ON SOME FINITE GROUPS AND THEIR COHOMOLOGY

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The purposes of this paper are: (I) to characterize the finite groups whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group and which have periodic cohomology of period 4, (II) to show that all possible cohomologies of such a group G can be realized by direct sums of G-modules which belong to a specific finite family of G-modules.

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The reader is referred to [1, Ch. XII] for basic notions, definitions and notations concerning cohomology of finite groups. The only departure from [1, Ch. XII] is the following: we shall say that a finite group G has periodic cohomology of period k if k is the *least* positive integer such that $\hat{H}^k(G, Z)$ contains a maximal generator [1, pp. 260–261]. And to avoid typographical difficulties we will denote by $Z(l)$ the cyclic group of order l.

PROPOSITION I. *Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group. Then G has periodic cohomology of period 4 if and only if G has a presentation*

$$G = \{\sigma, \tau\colon \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\}, \textit{ with the conditions}$$

(i) *s is an odd integer >1,*
(ii) *t is a positive even integer prime to s.*

Proof. Let G be a finite group whose 2-Sylow subgroups are not isomorphic to a generalized quaternion group and which has periodic cohomology of period 4. It is well-known [1, Theorem 11.6, p. 262] that if a finite group has periodic cohomology (of finite period) every Sylow subgroup of the group is either cyclic or is a generalized quaternion group. Since we assume that the 2-Sylow subgroups of G are not isomorphic to a generalized quaternion group, every Sylow subgroup of G is cyclic. It is also well-known [6, Theorem 11, p. 175] that a finite group G containing only cyclic Sylow subgroups is metacyclic and has a presentation

$$G = \{\sigma, \tau\colon \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^r\}, \text{ with the conditions}$$

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(1) $0 < s$, ($st =$ the order of the group G),
(2) $((r-1)t, s) = 1$
(3) $r^t \equiv 1 (\bmod s)$, and conversely.
We observe that if $s = 1$ or $t = 1$ or $r = 1$ the finite group G is cyclic and G has periodic cohomology of period 2 (or 0). These cases are therefore excluded. On the other hand, once these exceptional cases are excluded G is no more a cyclic group and it will have periodic cohomology of period $\geqq 4$.

Notice that (1), (2) and (3) imply (i)

Let H be the subgroup of G generated by the element σ. H is clearly a cyclic normal subgroup of order s. And G/H is cyclic of order t. By condition (2), s and t are relatively prime to each other. We can therefore apply the decomposition theorem of Hochschild-Serre [2, Theorem 1, p. 127] and obtain

$$\hat{H}^k(G, K) \cong \hat{H}^k(G/H, K^H) \oplus (\hat{H}^k(H, K))^{G/H} , \tag{4}$$

for all $k > 0$ and for all G-module K. (For $k > 0$, $\hat{H}^k(G, K) = H^k(G, K)$). In particular, we have

$$\hat{H}^k(G, Z) \cong \hat{H}^k(G/H, Z) \oplus (\hat{H}^k(H, Z))^{G/H} ,$$

for $k > 0$. The G/H-operators on $\hat{H}^k(H, K)$ are explicitly described in [2, p. 117]. In particular, G/H-operators on $\hat{H}^k(H, Z)$ are induced by the automorphisms of H which are themselves induced, on H, by inner automorphisms of G. In the present situation, all such automorphisms of H are generated by the automorphism $f(\rho) = \rho^r (= \tau\rho\tau^{-1})$, where $\rho \in H$. The automorphism $f\colon H \to H$ induces an automorphism f^* of $\hat{H}^k(H, Z)$ [4, Lemma 3, p. 343] such that if $g_{2k} \in \hat{H}^{2k}(H, Z)$, then $f^*(g_{2k}) = r^k g_{2k}$. Therefore $\hat{H}^4(G, Z)$ has a maximal generator, i.e. G has periodic cohomology of period $\leqq 4$ if and only if $f^*(g) = g$ for all $g \in \hat{H}^4(H, Z)$. This is equivalent to
(5) $r^2 \equiv 1 (\bmod s)$.
(We recall that $r = 1$ we excluded). An elementary number theoretic calculation shows that the only solution for r in (2) and (5) is $r \equiv -1 (\bmod s)$. Therefore the number t in (3) is an even positive integer (if it is negative, we can present G by letting $\tau' = \tau^{-1}$). This shows that the finite group G has a presentation as mentioned above.

The converse of the proposition is obvious.

We know that if l is the order of the group G then for any G-module K all the cohomology groups $\hat{H}^k(G, K)$ $(-\infty < k < \infty)$ are of exponent l—that is, for all $g \in \hat{H}^k(G, K)$, $lg = 0$. Let

$$s = p_1^{u_1} \cdots p_h^{u_h}, P_1 = \{p_1, \cdots, p_h\} \quad \text{and} \quad t = q_1^{v_1} \cdots q_e^{v_e}, P_2 = \{q_1, \cdots, q_e\}$$

be decompositions of s and t into products of prime powers (where

$q_1 = 2$ and $v_1 \geqq 1$). It is obvious from (4) that a group with periodic cohomology of period 4 has P_2-period [1, Exercise 11, p. 265] equal to 2. Conversely, we have

PROPOSITION II. *Let G be a group having a presentation*

$$G = \{\sigma, \tau : \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^{-1}\} \text{ with the conditions}$$

(i) *s is an odd integer >1.*

(ii) *t is a positive even integer prime to s.*

Let P_1, P_2 be as defined above. Then there exists a finite family of G-modules $\mathscr{F}$ such that given any sequence of abelian groups $A_k(-\infty < k < \infty)$ satisfying

(a) *each A_k is of exponent st,*

(b) *the sequence is periodic of period 4,*

(c) *the P_2-period of the sequence is equal to 2, then there exists a G-module M which is a direct sum of G-modules of $\mathscr{F}$ such that $\hat{H}^k(G, M) = A_k(-\infty < k < \infty)$.*

First we observe the following

LEMMA. *Let G be a finite group and let K be a G-module. Let S be a set of primes in the ring of integers Z and let $Q(S)$ be the quotient ring [5, p. 46] of Z with respect to the multiplicative system generated by S. (As usual when $Q(S)$ is considered as a G-module it is to be understood that G operates trivially on (the additive group of) $Q(S)$). Then*

$$\hat{H}^k(G, K \otimes Q(S)) \cong \hat{H}^k(G, K) \otimes Q(S)(-\infty < k < \infty) ,$$

where $\otimes = \otimes_Z$

The proof is immediate.

Proof of Proposition II. Let s, t, P_1, P_2 be as before. Let

$$s(i, \mu) = s/p_i^{\mu}(i = 1, \cdots, h, 0 \leqq \mu \leqq u_i),$$
$$t(i, \nu) = t/q_i^{\nu}(i = 1, \cdots, e, 0 \leqq \nu \leqq v_i) .$$

Let $K^1(i, \mu) = \sum_{j=1}^{s(i, \mu)} Zx_j^{(i,\mu)}$ (direct sum on the symbols $x_j^{(i, \mu)}$)

$$K^2(i, \nu) = \sum_{j=1}^{t(i, \nu)} Zy_j^{(i,\nu)} \text{ (direct sum on the symbols } y_j^{(i,\nu)}) .$$

Define G-operators on $K^1(i, \mu)$ and $K^2(i, \nu)$ by

$$\begin{aligned} \sigma x_j^{(i,\mu)} &= x_{j+1}^{(i,\mu)} \\ \tau x_j^{(i,\mu)} &= x_{-j}^{(i,\mu)} , \end{aligned} \quad \text{(subscripts are modulo } s(i, \mu))$$

$$\begin{aligned} \sigma y_j^{(i,\nu)} &= y_j^{(i,\nu)} \\ \tau y_j^{(i,\nu)} &= y_j^{(i,\nu)} . \end{aligned} \quad \text{(subscripts are modulo } t(i, \nu)) .$$

Let

$$M^1(i, \mu) = K^1(i, \mu) \otimes Q((P_1 - \{p_i\}) \cup P_2),$$
$$M^2(i, \nu) = K^2(i, \nu) \otimes Q(P_1 \cup (P_2 - \{q_i\})) .$$

By (4), the above lemma and the fact that $(\hat{H}^{4k+2}(H, K^1(i, \mu))^{G/H} = (0)$, one shows

$$\begin{array}{ll} \hat{H}^{4k}(G, M^1(i, \mu)) = Z(p_i^\mu) & \hat{H}^{4k}(G, M^2(i, \nu)) = Z(q_i^\nu) \\ \hat{H}^{4k+1}(G, M^1(i, \mu)) = (0) & \hat{H}^{4k+1}(G, M^2(i, \nu)) = (0) \\ \hat{H}^{4k+2}(G, M^1(i, \mu)) = (0) & \hat{H}^{4k+2}(G, M^2(i, \nu)) = Z(q_i^\nu) \\ \hat{H}^{4k+3}(G, M^1(i, \mu)) = (0) & \hat{H}^{4k+3}(G, M^2(i, \nu)) = (0) \end{array}$$

The calculation is purely mechanical.

Now, let $0 \to I \to Z[G] \xrightarrow{\varepsilon} Z \to 0$, where $\varepsilon(\sum_{\sigma \in G} l_\sigma \sigma) = \sum_{\sigma \in G} l_\sigma$, $I = \mathrm{Ker}(\varepsilon)$, and let $\mathscr{F}$ consist of

$$I^k \otimes M^1(i, \mu)(k = 0, 1, 2, 3, i = 1, \cdots, h, 0 \leqq \mu \leqq u_i)$$
$$I^k \otimes M^2(i, \nu)(k = 0, 1, i = 1, \cdots, e, 0 \leqq \nu \leqq v_i) ,$$

where $I^k = I \otimes \cdots \otimes I\,(k$ times), $I^\circ = Z$.

Suppose we are given a sequence of abelian groups $A_k(-\infty < k < \infty)$ satisfying conditions (a), (b), (c). Since by (a) each A_k is of exponent st, it follows from [3, Theorem 6, p. 17] that A_k is a direct sum of cyclic groups. Let nA denote the direct sum of n copies of A, where A is either an abelian group or a G-module and n is a cardinal number. Then we can write

$$A_k = \sum_{i=1}^{h} \sum_{0 \leqq \mu \leqq u_i} m(k, i, \mu) Z(p_i^\mu) \oplus \sum_{i=1}^{e} \sum_{0 \leqq \nu \leqq v_i} n(k, i, \nu) Z(q_i^\nu) ,$$

where $m(k, i, \mu) = m(k + 4, i, \mu)(i = 1, \cdots, h, 0 \leqq \mu \leqq u_i)$, $n(k, i, \nu) = n(k + 2, i, \nu)(i = 1, \cdots, e, 0 \leqq \nu \leqq v_i)$ and $m(k, i, \mu)$, $n(k, i, \nu)$ are cardinal numbers. Take

$$\begin{aligned} M = &\sum_{k=0}^{3} \sum_{i=1}^{h} \sum_{0 \leqq \mu \leqq u_i} m(k, i, \mu) I^k \otimes M^i(i, \mu) \\ &\oplus \sum_{k=0}^{1} \sum_{i=1}^{e} \sum_{0 \leqq \nu \leqq v_i} n(k, i, \nu) I^k \otimes M^2(i, \nu) . \end{aligned}$$

Observe that $\hat{H}^{k-l}(G, K) \cong \hat{H}^k(G, I^l \otimes K)$. Clearly $\hat{H}^k(G, M) = A_k$ $(-\infty < k < \infty)$.

REMARK. In a similar but much simpler fashion one can show that all possible cohomology of a cyclic group G can also be realized by direct sums of G-modules of a certain finite family of G-modules $\mathscr{F}'$.

Addendum to the paper

"On Some Finite Groups And Their Cohomology"

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Let group G have a presentation

$$(*)\qquad G = \{\sigma, \tau: \sigma^s = 1, \tau^t = 1, \tau\sigma\tau^{-1} = \sigma^r\}\ ,$$

with the conditions

(i) $0 < s$

(ii) $((r-1)t, s) = 1$

(iii) $r^t \equiv 1 (\operatorname{mod} s)$

(iv) there exists a positive integer n such that n is the order to which r belongs to moduli p_i $(i = 1, \cdots, h)$ (i.e. n is the smallest positive integer such that $r^n \equiv 1(\operatorname{mod} p_i)$), where $s = p_1^{u_1} \cdots p_h^{u_h}$. Let s, t, P_1, P_2, be as defined before (here q_1 is not necessarily $=2$). It is clear from condition (iv) that G has P_1-period equal to $2n$ and P_2-period equal to 2.

PROPOSITION III. *Let G be a group having a presentation* $(*)$ *with the conditions (i), (ii), (iii), (iv). Then there exists a finite family of G-modules $\mathscr{F}$ such that given any sequence of abelian groups $A_k(-\infty < k < \infty)$ satisfying the following conditions*:

(a) *each A_k is of exponent st*

(b) *the P_1-period (in the obvious sense) of the sequence is $2n$*

(c) *the P_2-period of the sequence is 2,*

there exists a G-module M, which is a direct sum of G-modules of $\mathscr{F}$ such that $\hat{H}^k(G, M) = A_k(-\infty < k < \infty)$.

Proof. Let $s(i, \mu)$, $t(i, \nu)$, $K^1(i, \mu)$, $K^2(i, \nu)$, be as defined in Proposition II, Define G-operators on $K^1(i, \mu)$ and $K^2(i, \nu)$ by

$$\left.\begin{aligned} \sigma x_j^{(i,\mu)} &= x_{j+1}^{(i,\mu)} \\ \tau x_j^{(i,\mu)} &= x_{r\cdot j}^{(i,\mu)} \end{aligned}\right. ,\quad \text{(subscripts are modulo } s(i, \mu))$$

$$\left.\begin{aligned} \sigma y_j^{(i,\nu)} &= y_j^{(i,\nu)} \\ \tau y_j^{(i,\nu)} &= y_{j+1}^{(i,\nu)} \end{aligned}\right. \quad \text{(subscripts are modulo } t(i, \nu))\ .$$

By condition (iv) we have

$$\hat{H}^{2nk+i}(H, K^1(i, \mu))^{G/H} = (0)(i = 1, 2, \cdots, 2n-1)\ .$$

The rest of the proof is parallel to that of Proposition II. $\mathscr{F}$ consists of G-modules

$$I^k \otimes M^1(i, \mu)(k = 0, 1, \cdots, 2n-1; i = 1, \cdots, h; \mu = 0, 1, \cdots, u_i)$$
$$I^k \otimes M^2(i, \nu)(k = 0, 1; i = 1, 2, \cdots, e; \nu = 0, 1, \cdots, v_i)\ .$$

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