2]$ and let

$$T^* = \{(x, y, y') \mid x_1 \leqq x \leqq x_2, |y| \leqq R + 1, |y'| \leqq R + 1\} .$$

Let $K > 0$ be such that $|f(x, y, y_1') - f(x, y, y_2')| \leqq K|y_1' - y_2'|$ for all (x, y, y_1'), $(x, y, y_2') \in T^*$. Let $z(x)$ be a solution of $z'' = (K + 1)z'$ on $[x_1, x_2]$ which satisfies the conditions $0 \leqq z(x) \leqq \varepsilon$ and $-1 \leqq z'(x) < 0$ on $[x_1, x_2]$. Then on (x_1, x_2)

$$\begin{aligned} &\underline{D}[\phi'(x) - z'(x)] - f(x, \phi(x) - z(x), \phi'(x) - z'(x)) \\ &\qquad \geqq f(x, \phi(x), \phi'(x)) - f(x, \phi(x), \phi'(x) - z'(x)) - z''(x) \\ &\qquad \geqq -K|z'(x)| - (K + 1)z'(x) \\ &\qquad = -z'(x) > 0 . \end{aligned}$$

Hence, $\phi_1(x) \equiv x(\phi) - z(x)$ is the desired function.

A dual statement holds by reversing the inequalities. We shall refer to the dual of a result by using an asterisk (for example, Lemma 2.2*).

THEOREM 2.2 (Maximum Principle). *If* $f(x, y, y')$ *satisfies* A_0, A_1, *and* A_2, *and if there exist functions* ϕ *and* ψ *satisfying*:

(i) $\phi, \psi \in C'(c, d) \cap C[c, d]$ *for some* $[c, d] \subset [a, b]$,

(ii) $\underline{D}\phi'(x) \geqq f(x, \phi(x), \phi'(x))$ *and* $\bar{D}\psi'(x) \leqq f(x, \psi(x), \psi'(x))$ *on* (c, d),

(iii) $\phi(c) - \psi(c) \leqq M$ *and* $\phi(d) - \psi(d) \leqq M$ *for some* $M \geqq 0$, *then* $\phi(x) - \psi(x) \leqq M$ *on* $[c, d]$.

Proof. It suffices to consider only the case where $M = 0$. Let $M = 0$ and assume conclusion is false. Then there exist $x \in (c, d)$ such that $\phi(x) - \psi(x) > 0$. Let $N = \max[\phi(x) - \psi(x)] > 0$, $x \in [c, d]$, and let $x_0 \in (c, d)$ be such that $\phi(x_0) - \psi(x_0) = N$. There exist $x_1, x_2 \in (c, d)$ such that $\phi(x_1) - \psi(x_1) \leqq N/2$, $\phi(x_2) - \psi(x_2) \leqq N/2$ and $x_1 < x_0 < x_2$. Choose ε such that $0 < \varepsilon \leqq N/2$. By Lemma 2.2*, there exists a function $\psi_1(x) \in C'[x_1, x_2]$ with

$$\psi(x) \leqq \psi_1(x) \leqq \psi(x) + \varepsilon \quad \text{and} \quad \bar{D}\psi_1'(x) < f(x, \psi_1(x), \psi_1'(x)) \quad \text{on } (x_1, x_2) .$$

Then $\phi(x) - \psi_1(x) < N/2$ on (x_1, x_2) by Lemma 2.1. In particular, $\phi(x_0) - \psi_1(x_0) < N/2$ so

$$\phi(x_0) < \psi_1(x_0) + N/2 \leqq \psi(x_0) + \varepsilon + N/2 \leqq \psi(x_0) + N$$

which is a contradiction.

Uniqueness of solutions of the BVP is immediate from the Maximum Principle.

COROLLARY 2.2. *Let $f(x, y, y')$ satisfy A_0, A_1, and A_2. Then the solution of the* BVP: $y'' = f(x, y, y')$, $y(x_1) = y_1$, $y(x_2) = y_2$, *for* $[x_1, x_2] \subset [a, b]$, *if it exists, is unique.*

The following example shows that the Maximum Principle does not remain valid if we replaced A_2 by the following weaker assumption:

(A_2^*) $f(x, y, y')$ satisfies a Hölder condition with exponent α, $0 < \alpha < 1$, with respect to y' on each fixed compact subset of T. Consider the BVP: $y'' = f(x, y, y') = A^{1-\alpha} | a |^{p(\alpha-1)} p^{1-\alpha}(p-1) | y' |^{\alpha}$ with $y(-a) = y(a) = A > 0$ where $\alpha = (p-2)(p-1)^{-1}$, $p > 2$. $f(x, y, y')$ satisfies A_0, A_1, and A_2^*, but $y_1(x) = A$ and $y_2(x) = A | a |^{-p} | x |^p$ are distinct solutions of the BVP.

3. Subfunctions.

The concept of sub- and superfunctions and their properties are fundamental for the remainder of this paper. For this reason some of the known results for subfunctions due to Fountain and Jackson [3] will be stated without proof. In [3] these results were proven assuming $f(x, y, y')$ satisfies A_0, A_1, and A_2'. If $f(x, y, y')$ satisfies A_0, A_1, and A_2, the proofs can be carried through as in [3] with only slight modifications.

Throughout this section, we shall assume that $f(x, y, y')$ satisfies A_0 although no further explicit mention of this fact will be made. I will denote an interval of $[a, b]$, $cl(I)$ the closure of I, and I° the interior of I.

DEFINITION 3.1. A real-valued function s defined on I is said to be subfunction on I in case $s(x) \leqq y(x)$ on $[x_1, x_2]$ for any $[x_1, x_2] \subset I$ and any solution y of (1.1) on $[x_1, x_2]$ with $s(x_1) \leqq y(x_1)$ and $s(x_2) \leqq y(x_2)$.

Superfunctions are defined dually by reversing inequalities in the preceding definition, and dual results hold.

Subfunctions satisfy the following properties.

(3.1). If $s(x)$ is a subfunction on I, then the right- and left-hand limits exist in the extended reals with appropriate limits existing at the endpoints of $cl(I)$, $s(x_0) \leqq \max [s(x_0 - 0), s(x_0 + 0)]$ for $x_0 \in I^\circ$, and the number of discontinuities of $s(x)$ on I is at most countable.

(3.2). If $s(x)$ is a bounded subfunction on I with $cl(I)$ having endpoints x_1 and x_2, then, if

$$d^+s(x_0) = \limsup [s(x) - s(x_0 + 0)]/(x - x_0)$$

and

$$d_+s(x_0) = \liminf [s(x) - s(x_0 + 0)]/(x - x_0) \quad \text{as } x \to x_0^+$$

with $d^-s(x_0)$ and $d_-s(x_0)$ analogously defined, $d^-s(x_0) = d_-s(x_0)$ for all $x_1 < x_0 \leqq x_2$ and $d^+s(x_0) = d_+s(x_0)$ for all $x_1 \leqq x_0 < x_2$ and, hence, $s(x)$ has a finite derivative almost everywhere on I.

(3.3). The supremum of a collection of subfunctions bounded above at each point of I is a subfunction on I.

(3.4). If $s(x)$ is a subfunction of class C' on I, then $\underline{D}s'(x) \geqq f(x, s(x), s'(x))$ on I°.

(3.5). If $f(x, y, y')$ satisfies A_1 and A_2, if $s(x) \in cl(I) \cap C'(I^\circ)$, and if $\underline{D}s'(x) \geqq f(x, s(x), s'(x))$ on I°, then $s(x)$ is a subfunction on I.

(3.6). Let s_1 be a subfunction on I and s_2 a subfunction on $[x_1, x_2] \subset cl(I)$. Assume further that $s_2(x_i) \leqq s_1(x_i)$, $i = 1, 2$, in case $x_i \in I^\circ$. Then

$$s(x) = \begin{cases} s_1(x) , & x \notin [x_1, x_2] \\ \max [s_1(x), s_2(x)] , & x \in [x_1, x_2] \end{cases}$$

is a subfunction on I.

(3.7). If $f(x, y, y')$ satisfies A_1 and A_2, if $s(x)$ and $S(x)$ are continuous sub- and superfunctions on $cl(I)$, if one of $s(x)$ and $S(x)$ is of class C' on I°, and if $s(x_i) \leqq S(x_i)$, $i = 1, 2$, where x_i are the endpoints of I, then $s(x) \leqq S(x)$ on $cl(I)$.

(3.8). If $f(x, y, y')$ satisfies A_1 and A_2 and if $s(x)$ is a continuous subfunction on I, then $s(x) - M$ is a continuous subfunction on I where M is an arbitrary nonnegative constant.

4. A generalized solution of the boundary value problem. A function $H(x)$ will be defined which will be referred to as the "generalized solution" of the BVP in the sense that if a solution of the BVP exists and if $f(x, y, y')$ satisfies certain to-be-determined conditions, then $H(x)$ is that solution.

DEFINITION 4.1. A function $\phi(x)$ is said to be an underfunction with respect to the BVP (1.2) in case $\phi(x)$ is a subfunction on $[a, b]$ with $\phi(a) \leqq \alpha$ and $\phi(b) \leqq \beta$.

Overfunctions are defined dually. For the construction of the "generalized solution," the following assumptions will be needed. Assume

(A_3) $f(x, y, y')$ is such that with respect to the given BVP there is an underfunction which is continuous on $[a, b]$ and there is an overfunction which is continuous on $[a, b]$ and of class C' on (a, b).

DEFINITION 4.2. Let $\{\phi\}$ be the family of all underfunctions with respect to the BVP (1.2) which are continuous on $[a, b]$. Define

$$H(x) = \sup [\phi(x) \mid \phi \in \{\phi\}]$$

for each $x \in [a, b]$. $H(x)$ is called the generalized solution of the BVP.

$H(x)$ satisfies the following properties the proofs of which are given in [3], assuming $f(x, y, y')$ satisfies A_0, A_1, A_2', and A_3. If $f(x, y, y')$ satisfies A_0, A_1, A_2, and A_3, the proofs of these results carry through with some modification. Assuming that $f(x, y, y')$ satisfies A_0, A_1, A_2, and A_3, then:

(4.1). $H(x)$ is both a bounded sub- and superfunction on $[a, b]$ with $H(x) = \min [H(x + 0), H(x - 0)]$, $x \in (a, b)$.

(4.2). $H(x)$ is a solution of $y'' = f(x, y, y')$ on an open subset of $[a, b]$, the complement of which is of measure 0.

(4.3). $DH(x_0 - 0) = DH(x_0 + 0)$ on (a, b) where

$$DH(x_0 \pm 0) = \lim [H(x) - H(x_0 \pm 0)]/(x - x_0) \quad \text{as } x - x_0^{\pm} .$$

More specifically, let E be the set of interior points of $[a, b]$ at which $H(x)$ does not have a finite derivative, then (i) if $H(x + 0) > H(x - 0)$, $DH(x + 0) = DH(x - 0) = +\infty$, (ii) if $H(x + 0) < H(x - 0)$, $DH(x + 0) = DH(x - 0) = -\infty$; (iii) if $x \in E$ is a point of discontinuity, $DH(x + 0) = DH(x - 0) = \pm\infty$.

(4.4) (i) if $DH(b - 0) \neq -\infty$, $H(b - 0) = H(b)$, (ii) if $H(b - 0) < \beta$, $DH(b - 0) = +\infty$, and (iii) if $DH(b - 0)$ is finite, $H(b - 0) = H(b) = \beta$. Similar results hold at $x = a$.

Consider the BVP: $y'' = f(x, y, y') = -18x(y')^4$ with $y(-1) = -1$ and $y(1) = 1$. $f(x, y, y')$ satisfies A_0, A_1, A_2, and A_3. However, no solution of this BVP exists for if it did the solution would be $y_0(x) = x^{1/3}$. It can be shown that $H(x) = y_0(x)$ on $[-1, 1]$. Although the generalized solution exists, this example points out the fact that more stringent conditions must be imposed on $f(x, y, y')$ to guarantee that a solution of the BVP exists.

DEFINITION 4.3. A differential equation $y'' = f(x, y, y')$ is said to satisfy property (P) in case given any $A > 0$ and any compact subset $S \subset \{(x, y) \mid a \leqq x \leqq b, |y| < \infty\}$, there exists $B_{(A, S)} > 0$ such that for any solution $y(x)$ of $y'' = f(x, y, y')$ with the initial condition $y(x_0) = y_0$ and $|y'(x_0)| \leqq A$ where $(x_0, y_0) \in S$ the inequality $|y'(x)| < B_{(A,S)}$ holds as long as $(x, y(x)) \in S$.

THEOREM 4.1. *Let $f(x, y, y')$ satisfy A_i, $i = 0, \cdots, 3$, and let* (1.1) *satisfy property* (P), *then $H(x)$ is a solution of the* BVP (1.2) *of class C^2 on $[a, b]$.*

The proof follows from (4.2), (4.3), and (4.4).

5. **"Natural" conditions for $f(x, y, y')$.** By Theorem 4.1, a solution of the BVP (1.2) exists provided assumptions A_i, $i = 0, \cdots, 3$, and property (P) are satisfied. Of these, assumption A_3 and property (P) are unnatural in the sense that they are not imposed directly on $f(x, y, y')$. "Natural" conditions will be given in this section which imply A_3 and P.

Sufficient conditions to assure that $y'' = f(x, y, y')$ satisfies property (P) are given in the following theorem which is similar to a result due to Nagumo [5, pp. 861–863].

THEOREM 5.1. *Let*

(1) *$f(x, y, y')$ satisfy A_0*

(2) *$\phi(u)$ be a positive continuous function of $u (u \geqq 0)$ such that*

$$|f(x, y, y') - f(x, y, 0)| \leqq K_S \phi(|y'|) \tag{5.1}$$

where K_S is a constant depending on compact subsets

$$S \subset \{(x, y) \mid a \leqq x \leqq b, |y| < \infty\}, \quad (x, y) \in S, \quad |y'| < \infty,$$

and

$$\int_0^\infty \frac{u du}{\phi(u) + 1} = +\infty, \tag{5.2}$$

then $y'' = f(x, y, y')$ satisfies property (P).

Proof. Let S be any compact subset of $\{(x, y) \mid a \leqq x \leqq b, |y| < \infty\}$, then for any $(x_1, y_1), (x_2, y_2) \in S$ there is an $L > 0$ such that $|y_1 - y_2| < L$.

By (5.2), given any $A > 0$ there exists $B_{(A,S)} > A$ such that

$$\int_A^{B(A,S)} \frac{udu}{\phi(u) + 1} > L \cdot K \tag{5.3}$$

where $K = \max [K_S, \max_{(x,y) \in S} |f(x, y, 0)|]$.

Let $y(x)$ be an arbitrary solution of $y'' = f(x, y, y')$ with initial conditions $y(x_0) = y_0$, $y'(x_0) = A$ where $(x_0, y_0) \in S$. Let $x_1 \leqq x \leqq x_2$ be an interval such that $(x, y(x)) \in S$ and $x_1 \leqq x_0 \leqq x_2$. Claim $y'(x) < B_{(A,S)}$ on $[x_1, x_2]$. If not, there would be two points t_1 and t_2 in $[x_1, x_2]$ such that $y'(t_1) = A$, $y'(t_2) = B_{(A,S)}$, and $A \leqq y'(x) \leqq B_{(A,S)}$ for $x \in [t_1, t_2]$ assuming without loss of generality that $t_1 < t_2$.

By (5.1), $f(x, y, y') \leqq K(\phi(|y'|) + 1)$ and thus

$$\frac{y'y''}{\phi(|y'|) + 1} \leqq Ky' .$$

Therefore,

$$\int_A^{B(A,S)} \frac{udu}{\phi(u) + 1} = \int_{t_1}^{t_2} \frac{y'y''dx}{\phi(|y'|) + 1} \leqq K \int_{t_1}^{t_2} y'dx \leqq K \cdot L$$

which contradicts (5.3), and hence $y'(x) < B_{(A,S)}$ on $[x_1, x_2]$. Analogously, one can show that $y'(x) > -B_{(A,S)}$ on $[x_1, x_2]$ whenever $y'(x_0) \geqq -A$.

The following theorem imposes sufficient conditions which imply A_3 and, in fact, more.

THEOREM 5.2. *Let $f(x, y, y')$ satisfy A_i, $i = 0, 1, 2$, and*

$$|f(x, 0, y') - f(x, 0, 0)| \leqq K |y'| \tag{5.4}$$

for all $a \leqq x \leqq b$ and $|y'| < +\infty$, then there exists an overfunction $\psi(x)$ of class C^2 on $[a, b]$ and an underfunction $\phi(x)$ of class C^2 on $[a, b]$.

Proof. Let $R > 0$ be such that $|f(x, 0, 0)| \leqq R$ on $[a, b]$. Define

$$\psi(x) \equiv \varepsilon_1(N - e^{m(x-a)}) \text{ on } [a, b]$$

where $m = K + 1$, $\varepsilon_1 = Rm^{-1} + 1$, and

$$N = \max [\varepsilon_1^{-1} \cdot \alpha + 1, \varepsilon_1^{-1}\beta + e^{m(b-a)}, e^{m(b-a)}] .$$

Using (3.5), it is easily verified that $\psi(x)$ is an overfunction of class $C^2[a, b]$.

Similarly, $\phi(x) = \varepsilon_2(N - e^{m(x-a)})$ where $\varepsilon_2 = -(Rm^{-1} + 1)$ and $N = \max [\alpha\varepsilon_2^{-1} + 1, \beta\varepsilon_2^{-1} + e^{m(b-a)}, e^{m(b-a)}]$ is an underfunction of class $C^2[a, b]$.

The following slightly stronger result can be proved: If $f(x, y, y')$ satisfies A_0, A_1, A_2, and if

$$|f(x, 0, y') - f(x, 0, 0)| \leqq K[|y'| \cdot ln|y'| + 1]$$

where $x \in [a, b]$ and $|y'| < \infty$, then there exists an underfunction ϕ and an overfunction ψ of class C^2 on $[a, b]$ with respect to the BVP (1.2).

The proof of this result follows by observing that any solution of $y'' = K_2 y' ln y'$ where $y(x) \geqq 0$ and $y'(x) \geqq e$ on $[a, b]$ is a superfunction with respect to (1.1) provided $K_2 \leqq -(2K + M)$ where $M = \max|f(x, 0, 0)|$.

In general, one cannot replace (5.4) by the weaker condition:

$$|f(x, 0, y') - f(x, 0, 0)| \leqq K|y'|^{1+v} \tag{5.5}$$

where

$$x \in [a, b], \quad |y'| < \infty, \quad \text{and } v > 0;$$

for consider the BVP:

$$y'' = 1 + (y')^2$$
$$y(0) = 0 \qquad y(\pi/2) = \beta \neq 0 .$$

There exists no continuous underfunction on $[0, \pi/2]$. However, if the length of the interval is suitably restricted, the following result holds.

THEOREM 5.3. *Let $f(x, y, y')$ satisfy A_i, $i = 0, 1, 2$, and* (5.5), *then, if $b - a < (vK_1)^{-1}$, there exists an overfunction $\psi(x)$ and an underfunction $\phi(x)$ of class $C^2[a, b]$ with respect to the BVP* (1.2).

Proof. Observe that any solution of the differential equation

$$y'' = -K_1(y')^{1+v} \tag{5.6}$$

where $K_1 = K + \max|f(x, 0, 0)|$, $x \in [a, b]$, satisfying the conditions $y(x) \geqq 0$ and $y'(x) \geqq 1$ on $[a, b]$ is a superfunction on $[a, b]$ with respect to (1.1).

If $v \neq 1$, the function $\psi(x)$ defined by

$$\psi(x) = [K_1(v-1)]^{-1} \cdot ([vK_1(x-b) + 1]^{(v-1)/v} - [vK_1(a-b) + 1]^{(v-1)/v}) + \eta$$

where $\eta = \max[\alpha, \beta + [K_1(v-1)]^{-1} \cdot [(vK_1(a-b) + 1)^{(v-1)/v} - 1], 0]$ is a solution of (5.6) and, if $b - a < (vK_1)^{-1}$, then $\psi(x) \geqq 0$ and $\psi'(x) \geqq 1$ on $[a, b]$ with $\psi(a) \geqq \alpha$ and $\psi(b) \geqq \beta$. Thus, $\psi(x)$ is an overfunction of class $C^2[a, b]$ with respect to the BVP (1.2) for $v \neq 1$.

If $v = 1$, the function

$$\psi(x) = K_1^{-1} ln|K_1(x-b) + 1| + \eta$$

where

$$\eta = \max [0, \alpha - K_1^{-1} \cdot ln \, | K(a - b) + 1 |, \beta]$$

is an overfunction of class $C^2[a, b]$, provided $b - a < (vK_1)^{-1}$.

An underfunction $\phi(x)$ can similarly be constructed.

6. Some existence theorems. Having found sufficient conditions which when imposed upon $f(x, y, y')$ imply assumptions A_3 and P, we proceed to state and prove some existence theorems for the BVP (1.2).

THEOREM 6.1. *If $f(x, y, y')$ satisfies A_i, $i = 0, 1, 2$, and if*

$$(6.1) \quad |f(x, 0, y') - f(x, 0, 0)| \leqq K |y'| \quad \textit{where } x \in [a, b] \textit{ and } |y'| < \infty ,$$

then $H(x)$, the generalized solution of the BVP (1.2), *has a finite derivative at each point of (a, b). In fact, $|H'(x_0)| \leqq M$ where the constant M depends on $x_0, a, b, \alpha, \beta, K$, and $R = \max |f(x, 0, 0)|$ on $[a, b]$.*

Proof. $H(x)$ exists on $[a, b]$. Let $x_0 \in (a, b)$ and consider the two possible cases: (i) $H(x_0) \leqq 0$ and (ii) $H(x_0) > 0$.

(i) Let $H(x_0) \leqq 0$. By the known properties of $H(x)$, $H(x_0 - 0)$ and $H(x_0 + 0)$ exist and $H(x_0) = \min [H(x_0 - 0), H(x + 0)]$. Assume without loss of generality that $H(x_0 - 0) = H(x_0)$. Any solution of

$$(6.2) \qquad y'' = K_1 y'$$

where $K_1 = K + R$ with $y(x) \leqq 0$ and $y'(x) \geqq 1$ is a subfunction by (3.5). Let

$$A = \min [\phi(a), H(x_0) - (1 - \exp [K_1(a - x_0)])(K_1 \exp [K_1(a - x_0)])^{-1}] ,$$

then the function $y_1(x)$ defined by

$$\begin{aligned} y_1(x) = & \frac{H(x_0)}{1 - \exp [K_1(a - x_0)]} [\exp [K_1(x - x_0)] - \exp [K_1(a - x_0)]] \\ & + \frac{A}{1 - \exp [K_1(a - x_0)]} [1 - \exp [K_1(x - x_0)]] \end{aligned}$$

satisfies differential equation (6.2) with $y_1(a) = A$, $y_1(x_0) = H(x_0)$, $y_1(x) \leqq 0$, and $y_1'(x) \geqq 1$.

Since $y_1(x)$ is a subfunction of class C^2 on $[a, x_0]$ with $H(a) \geqq y_1(a)$ and $H(x_0) = y_1(x_0)$, we assert that $y_1(x) \leqq H(x)$ on $[a, x_0]$. Assume not, then there exists an $x_1 \in (a, x_0)$ such that $y_1(x_1) > H(x_1)$. Let $\varepsilon = y_1(x_1) - H(x_1) > 0$, then $y_1(x) - \varepsilon/2$ is a subfunction on $[a, x_0]$ by (3.8). By the definition of $H(x)$, there are continuous underfunctions ϕ_1 and ϕ_2 such that $H(a) - \phi_1(a) \leqq \varepsilon/4$ and $H(x_0) - \phi_2(x_0) \leqq \varepsilon/4$.

Define $\phi_3(x) = \max [\phi_1(x), \phi_2(x)]$ for $x \in [a, b]$, then $\phi_3(x)$ is a continuous underfunction on $[a, b]$ with $H(a) - \phi_3(a) \leqq \varepsilon/4$ and $H(x_0) - \phi_3(x_0) \leqq \varepsilon/4$. Define ϕ^* on $[a, b]$ by:

$$\phi^*(x) = \begin{cases} \phi_3(x) , & x \notin [a, x_0] \\ \max [\phi_3(x), y_1(x) - \varepsilon/2] , & x \in [a, x_0] . \end{cases}$$

$\phi^*(x)$ is a continuous underfunction by (3.6), but $\phi^*(x_1) \geqq y_1(x_1) - \varepsilon/2 = H(x_1) + \varepsilon/2$ which contradicts the definition of $H(x)$. Therefore, $y_1(x) \leqq H(x)$ for all $x \in [a, x_0]$.

Hence,

$$\frac{H(x_0 - 0) - H(x)}{x_0 - x} \leqq \frac{y_1(x_0) - y_1(x)}{x_0 - x} \quad \text{for } a \leqq x < x_0$$

and

$$\begin{aligned} DH(x_0 - 0) \leqq y_1'(x_0) \leqq K_1[1 - \exp [K_1(a - x_0)]]^{-1} \cdot [H(x_0) - A] \\ \cdot \exp [K_1 \max [b - x_0, x_0 - a]] \\ = M_1(a, b, \alpha, \beta, x_0, K, R) . \end{aligned}$$

$DH(x_0 - 0) < M_1 < +\infty$ implies, by (4.3), that $H(x_0 - 0) \geqq H(x_0 + 0)$. Hence $H(x_0 + 0) = H(x_0 - 0) = H(x_0)$ and $H'(x_0) \leqq M_1$.

By a similar argument applied to the right-hand side of x_0, there exists a constant M_2 depending on $a, b, \alpha, \beta, x_0, K$, and R such that $M_2 \leqq H'(x_0)$.

Therefore,

$$M_2 \leqq H'(x_0) \leqq M_1 .$$

(ii) If $H(x_0) > 0$, proof is similar to case (i) and will be omitted.

COROLLARY 6.1. *Under the hypotheses of Theorem* 6.1, $H(x)$ *is of class* C^2 *and a solution of* (1.1) *on* (a, b).

This follows from (4.1), (4.2), (4.3), and Theorem 6.1.

COROLLARY 6.2. *If*

(1) $f(x, y, y')$ *satisfies* A_0, A_1, *and* A_2,

(2) $f(x, y, y') = g(x, y, y') + h(x, y, y')$ *on* T,

(3) $h(x, 0, y') = 0$ *for* $x \in [a, b]$ *and* $|y'| < \infty$,

(4) $|g(x, 0, y') - g(x, 0, 0)| \leqq K|y'|$ *for* $x \in [a, b]$ *and* $|y'| < \infty$, *then* $H(x)$ *is of class* C^2 *and a solution of* (1.1) *on* (a, b).

Our primary result is the following theorem.

THEOREM 6.2. *If*

(1) $f(x, y, y')$ *satisfies* A_0, A_1, *and* A_2,

(2) *there exists a positive continuous function* $\phi(u)$ *defined for* $u \geqq 0$ *such that*

$$|f(x, y, y') - f(x, y, 0)| \leqq K_S\phi(|y'|)$$

where K_S *is a constant depending on compact subsets* S *of*

$$\{(x, y) \mid x \in [a, b], |y| < \infty\}, \quad (x, y) \in S, \quad |y'| < \infty,$$

and

$$\int_0^\infty \frac{udu}{\phi(u) + 1} = +\infty,$$

(3) $|f(x, 0, y') - f(x, 0, 0)| \leqq K|y'|$ *for* $x \in [a, b]$ *and* $|y'| < \infty$, *then* $H(x)$ *is the solution of the* BVP (1.2) *of class* $C^2[a, b]$.

Proof. By Theorems 4.1, 5.1, and 5.2, $H(x)$ is a solution of the BVP (1.2) in the sense defined. By standard arguments, $H(x)$ is in fact of class $C^2[a, b]$.

COROLLERY 6.3. *If*
(1) $f(x, y, y')$ *satisfies* A_0 *and* A_1,
(2) $|f(x, y, y_1') - f(x, y, y_2')| \leqq K_S|y_1' - y_2'|$ *where* K_S *is a constant depending on compact subsets* S *of* $\{(x, y) \mid x \in [a, b], |y| < \infty\}$, $(x, y) \in S$, *and* $|y'| < \infty$,
then $H(x)$ *is the solution of the* BVP (1.2) *of class* $C^2[a, b]$.

COROLLARY 6.4. *If*
(1) $f(x, y, y')$ *satisfies* A_0 *and* A_1, *and*
(2) $|f(x,y,y_1') - f(x,y,y_2')| \leqq K|y_1' - y_2'|$ *for all* $(x, y, y_1'), (x, y, y_2') \in T$,
then $H(x)$ *is the solution of the* BVP (1.2) *of class* $C^2[a, b]$.

The above corollary was proven by Lees [4] using the method of finite differences.

If the Nagumo condition (Assumption 2 of Theorem 6.2) is dropped, it is still possible to assert the existence of the solution of BVP (1.2) provided some rather severe limitations are placed on the boundary values.

THEOREM 6.3. *If given the* BVP: $y'' = f(x, y, y')$, $y(a) = 0 = y(b)$, *with*
(1) $f(x, y, y')$ *satisfying* A_0, A_1, A_2, *and*
(2) $|f(x, 0, y') - f(x, 0, 0)| \leqq K|y'|$ *for all* $x \in [a, b]$ *and* $|y'| < \infty$,
then $H(x)$ *is the solution of the* BVP (1.2).

Proof. By Corollary 6.1, $H(x)$ is of class C^2 and a solution of (1.1) on (a, b). Thus, it suffices to show, that under the additional stipulation of zero boundary values, $H(x)$ is continuous on $[a, b]$ and assumes the zero boundary values. This is accomplished by constructing an overfunction ψ and an underfunction ϕ both of which are continuous on $[a, b]$ and such that $\phi(a) = \psi(a) = 0$ and $\phi(b) = \psi(b) = 0$.

Let $\phi_1(x)$ be a solution of

$$y'' = K_1 y' \tag{6.3}$$

where $K_1 = K + \max|f(x, 0, 0)|$ for $x \in [a, b]$ satisfying the condition $\phi_1(x) \leqq 0$, $\phi_1'(x) \geqq 1$, and $\phi_1(b) = 0$, and let $\phi_2(x)$ be a solution of

$$y'' = -K_1 y' \tag{6.4}$$

satisfying the conditions $\phi_2(x) \leqq 0$, $\phi_2'(x) \leqq -1$, and $\phi_2(a) = 0$, then $\phi_1(x)$ and $\phi_2(x)$ are subfunctions with respect to (1.1) on $[a, b]$. Let $\phi(x) = \max[\phi_1(x), \phi_2(x)]$, then $\phi(x)$ is a continuous underfunction such that $\phi(a) = 0 = \phi(b)$.

Similarly, there exists a continuous overfunction $\psi(x)$ such that $\psi(a) = 0 = \psi(b)$.

Thus, $\phi(x) \leqq H(x) \leqq \psi(x)$ on $[a, b]$ and result follows.

COROLLARY 6.5. *If*

(1) *$g(x, y, y')$ satisfies* A_0, A_1, *and* A_2,

(2) *$g(x, 0, y') = 0$ for all $a \leqq x \leqq b$ and $|y'| < \infty$,*

(3) *$f(x, y, y') = h_1(x) + h_2(x)y' + h_3(y) + g(x, y, y')$ where h_i, $i = 1, 2, 3$, are continuous on their respective domains and h_3 is nondecreasing,*

then there exists a solution of the BVP: (1.1) *with* $y(a) = 0 = y(b)$.

REFERENCES

1. E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

2. N. Dunfold and J. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, 1958.

3. L. Fountain and L. Jackson, *A Generalized Solution for the Boundary Value Problem for $y'' = f(x, y, y')$*, Pacific J. Math., **12** (1962), 1251–1272.

4. M. Lees, *A Boundary Value Problem for Nonliniar Ordinary Differential Equations*, J. Math., Mech., **10** (1961), 423–430.

5. M. Nagumo, *Über die Differentialgleichung $y'' = f(x, y, y')$*, Proc. Physics-Math. Soc. Japan, Series 3, **19** (1937), 861–866.

6. B. Picard, *Leçons sur Quelques Problems aux Limites de la Theorie des Equations Differentielles*, Gauthier-Villars, Paris, 1930.

UNIVERSITY OF COLORADO