ON SOME MAPPINGS RELATED TO GRAPHS

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Let N denote a set of n distinct elements $a_1, a_2, \cdots, a_n$ and let $\mathscr{S}(h) = \{S_1, S_2, \cdots, S_m\}$, $m = \binom{n}{h}$ be the collection of all sets formed by selecting h elements at a time from N. If $S_i = \{a_{i_1}, a_{i_2}, \cdots, a_{i_h}\}$ is any set in $\mathscr{S}(h)$ and if Γ is any mapping of N onto itself, then Γ induces a mapping Ψ of $\mathscr{S}(h)$ onto itself defined by $S_i\Psi = \{a_{i_1}\Gamma, a_{i_2}\Gamma, \cdots, a_{i_h}\Gamma\}$. We seek conditions under which, conversely, a mapping of $\mathscr{S}(h)$ onto itself must be of this induced type.

If Ψ is a mapping of $\mathscr{S}(h)$ onto itself, it will be said to "preserve maximal intersections" if each two of its sets which intersect on $h-1$ elements are mapped to two sets which also have $h-1$ elements in common. It will be shown that if $n \neq 2h$ this is sufficient to imply that Ψ is induced by a mapping of N onto itself.

We observe first that to each set S_i in $\mathscr{S}(h)$ there corresponds a set S_i^* in $\mathscr{S}(n-h)$ and which consists of those elements of N not in S_i. And to any mapping Ψ of $\mathscr{S}(h)$ onto itself there corresponds a mapping Ψ^* of $\mathscr{S}(n-h)$ onto itself defined by $S_i^*\Psi^* = (S_i\Psi)^*$, $i = 1, 2, \cdots, m$. Clearly, if Ψ preserves maximal intersections so does Ψ^* and both Ψ and Ψ^* are induced mappings or neither is. Thus it suffices always to consider the case $h \leqq n-h$, that is, $h \leqq n/2$.

THEOREM 1. *If $n \neq 2h$ and if Ψ is a mapping of $\mathscr{S}(h)$ onto itself which preserves maximal intersections, then Ψ is induced by a mapping of N onto itself.*

Proof. The theorem is trivially correct for $h = 1$. For a proof by induction, we suppose the theorem true up to some value $h-1$ and consider Ψ to be a mapping of $\mathscr{S}(h)$ onto itself, where $1 < h < n/2$.

Each set in $\mathscr{S}(h-1)$ belongs to exactly $n-h+1$ sets in $\mathscr{S}(h)$ and we wish to show that these sets in $\mathscr{S}(h)$ must map under Ψ to $n-h+1$ sets which also have a set of $h-1$ elements in common. Suppose that this is not the case. Then there exists a set in $\mathscr{S}(h-1)$, which we may take to be $T = \{a_1, a_2, \cdots, a_{h-1}\}$, such that the sets in $\mathscr{S}(h)$ which contain T do not map under Ψ to a collection of sets with a common intersection of $h-1$ elements. Let

$$(1) \qquad S_i = \{a_1, a_2, \cdots, a_{h-1}, a_{h+i}\}\,, \qquad i = 0, 1, \cdots, h, \cdots, n-h$$

denote the sets of $\mathscr{S}(h)$ which contain T. There is no loss of gener-

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ality in supposing that it is the intersection of $S_0\Psi$ and $S_1\Psi$ which is not contained in $S_2\Psi$. Since Ψ preserves maximal intersections, we can denote

$$(2) \qquad S_0\Psi = \{b_1, b_2, \cdots, b_{h-1}, b_h\}\,, \qquad S_1\Psi = \{b_1, b_2, \cdots, b_{h-1}, b_{h+1}\}\,,$$

where each b_i is an element from N and $i \neq j$ implies $b_i \neq b_j$, $i, j = 1, 2, \cdots, h+1$. Because $S_2\Psi$ does not contain $\{b_1, b_2, \cdots, b_{h-1}\}$, but must intersect $S_0\Psi$ and $S_1\Psi$ on $h-1$ elements, $S_2\Psi$ must contain both b_h and b_{h+1} and fail to possess just one elements from $b_1, b_2, \cdots, b_{h-1}$. Since there is nothing to distinguish the possibilities, we may suppose that $S_2\Psi$ does not possess b_1, and hence that

$$(3) \qquad S_2\Psi = \{b_2, \cdots, b_{h-1}, b_h, b_{h+1}\}\,.$$

Because $n > 2h$, there are at least $h+2$ sets S_i defined by (1) and so at least $h-1$ sets S_i, where $2 < i \leqq n-h$. And the Ψ images of all these sets must possess b_1, b_h, and b_{h+1}. For suppose $b_1 \notin S_i\Psi$. Since $S_i\Psi$ intersects $S_0\Psi$ on $h-1$ elements and not on b_1 then $\{b_2, b_3, \cdots, b_h\} \subset S_i\Psi$. And since $S_i\Psi$ intersects $S_1\Psi$ on $h-1$ elements and not on b_1, then $\{b_2, \cdots, b_h, b_{h+1}\} \subset S_i\Psi$. But then $S_i\Psi = \{b_2, \cdots, b_h, b_{h+1}\} = S_2\Psi$, which is impossible for $i \neq 2$. In the same way, $b_h \notin S_i\Psi$ implies $S_i\Psi = S_1\Psi$ and $b_{h+1} \notin S_i\Psi$ implies $S_i\Psi = S_0\Psi$, neither of which is possible for $2 < i \leqq n-h$.

From the last argument it follows that for $i > 2$, $S_i\Psi$ must be of the form

$$(4) \qquad S_i\Psi = \{b_1, b_h, b_{h+1}, x_1, \cdots, x_{h-3}\}\,,$$

where $\{x_1, x_2, \cdots, x_{h-3}\}$ is a subset of $\{b_2, b_3 \cdots, b_{h-1}\}$, which is clearly impossible if $h = 2$. But in any case, there are at least $h-1$ different sets $S_i\Psi$, where $i > 2$, and each of these is determined by the $h-3$ order subset of $\{b_2, \cdots, b_{h-1}\}$ which it contains. And since there are only $h-2$ mutually different such subsets, the sets $S_i\Psi$, $i > 2$, cannot all be distinct, which contradicts the fact that Ψ is a one-to-one mapping.

It is now established that for each set T in $\mathscr{S}(h-1)$ there exists a set T' in $\mathscr{S}(h-1)$ such that all the sets in $\mathscr{S}(h)$ which contain T are mapped under Ψ to all the sets in $\mathscr{S}(h)$ which contain T'. But then the correspondence $T \to T'$ is clearly a mapping of $\mathscr{S}(h-1)$ onto itself, say the mapping Φ.

For $h = 2$, Φ is a mapping of N onto itself. If $\{a_i, a_j\}$ is any set in $\mathscr{S}(2)$, then $a_i\Phi$ belongs to the Ψ images of all sets which possess a_i, so $a_i\Phi$ belongs to $\{a_i, a_j\}\Psi$. By the same argument, $a_j\Phi$

belongs to $\{a_i, a_j\}\Psi$. Since $a_i\Phi \neq a_j\Phi$, it follows that $\{a_i, a_j\}\Psi = \{a_i\Phi, a_j\Phi\}$ and hence that Ψ is induced by Φ.

If $h > 2$, consider any two sets in $\mathscr{S}(h-1)$, whose intersection is maximal, say

$$(5) \qquad T_1 = \{a_1, a_2, \cdots, a_{h-2}, a_{h-1}\}\,, \qquad T_2 = \{a_1, a_2, \cdots, a_{h-2}, a_h\}\,.$$

The set $S = \{a_1, a_2, \cdots, a_h\}$ in $\mathscr{S}(h)$ maps to a set $S\Psi = \{b_1, b_2, \cdots, b_h\}$. Since T_1 and T_2 are contained in S, $T_1\Phi$ and $T_2\Phi$ are $h-1$ order subsets of $S\Psi$. Since $T_1 \neq T_2$, and Φ is a one-to-one mapping, $T_1\Phi \neq T_2\Phi$, so the order of $T_1\Phi \cap T_2\Phi$ is $h-2$. Thus Φ preserves maximal intersections and so, by the inductive hypothesis, Φ is induced by some mapping Γ of N onto itself.

Now $S = \{a_1, a_2, \cdots, a_h\}$ contains T_1 and T_2 defined in (5) so $S\Psi$ contains $T_1\Phi$ and $T_2\Phi$. But $T_1\Phi = \{a_1\Gamma, a_2\Gamma, \cdots, a_{h-1}\Gamma\}$, and $T_2\Phi = \{a_1\Gamma, \cdots, a_{h-1}\Gamma, a_h\Gamma\}$. Since $a_i\Gamma \neq a_j\Gamma$ if $i \neq j$, it follows that $S\Psi = \{a_1\Gamma, a_2\Gamma, \cdots, a_h\Gamma\}$, and hence that Ψ is induced by Γ.

The theorem is not true for $n = 2h$, since then the correspondence of S_i and S_i^* is a non-induced mapping of $\mathscr{S}(h)$ onto itself which preserves all orders of intersection.[1]

Consider next an ordinary, finite graph G, that is, one with n vertices $\{p_1, p_2, \cdots, p_n\}$ where each two vertices have at most one join and none is joined to itself. Let $c(p_i, p_j, p_k)$ denote the subgraph of G induced by G on the set of vertices which does not include p_i, p_j, p_k, and let $m(G)$ be the notation for the join-measure of G, that is the number of joins in G.

THEOREM 2. *If G and H are ordinary nth order graphs and if there is a mapping of the vertices of G onto those of H such that for some integer h, $1 < h < n-1$, all corresponding subgraphs of order h have the same join measure, then the mapping is an isomorphism of G and H.*

Proof. For $h = 2$ the condition becomes the definition of an isomorphism, so assume that $2 < h < n-1$. Let $\{p_1, p_2, \cdots, p_n\}$ be the vertices of G and let the vertices $\{q_1, q_2, \cdots, q_n\}$ of H be labeled so that q_i is the image of p_i under the given mapping ψ, $i = 1, 2, \cdots, n$.

Let $\{p_{i_1}, p_{i_2}, \cdots, p_{i_{h+1}}\}$ be the vertices of any subgraph G_i of order $h+1$ in G, and let $c(p_{i_k}; G_i)$ denote the subgraph of G_i defined on all the vertices of G_i except p_{i_k}. Since any join in G_i belongs to all the h-order subgraphs of G_i except two, we have,

$$(1) \qquad m(G_i) = \frac{1}{h-1} \sum_{k=1}^{k=h+1} m[c(p_{i_k}; G_i)]\,.$$

[1] This general exception was pointed out to the writer by P. Erdös.

By the same reasoning,

$$(2) \qquad m(G_i\Psi) = \frac{1}{h-1} \sum_{k=1}^{k=h+1} m[c(q_{i_k}; G_i\Psi)] \, .$$

Since, by assumption,

$$(3) \qquad m[c(p_{i_k}; G_i)] = m[c(q_{i_k}; G_i\Psi)] \, , \qquad \text{for all } p_{i_k} \text{ and } q_{i_k} \, ,$$

it follows that $m(G_i) = m(G_i\Psi)$.

Thus if Ψ preserves the join measure on h-order subgraphs it does so on $h+1$ order subgraphs, and, by the same reasoning, preserves the join measure on all subgraphs of order equal to or greater than h. In particular, $m(G) = m(H)$. Then if $\rho(p_i)$ denotes the degree of p_i, it follows from

$$(4) \qquad \rho(p_i) = m(G) - m[c(p_i)] \, , \qquad i = 1, 2, \cdots, n$$

and

$$(5) \qquad \rho(q_i) = m(H) - m[c(q_i)] \, , \qquad i = 1, 2, \cdots, n$$

that

$$(6) \qquad \rho(p_i) = \rho(q_i) \, , \qquad i = 1, 2, \cdots, n \, ,$$

since $m[c(p_i)] = m[c(q_i)]$.

Now, corresponding to p_i and p_j in G, let ε_{ij} be 1 or 0 according as p_i and p_j are or are not joined. Let ε'_{ij} be defined in a similar way with respect to q_i and q_j. Then, by simple counting,

$$(7) \qquad m(G) = m[c(p_i, p_j)] + \rho(p_i) + \rho(p_j) - \varepsilon_{ij} \, , \qquad i \neq j \, ,$$

and

$$(8) \qquad m(H) = m[c(q_i, q_j)] + \rho(q_i) + \rho(q_j) - \varepsilon'_{ij} \, , \qquad i \neq j \, .$$

Comparing the terms in (7) and (8) it follows that $\varepsilon_{ij} = \varepsilon'_{ij}$ for all i, j, $i \neq j$, and hence that Ψ is an isomorphism of G and H.

As a corollary of these theorems it follows that two nth order graphs are isomorphic if and only if there is a one-to-one correspondence of their subgraphs of some order h, $1 < h < n - 1$, in which corresponding subgraphs have equal join measure and the correspondence preserves maximal intersections.

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