TRANSFORMATIONS OF DOMAINS IN THE PLANE AND APPLICATIONS IN THE THEORY OF FUNCTIONS

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In this paper we shall consider a family of transformations S_n $(n = 1, 2, \cdots)$ operating on open or closed sets in the complex plane z. S_n is defined relatively to a fixed point called the center of transformation, and it transforms an open set into a starlike domain which, for $n > 1$, is also n-fold symmetric with respect to this point. Therefore, for $n > 1$, S_n may be classified as a method of symmetrization. This method of symmetrization was already defined by Szegö [4] for domains which are starlike with respect to the center of transformation.

The definition of S_n will be extended (in the way usually used for symmetrizations) so that S_n will operate also on a certain class of functions and a family of condensers, in the plane. It will be proved that S_n diminishes the capacity of a condenser and this result will be used in order to obtain certain theorems in the theory of functions.

1. Definitions and notations.

The transformations S_n are defined as follows.

DEFINITION 1. *Let Ω be an open set in the plane z, which does not contain the point at infinity, and let z_0 be a point of Ω. If $|z - z_0| < \rho$, $(0 < \rho)$, is a circle contained in Ω, we define*:

$$L_\rho(\varphi) = \int_E \frac{dr}{r}\,, \tag{1}$$

where $|z - z_0| = r$ and

$$E = \{z \mid z \in \Omega, |z - z_0| > \rho, \arg(z - z_0) = \varphi\}\,;$$

$$L_\rho^{(n)}(\varphi) = \frac{1}{n}\sum_{k=0}^{n-1} L_\rho\Big(\varphi + \frac{2\pi k}{n}\Big)\,; \tag{2}$$

$$\begin{cases} R(\varphi) = \rho \exp\{L_\rho(\varphi)\} \\ R^{(n)}(\varphi) = \Big[\prod_{k=0}^{n-1} R\Big(\varphi + \frac{2\pi k}{n}\Big)\Big]^{1/n} = \rho\exp\{L_\rho^{(n)}(\varphi)\}\,. \end{cases} \tag{3}$$

Evidently, $R^{(n)}(\varphi)$ does not depend on ρ.

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Now, the set obtained from Ω by the transformation $S_n = S_n(z_0)$, with center z_0 is defined as follows:

$$S_n\Omega = \{z \mid z - z_0 = re^{i\varphi},\ 0 \leqq r < R^{(n)}(\varphi),\ 0 \leqq \varphi < 2\pi\}\ . \tag{4}$$

If instead of Ω we have a compact set H, which has an interior point z_0, we define:

$$S_nH = \{z \mid z - z_0 = re^{i\varphi},\ 0 \leqq r \leqq R^{(n)}(\varphi),\ 0 \leqq \varphi < 2\pi\}\ . \tag{4'}$$

It is easily verified that $S_n\Omega$ is a simply-connected domain and that S_nH is a connected compact set. Both sets are starlike with respect to z_0.

We shall extend the definition of S_n over a family of functions $\mathscr{G}$ which will now be defined. A non-constant real function $g(z)$ belongs to $\mathscr{G}$ if it is continuous over the extended plane z, if it takes its maximum value at infinity and if its minimum is assumed on a set of points, the interior of which is not empty. Let $g(z)$ be a function of $\mathscr{G}$ and let m and M be its minimum and maximum values, respectively. We define the following sets:

$$\begin{cases} G_m = \{z \mid g(z) = m\}\ , & \\ G_c = \{z \mid g(z) < c\}\ , & \text{for } m < c \leqq M\ . \end{cases} \tag{5}$$

G_c (for $m < c < M$) is an open bounded set while G_m is a compact set. Let z_0 be an interior point of G_m and suppose that the circle $|z - z_0| \leqq \rho$, $(0 < \rho)$, is contained in G_m. Denote by $L_\rho(c, \varphi)$, $L_\rho^{(n)}(c, \varphi)$, $R^{(n)}(c, \varphi)$ the functions defined by (1), (2), (3) with G_c replacing Ω. Clearly, for a fixed φ, $L_\rho(c, \varphi)$ is strictly monotonic increasing, for $m \leqq c \leqq M$. We also have:

$$\begin{cases} \lim_{c \to d^-} L_\rho(c, \varphi) = L_\rho(d, \varphi), & \text{for } m < d \leqq M\ ; \\ \lim_{c \to m} L_\rho(c, \varphi) = L_\rho(m, \rho)\ . & \end{cases} \tag{6}$$

Let $S_n = S_n(z_0)$. From these properties of $L_\rho(c, \varphi)$, it follows that:

$$S_nG_c \subset S_nG_d\ , \qquad \text{for } m \leqq c < d \leqq M\ ; \tag{7}$$

$$S_nG_c = \bigcup_{m \leqq d < c} S_nG_d\ , \qquad \text{for } m < c \leqq M\ ; \tag{8}$$

$$S_nG_m = \bigcap_{m < d < M} S_nG_d\ . \tag{9}$$

Since $\bar{G}_c \subseteqq \bigcap_{c<d<M} G_d$ we also have:

$$S_n\bar{G}_c \subseteqq \bigcap_{c<d<M} S_nG_d\ , \qquad m \leqq c < M\ . \tag{10}$$

DEFINITION 2. *Let $g(z) \in \mathscr{G}$. Using the notations introduced* *above, we define the function $g^{(n)}(z)$ obtained from $g(z)$ by the transformation $S_n = S_n(z_0)$, as follows:*

$$S_n g \equiv g^{(n)}(z) = \begin{cases} \inf \{c \mid z \in S_n G_c\} , & \text{for } z \in S_n G_M , \\ M , & \text{otherwise} . \end{cases} \tag{11}$$

From (8) and (9) we now conclude:

$$\begin{cases} S_n G_c = \{z \mid g^{(n)}(z) < c\} , & \text{for } m < c \leqq M , \\ S_n G_m = \{z \mid g^{(n)}(z) = m\} . \end{cases} \tag{12}$$

2. A lemma concerning the function $g^{(n)}(z)$.

LEMMA 1. *The function $g^{(n)}(z)$ is continuous over the extended plane z. If moreover $g(z)$ is Lip on every compact subset of G_M*[1] *then $g^{(n)}(z)$ is Lip on every compact subset of $S_n G_M$.*

Proof. We begin by proving the continuity of $g^{(n)}(z)$. If $z^* \in S_n G_m$ and $g^{(n)}(z^*) = d > m$ then by (10) and (12), the set $S_n G^*_{d+\varepsilon} - S_n \bar{G}^*_{d-\varepsilon}$ (where $m < d^* - \varepsilon < d^* + \varepsilon < M$) is an open neighbourhood of z^* in which $|g^{(n)}(z) - g^{(n)}(z^*)| \leqq \varepsilon$. If z^* belongs to $S_n G_m$ or z^* belongs to the complement of $S_n G_M$, then the set $S_n G_{m+\varepsilon} (m < m + \varepsilon < M)$, and the complement of $S_n \bar{G}_{M-\varepsilon} (m < M - \varepsilon < M)$ respectively, are open neighbourhoods of z^* in which $|g^{(n)}(z) - g^{(n)}(z^*)| \leqq \varepsilon$.

In order to prove the second assertion of the lemma it is sufficient to show that $g^{(n)}(z)$ is Lip on every set $S_n G_c (m < c < M)$. Without loss of generality we may suppose that $z_0 = 0$ and that $\rho = 1$. (And in this case we shall write $L^{(n)}(c, \varphi)$ instead of $L_1^{(n)}(c, \varphi)$.) We now map the z plane, cut along the positive real axis from zero to infinity, by a branch of $w = \log z$, $(w = u^+ iv)$, onto the strip $0 < v < 2\pi$. (The points of the positive real axis will be mapped both on $v = 0$ and $v = 2\pi$). We denote by H_c and H_c^n the images of G_c and $S_n G_c$ by this mapping, and we put $h(w) = g(e^w)$ and $h^{(n)}(w) = g^{(n)}(e^w)$.

Let c be a fixed number in the open interval (m, M). Since $g(z)$ is Lip on G_c, the function $h(w)$ is Lip on H_c, and if it is shown that $h^{(n)}(w)$ is Lip on H_c^n, the required result follows.

Since $h(w)$ is Lip on H_c, there exists a number $p > 0$ such that: $|h(w_1) - h(w_2)| \leqq p |w_1 - w_2|$, for any $w_1, w_2 \in H_c$.

We need the following assertion:

If δ is a positive number and v_1, v_2, c_1, c_2 are real numbers such that:

$$|v_1 - v_2| < \delta, m < c_1 < c_2 - p\delta < c - p\delta , \tag{13}$$

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[1] A function $g(z)$ is Lip on a set E if there exists a constant p, such that for any two points $z_1, z_2 \in E$, we have $|g(z_1) - g(z_2)| \leqq p |z_1 - z_2|$.

then

$$L^{(n)}(c_2, v_2) \geqq L^{(n)}(c_1, v_1) + [\delta^2 - (v_1 - v_2)^2]^{1/2} . \tag{14}$$

Because of the definition of $L^{(n)}(c, v)$, it is enough to prove (14) for $n = 1$. Without loss of generality we may suppose that $0 \leqq v_k < 2\pi$, $(k = 1, 2)$.

Denote by J_k the intersection of the half line $Im w = v_k$, $Re w \geqq 0$, with the set H_{c_k}, for $k = 1, 2$. The Lebesgue measure of J_k is $L(c_k, v_k)$. Using (5) and (13) the following is easily verified:

Let $w_1 \in J_1$. If $w_2 = u_2 + iv_2$, $u_2 \geqq 0$ and $|w_1 - w_2| \leqq \delta$, then $w_2 \in J_2$. From this and the fact that J_1 is bounded on the right, (14) follows for $n = 1$.

It will now be shown that

$$|h^{(n)}(w') - h^{(n)}(w'')| \leqq p\,|w' - w''| , \quad \text{for any } w', w'' \in H_c^n .$$

Suppose that there are two points w_1, w_2 in H_c^n for which this inequality does not hold, and let δ be a number such that:

$$|h^{(n)}(w_1) - h^{(n)}(w_2)| > p\delta > p\,|w_1 - w_2| . \tag{15}$$

Let $h^{(n)}(w_1) < h^{(n)}(w_2)$. Then we can find numbers c_1, c_2 such that:

$$m \leqq h^{(n)}(w_1) < c_1 < c_2 - p\delta < h^{(n)}(w_2) - p\delta < c - p\delta . \tag{16}$$

Now the numbers c_1, c_2, $v_1 = Im w_1$, $v_2 = Im w_2$ satisfy (13), and therefore inequality (14) holds. Since, for $m < c < M$,

$$H_c^n = \{w \mid 0 \leqq Im w \leqq 2\pi, h^{(n)}(w) < c\} = \{w \mid 0 \leqq v \leqq 2\pi, u < L^{(n)}(c, v)\} ,$$

it follows (by (16)) that $w_1 \in H_{c_1}^n$ and $w_2 \notin H_{c_2}^n$; hence $u_1 = Re w_1 < L^{(n)}(c_1, v_1)$ and $u_2 = Re w_2 \geqq L^{(n)}(c_2, v_2)$. These inequalities together with (14) yield $|w_1 - w_2| > \delta$, which is in contradiction to (15). This completes the proof of the lemma.

REMARK. The following is a consequence of the second part of the lemma: If $g(z)$ is Lip on every compact subset of $G_M - G_m$, then $g^{(n)}(z)$ is Lip on every compact subset of $S_n G_M - S_n G_m$.

3. **On a class of functions** (C, z_0). Let $C = (D, E_0, E_1)$ be a condenser in the complex plane z, i.e. a system consisting of a domain D and two disjoint closed sets E_0 and E_1, such that D does not contain the point at infinity, E_0 is bounded, E_1 is unbounded and the union of E_0 and E_1 is equal to the complement of D.

Suppose that E_0 contains an interior point z_0, let $z - z_0 = re^{i\varphi}$ and denote by S_φ the ray $\arg(z - z_0) = \varphi$. Then a subclass (C, z_0) of $\mathscr{G}$ is defined as follows.

A real function $g(z)$, continuous over the extended plane z, belongs to (C, z_0) if:

(i) $g(z)$ possesses continuous first partial derivatives, in D.

(ii) $g(z) \equiv 0$ in E_0, $g(z) \equiv 1$ in E_1 and $0 < g(z) < 1$ in D.

(iii) The set of points on the ray S_φ, at which $g(z)$ assumes a given value $c\,(0 < c < 1)$, is finite.

(iv) Any compact set of points on S_φ, which is contained in D, contains only a finite number of points (possibly zero) at which $\partial g(r, \varphi)/\partial r = 0$.

Suppose that the Dirichlet problem of the equation $\Delta u = 0$, with continuous boundary values, always has a solution in D. Then there exists a real function $\omega(z)$, continuous over the extended plane z, which is harmonic in D, vanishes on E_0 and assumes the value 1 on E_1. This function is the potential functions of C. Evidently, it belongs to (C, z_0).

Let $g(z) \in (C, z_0)$. Using property (iii) we find that (6) may be replaced by

$$\lim_{c \to c_0} L_\rho(c, \varphi) = L_\rho(c_0, \varphi), \qquad \text{for } 0 \leqq c_0 \leqq 1 . \tag{17}$$

Therefore in this case, the function $g^{(n)}(z) \equiv S_n(z_0)g$ may be defined in the following way:

$$g^{(n)}(z) = g^{(n)}(r, \varphi) = \begin{cases} 0, & \text{for } r \leqq R^{(n)}(0, \varphi), \\ c, & \text{for } r = R^{(n)}(c, \varphi),\ 0 < c < 1, \\ 1, & \text{for } r \geqq R^{(n)}(1, \varphi). \end{cases} \tag{18}$$

Since, for a fixed φ, $g^{(n)}(r, \varphi)$ is a strictly monotonic increasing function of r in the interval $R^{(n)}(0, \varphi) < r < R^{(n)}(1, \varphi)$ and since $g^{(n)}(r, \varphi)$ is continuous over the entire plane, it follows that $R^{(n)}(c, \varphi)$ is continuous in both variables for $0 < c < 1$, $0 \leqq \varphi < 2\pi$.

The following definition extends the transformation S_n over a family of condensers $\{C\}$.

DEFINITION 3. *Let $C = (D, E_0, E_1)$ be a condenser in the complex plane z, such that E_0 contains an interior point z_0. Put $G_1 = D \cup E_0$ and suppose that S_nG_1 (with $S_n = S_n(z_0)$) does not contain the entire open plane. Then, the condenser $C^{(n)}$ obtained from C by the transformation $S_n = S_n(z_0)$ is defined as follows:*

$$C^{(n)} = (D^{(n)}, E_0^{(n)}, E_1^{(n)}) ,$$

where $D^{(n)} = S_nG_1 - S_nE_0$, $E_0^{(n)} = S_nE_0$ and $E_1^{(n)} =$ the complement of S_nG_1.

4. A theorem concerning the Dirichlet integral of functions belonging to (C, z_0).

THEOREM 1. *Let $C = (D, E_0, E_1)$ be a condenser in the complex plane z, such that E_0 contains an interior point z_0. Suppose that $g(z)$ belongs to (C, z_0) and that its Dirichlet integral over D is finite. If $S_n = S_n(z_0)$, $(n = 1, 2, 3, \cdots)$, $g^{(n)}(z) = S_n g$, and $D^{(n)}$ is the domain mentioned in Definition 3, then:*

$$\iint_{D^{(n)}} (\nabla g^{(n)})^2 dxdy \leqq \iint_D (\nabla g)^2 dxdy \,. \tag{19}$$

REMARK. This theorem was proved by Szegö [4], for $n = 2, 3, \cdots$, in the special case where, D is a doubly-connected domain bounded by two smooth curves which are starlike with respect to z_0; E_0 and E_1 are connected sets; and the function $g(z)$ is the potential function of the condenser C.

Proof. By property (i) of $g(z)$ and by the remark at the end of Lemma 1 it follows that $g^{(n)}(z)$ is Lip on every compact subset of $D^{(n)}$. Therefore the first partial derivatives of $g^{(n)}(x, y)$ exist almost everywhere in $D^{(n)}$ and are bounded in every compact subset of $D^{(n)}$.

Without loss of generality we may suppose that $z_0 = 0$ and that the circle $|z| \leqq \rho = 1$ is contained in E_0. Again we shall write $L^{(n)}(c, \varphi)$ instead of $L_\rho^{(n)}(c, \varphi)$. We also introduce the following notations:

$$D(a, b) = \{z \mid a < g(z) < b\} \,,$$
$$D^{(n)}(a, b) = \{z \mid a < g^{(n)}(z) < b\} \,, \qquad \text{for } 0 < a < b < 1 \,.$$

The sets $D(a, b)$ and $D^{(n)}(a, b)$ will be mapped by $w = \log z$ $(0 \leqq Im w < 2\pi)$ on two sets which we denote by $H(a, b)$ and $H^{(n)}(a, b)$, respectively. Finally we define: $h(w) = g(e^w)$, $h^{(n)}(w) = g^{(n)}(e^w)$ and

$$\gamma_c = \{w \mid 0 < Im\, w < 2\pi, h(w) = c\} \,, \qquad \text{for } 0 < c < 1 \,.$$

The proof of the theorem rests on the following inequality:

$$\iint_{H^{(n)}(a,b)} [1 + \varepsilon^2 (\nabla h^{(n)})^2]^{1/2} dudv \leqq \iint_{H(a,b)} [1 + \varepsilon^2 (\nabla h)^2]^{1/2} dudv \,, \tag{20}$$

where $w = u + iv$, $0 < a < b < 1$ and $\varepsilon > 0$.

Inequality (19) is derived from (20) by a standard argument which we shall briefly describe.

The closures of the sets $D(a, b)$ and $D^{(n)}(a, b)$ are compact sets contained in D and $D^{(n)}$, respectively. Therefore the first partial derivatives of $h(u, v)$ $(h^{(n)}(u, v))$ are bounded in $H(a, b)$ $(H^{(n)}(a, b))$. It is evident from the definitions that the area of $H(a, b)$ equals that of $H^{(n)}(a, b)$. Taking into account these facts and using the binomial expansion of the integrands in (20), (for ε small enough), we obtain:

$$\frac{\varepsilon^2}{2}\iint_{H^{(n)}(a,b)}(\nabla h^{(n)})^2 dudv + O(\varepsilon^4) \leqq \frac{\varepsilon^2}{2}\iint_{H(a,b)}(\nabla h)^2 dudv + O(\varepsilon^4) .$$

Dividing by ε^2 and letting ε tend to zero we find that

$$\iint_{H^{(n)}(a,b)}(\nabla h^{(n)})^2 dudv \leqq \iint_{H(a,b)}(\nabla h)^2 dudv .$$

Since the Dirichlet integral is invariant under a simple conformal mapping, it follows that

$$\iint_{D^{(n)}(a,b)}(\nabla g^{(n)})^2 dxdy \leqq \iint_{D(a,b)}(\nabla g)^2 dxdy .$$

Hence, letting a tend to zero and b tend to one, we obtain the required inequality.

In the proof of (20) we may suppose that $\varepsilon = 1$.

The first step is the following assertion. Suppose that $w^* = u^* + iv^* \in H^{(n)}(a, b)$ and $0 < v^* < (2\pi/n)$. Put $h^{(n)}(u^*, v^*) = c^*$. If $\partial h/\partial u \neq 0$ at all the points of intersection of the set γ_{c^*} and the lines $Im\, w = v^* + (2\pi m/n)$ $(m = 0, \cdots, n-1)$, then there exists a neighbourhood of w^* in which $h^{(n)}(u, v) \in C^1$.

In order to prove this assertion we shall show first that $L(c, v) \in C^1$ in a neighbourhood of (c^*, v^*). By property (iii) the set γ_{c^*} intersects the line $Im\, w = v^*$ in a finite number of points, which we denote by $w_1, \cdots, w_p$, where $Re\, w_1 < Re\, w_2 < \cdots < Re\, w_p$. By hypothesis, $\partial h/\partial u \neq 0$ at these points. Let q be a positive number such that the circles $K_j : |w - w_j| \leqq q$, $(j = 1, \cdots, p)$, are contained in $H(a, b)$ and $\partial h/\partial u \neq 0$ in them. Then the following is easily verified:

There exists a rectangle

$$P = \{(c, v) \mid |c - c^*| \leqq \delta, |v - v^*| \leqq \delta\} ,$$

(where $a < c^* - \delta < c^* + \delta < b$, $0 < v^* - \delta < v^* + \delta < (2\pi/n)$), such that:

(a) If $(c, v) \in P$ then γ_c intersects the line $Im\, w = v$ in exactly p points, one point in each circles K_j.

(b) The set $H(c^* - \delta, c^* + \delta)$ intersects the strip $v^* - \delta < Im w < v^* + \delta$ in exactly p domains Q_j, where $Q_j \subset K_j$, $(j = 1, \cdots, p)$.

Solving $c = h(u, v)$ for u in Q_j we obtain a function $u = u_j(c, v)$. This function belongs to C^1 in the rectangle P where

$$\frac{\partial u_j}{\partial c} = \left(\frac{\partial h}{\partial u}\right)^{-1}, \qquad \frac{\partial u_j}{\partial v} = -\left(\frac{\partial h}{\partial v}\right) \times \left(\frac{\partial h}{\partial u}\right)^{-1} . \tag{21}$$

Since by definition:

(22) $$L(c, v) = \sum_{j=1}^{p} (-1)^{j+1} \times u_j(c, v)$$

it follows that $L(c, v) \in C^1[P]$. We observe that in Q_j we have $\partial h/\partial u = (-1)^{j+1} \times |\partial h/\partial u|$ so that

(23) $$\frac{\partial L}{\partial c} = \sum_{j=1}^{p} \left| \frac{\partial u_j}{\partial c} \right| , \qquad \text{in } P .$$

Evidently, similar results hold for any of the points $c = c^*$, $v = v^* + (2\pi m/n)$, for $m = 0, \cdots, n-1$. Therefore it is possible to find a positive number $\eta(\eta \leqq \delta)$ such that $L^{(n)}(c, v) \in C^1$ and $(\partial L^{(n)}/\partial c) > 0$ in the rectangle $|c - c^*| < \eta$, $|v - v^*| < \eta$. By (18), for any fixed v, $c = h^{(n)}(u, v)$ is the inverse function of $u = L^{(n)}(c, v)$ in the interval $0 < c < 1$. Hence it follows that in a certain neighbourhood of (u^*, v^*), $h^{(n)}(u, v) \in C^1$ and

(24) $$\frac{\partial h^{(n)}}{\partial u} = \left(\frac{\partial L^{(n)}}{\partial c}\right)^{-1} , \qquad \frac{\partial h^{(n)}}{\partial v} = -\left(\frac{\partial L^{(n)}}{\partial v}\right) \times \left(\frac{\partial L^{(n)}}{\partial c}\right)^{-1} .$$

Denote by $A(v)$ and $A_n(v)$ the intersections of the line $Im\, w = v$ with the sets $H(a, b)$ and $H^{(n)}(a, b)$ respectively. Let $w \in A(v)$ and $h(w) = c$, $(0 < v < 2\pi)$. If at one of the points of intersection of γ_c with the line $Im\, w = v$, $\partial h/\partial u$ vanishes then we shall say that w is a critical point of $A(v)$. Let $w \in A_n(v)$ and $h^{(n)}(w) = c$. If the intersection of γ_c with one of the sets $A(v + 2\pi m/n)$, $(m = 0, \cdots, n-1)$, contains a crititical point of that set, we shall say that w is a critical point of $A_n(v)$. By properties (iii) and (iv) the set of critical points of $A(v)$ is finite, and consequently, the set of critical points of $A_n(v)$ is finite.

We shall prove now that

(25) $$\int_{A_1(v)} [1 + (\nabla h^{(1)})^2]^{1/2} du \leqq \int_{A(v)} [1 + (\nabla h)^2]^{1/2} du ,$$

for $0 < v < 2\pi$. Inequality (20) for $n = 1$, follows from (25).

Let v_0 be a fixed point in the interval $(0, 2\pi)$ and let $\{c_1, \cdots, c_{k-1}\}$ be the set of values (possibly void) taken by $h(w)$ at the critical points of $A(v_0)$. We assume that these values are ordered as follows:

$$a = c_0 < c_1 < \cdots < c_{k-1} < c_k = b .$$

Denote by B_l that subset of $A(v_0)$ which consists of open segments, free from critical points, such that at the endpoints of each segment $h(w)$ assumes the values c_l and c_{l+1}. Evidently, for any l $(l = 0, \cdots, k-1)$ the set B_l is not void and $A(v_0) = \bigcup_{l=0}^{k-1} B_l$.

Now let m be a fixed integer, $0 \leqq m \leqq k-1$, and denote by $\alpha_1, \cdots, \alpha_p$

the segments contained in B_m, which were described above. We shall assume that α_j is at the left of α_{j+1}, $(j = 1, \cdots, p-1)$. In some neighbourhood of α_j it is possible to solve $c = h(u, v)$ for u and thereby obtain a function $u = u_j(c, v)$. By (21) we obtain:

$$\int_{\alpha_j} [1 + (\nabla h(u, v_0))^2]^{1/2} du = \int_{c_m}^{c_{m+1}} [1 + (\nabla u_j(c, v_0))^2]^{1/2} dc \ , \tag{26}$$

for $j = 1, \cdots, p$.

Denote: $u'_j = L(c_j, v_0)$ and $w'_j = u'_j + iv_0$, $(j = 0, \cdots, k)$. Then w'_0 and w'_k are the endpoints of $A_1(v_0)$ while $w'_1, \cdots, w'_{k-1}$ are the critical points of $A_1(v_0)$. Denote by B'_m the open segment with endpoints w'_m, w'_{m+1}. By (22) and (24) (with $n = 1$) we get:

$$\begin{aligned} \int_{B'_m} [1 + (\nabla h^{(1)}(u, v_0))^2]^{1/2} du &= \int_{c_m}^{c_{m+1}} [1 + (\nabla L(c, v_0))^2]^{1/2} dc \\ &= \int_{c_m}^{c_{m+1}} \left\{1 + \left[\nabla \sum_{j=1}^{p} (-1)^{j+1} u_j(c, v_0)\right]^2\right\}^{1/2} dc \ . \end{aligned} \tag{27}$$

By (26), (27) and the well known inequality

$$\left\{\left(\sum_{j=1}^{p} x_j\right)^2 + \left(\sum_{j=1}^{p} y_j\right)^2 + \left(\sum_{j=1}^{p} t_j\right)^2\right\}^{1/2} \leqq \sum_{j=1}^{p} (x_j^2 + y_j^2 + t_j^2)^{1/2} \ , \tag{28}$$

(x_j, y_j, t_j being real numbers) we finally obtain:

$$\begin{aligned} \int_{B'_m} [1 + (\nabla h^{(1)}(u, v_0))^2]^{1/2} du &\leqq \int_{B_m} [1 + (\nabla h(u, v_0))^2]^{1/2} du \\ &= \sum_{j=1}^{p} \int_{\alpha_j} [1 + (\nabla h(u, v_0))^2]^{1/2} du \ . \end{aligned} \tag{29}$$

Since (29) holds for any m, $(m = 0, \cdots, k-1)$ inequality (25) follows.

It remains to prove inequality (20) for $n = 2, 3, \cdots$. Since this inequality is proved for $n = 1$, it is enough to show that

$$n \times \int_{A_n(v_0)} [1 + (\nabla h^{(n)}(u, v_0))^2]^{1/2} du \leqq \sum_{j=0}^{n-1} \int_{A_1(v_j)} [1 + (\nabla h^{(1)}(u, v_j))^2]^{1/2} du \ , \tag{30}$$

where $0 < v_0 < (2\pi/n)$ and $v_j = v_0 + (2\pi j/n)$.

Let $\{c_1^*, \cdots, c_{r-1}^*\}$ be the set of values (possibly void) assumed by $h^{(n)}(w)$ at the critical points of $A_n(v_0)$, these values being ordered as follows:

$$a = c_0^* < c_1^* < \cdots < c_{r-1}^* < c_r^* = b \ .$$

Put $u_m^* = L^{(n)}(c_m^*, v_0)$ and $u_{m,j}^* = L(c_m^*, v_j)$. By (24) we get:

$$\int_{u_m^*}^{u_{m+1}^*}[1+(\nabla h^{(n)}(u,v_0))^2]^{1/2}du=\int_{c_m^*}^{c_{m+1}^*}[1+(\nabla L^{(n)}(c,v_0))^2]^{1/2}dc$$
$$=\frac{1}{n}\int_{c_m^*}^{c_{m+1}^*}\Big[n^2+\Big(\sum_{j=0}^{n-1}\nabla L(c,v_j)\Big)^2\Big]^{1/2}dc\,; \tag{31}$$
$$\int_{u_{m,j}^*}^{u_{m+1\,j}^*}[1+(\nabla h^{(1)}(u,v_j))^2]^{1/2}du=\int_{c_m^*}^{c_{m+1}^*}[1+(\nabla L(c,v_j))^2]^{1/2}dc\,,$$

for $m=0,\cdots,r-1$ and $j=0,\cdots,n-1$. From (31) and (28), inequality (30) follows. This completes the proof of the theorem.

5. The transformation S_n diminishes the capacity of a condenser. Let $C=(D,E_0,E_1)$ be a condenser in the complex plane z, satisfying the conditions of Definition 3. It will be assumed that the Dirichlet problem for $\nabla u=0$, with continuous boundary values, always has a solution in D. (Sufficient conditions for the validity of this assumption are given, for example, in Hayman [2], Th. 4.2, pp. 63–64. Following Hayman's terminology we shall say that a domain is *admissible* if it satisfies these conditions.) The *capacity* of the condenser C is defined as the Dirichlet integral over D, of the potential function $\omega(z)$ of C, (see § 3).

Let $C^{(n)}=S_nC=(D^{(n)},E_0^{(n)},E_1^{(n)})$, (where $S_n=S_n(z_0)$). The domain $D^{(n)}$ is admissible so that the capacity of $C^{(n)}$ is defined. We now prove the following:

THEOREM 2. *Let C and $C^{(n)}$ be the condensers mentioned above and denote their capacities by I and I_n respectively. Then we have $I_n\leqq I$.*

Proof. Let $\omega^{(n)}(z)=S_n\omega(z)$, $(S_n=S_n(z_0))$. Since $\omega(z)\in(C,z_0)$, by Theorem 1 we have

$$\int_{D^{(n)}}(\nabla\omega^{(n)})^2dxdy\leqq\int_D(\nabla\omega)^2dxdy=I\,. \tag{32}$$

The function $\omega^{(n)}(z)$ is continuous over the extended plane z and Lip in every compact subset of $D^{(n)}$; it vanishes on E_0 and assumes the value 1 on E_1. Hence, by the Dirichlet minimum principle (see, Hayman [2], Th. 4.3, pp. 65–67) we have

$$I_n\leqq\int_{D^{(n)}}(\nabla\omega^{(n)})^2dxdy\,. \tag{33}$$

The required result follows from (32) and (33).

We shall apply Theorem 2 in order to obtain a result about the inner radius. Let D be a domain in the complex plane z, z_0 a point of D, and $r(D, z_0)$ the inner radius of D at z_0. (We refer here to the definition given, for example, in Hayman [2] pp. 78–80, where the inner radius is defined without any assumptions on D.) The domain D can be approximated from within by a series of bounded analytic domains $\{D_n\}$, which contain the point z_0, such that $\lim_{n\to\infty} r(D_n, z_0) = r(D, z_0)$. (An analytic domain is a domain bounded by a finite number of disjoint, simple closed, analytic curves.) By a well known method of Pólya and Szegö (see Pólya-Szegö [3] pp. 44–45; also Hayman [2] pp. 81–84) the following theorem is obtained as a consequence of Theorem 2.

THEOREM 3. *Let D be a domain in the complex plane z and let $z_0 \in D$. If $S_n = S_n(z_0)$, then*

$$r(D, z_0) \leqq r(S_n D, z_0) \,. \tag{34}$$

6. Applications in the theory of functions. In this section we denote by $w = f(z)$ a function which is regular in $|z| < 1$ and by D the domain of all values w assumed by this function at least once in $|z| < 1$. It is known that

$$|f'(0)| \leqq r(D, f(0)) \,, \tag{35}$$

equality holding if and only if $f(z)$ is a (1,1) mapping, (see Hayman [2], Th. 4.5, p. 80).

As a consequence of Theorem 3 we obtain the following:

THEOREM 4. *Let $S_n = S_n(f(0))$ and suppose that $S_n D$ does not contain the entire open plane. Let $w = F(z)$ be a (1,1) conformal mapping of $|z| < 1$ onto $S_n D$, such that $F(0) = f(0)$. Then we have $|f'(0)| \leqq |F'(0)|$.*

Proof. By (35) we get: $|f'(0)| \leqq r(D, f(0))$ and $|F'(0)| = r(S_n D, F(0))$. From these relations together with (34), the required inequality follows.

The following results are based on Theorem 4.

THEOREM 5. *Let $f(z) = a_1 z + a_2 z^2 + \cdots$. Define $R^{(n)}(\varphi)$ as in Definition 1, for the domain D and the point $w = 0$. Then,*

$$|a_1| \leqq \sqrt[n]{4}\, R^{(n)}(\varphi) \,, \qquad (0 \leqq \varphi < 2\pi) \tag{36}$$

and equality holds for the function

$$w = \psi_n(z) = t e^{i(\varphi+\theta)} z / (1 + e^{in\theta} z^n)^{2/n} \,, \qquad (t \text{ and } \theta \text{ real numbers}) \,.$$

Proof. Let φ_0 be a fixed real number and suppose that $R^{(n)}(\varphi_0) = d < \infty$. Denote by D_0 the domain containing the entire w plane, with the exception of n rays: $\arg w = \varphi_0 + (2\pi k/n)$, $d \leqq |w|$, $(k = 0, \cdots, n-1)$. The domain $S_n D(S_n = S_n(0))$ is contained in D_0. The function $w = \sqrt[n]{4}\, de^{i\varphi_0} f_n(z)$ where

$$f_n(z) = z/(1 + z^n)^{2/n} , \tag{37}$$

maps $|z| < 1$ conformally, (1,1) onto D_0. Therefore, by the principle of subordination and Theorem 4 it follows that $|a_1| \leqq \sqrt[n]{4}\, d$, and inequality (36) is proved. The assertion concerning the function $w = \psi_n(z)$ is evident.

The following theorem may be proved by the same method.

THEOREM 6. *Let $f(z) = a_1 z + a_2 z^2 + \cdots$. Suppose that $R^{(n)}(\varphi) \leqq M < \infty$ for $0 \leqq \varphi < 2\pi$ and that $R^{(n)}(\varphi_0) = \beta M\,(0 < \beta \leqq 1)$. Then*

$$|a_1| \leqq \beta M \cdot \sqrt[n]{4}/(1 + \beta^n)^{2/n} , \tag{38}$$

and equality holds for the function

$$w = \phi_n(z) = Me^{i\varphi_0} f_n^{-1}[q f_n(e^{i\theta} z)] ,$$

where $f_n(z)$ is defined by (37), $0 \leqq \theta < 2\pi$ *and* $q = \sqrt[n]{4}\,\beta/(1 + \beta^n)^{2/n}$.

We now prove

THEOREM 7. *Let $f(z) = a_1 z + a_2 z^2 + \cdots$ and define:*

$$R_0 = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log R(\varphi) d\varphi\right] = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log R^{(n)}(\varphi) d\varphi\right] . \tag{39}$$

Then $|a_1| \leqq R_0$, and equality holds for $w = a_1 z$.[2]

Proof. First suppose that $w = f(z)$ is regular in $|z| \leqq 1$ and that $f'(z) \neq 0$ on $|z| = 1$. Then $R(\varphi)$ is a continuous function of φ, and we have

$$\lim_{n\to\infty} R^{(n)}(\varphi) = \lim_{n\to\infty} \exp\left[\frac{1}{n}\sum_{k=0}^{n-1} \log R\left(\varphi + \frac{2\pi k}{n}\right)\right] = R_0 , \tag{40}$$

for any real φ. Therefore, if a positive ε is given and n is sufficiently large, the domain $S_n D$ (where $S_n = S_n(0)$) is contained in the circle $|z| < R_0 + \varepsilon$. Hence, by Theorem 4 and the principle of subordi-

[2] The author obtained this result in a weaker form, with $\bar{r}_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} R^{(n)}(\varphi) d\varphi$ instead of R_0. (By the geometric-arithmetic mean theorem $R_0 \leqq \bar{r}_n$ for every n). The stronger form written above was suggested by the referee, to whom our thanks are due.

nation, we get $|a_1| \leqq R_0 + \varepsilon$. In order to prove the theorem in the general case, we approximate the function $w = f(z)$ by functions $w = f(\rho z)$, with $0<\rho<1$.

Let Ω be an open set in the plane z and let $z_0 \in \Omega$. Denote by $m(\varphi)$ the linear (Lebesgue) measure of the set $E(\varphi) = \{z \mid \arg(z - z_0) = \varphi, z \in \Omega\}$, and define

$$m^{(n)}(\varphi) = \frac{1}{n} \sum_{k=0}^{n-1} m\left(\varphi + \frac{2\pi k}{n}\right) . \tag{41}$$

We shall show that Theorems 5, 6, 7, remain true if $R(\varphi)$ is replaced by $m(\varphi)$, and $R^{(n)}(\varphi)$ by $m^{(n)}(\varphi)$. This is a consequence of the following inequalities:

$$R(\varphi) \leqq m(\varphi) , \tag{42}$$

$$R^{(n)}(\varphi) \leqq m^{(n)}(\varphi) , \qquad \text{for } 0 \leqq \varphi < 2\pi . \tag{42'}$$

If $R(\varphi)$ is finite, equality holds in (42) if and only if the set $E(\varphi)$ is contained in a segment E^* such that $E^* - E(\varphi)$ is a set of measure zero. (We shall refer to this condition as the *MR* condition.) Inequality (42′) follows from (42) by the geometric-arithmetic mean theorem. Hence, if $R^{(n)}(\varphi)$ is finite, equality holds in (42′) if and only if

$$R(\varphi) = R\left(\varphi + \frac{2\pi k}{n}\right) = m(\varphi) = m\left(\varphi + \frac{2\pi k}{n}\right) , \ (k = 1, \cdots, n-1) .$$

From this it follows that when we replace $R(\varphi)$ by $m(\varphi)$ and $R^{(n)}(\varphi)$ by $m^{(n)}(\varphi)$, the functions mentioned at the end of Theorems 5, 6, 7, are in each case, the *only* functions for which equality holds.

In order to prove (42) we may suppose that $m(\varphi)$ is finite. In this case, for any $\varepsilon > 0$ we can find a subset F of $E(\varphi)$, consisting of a finite number of segments, such that the linear measure of $E(\varphi) - F$ is smaller than ε. Therefore it is enough to prove (42) in the case that $E(\varphi)$ consists of a finite number of segments. Suppose that these segments are not adjacent. Then, by shifting them toward z_0 (so that they do not overlap), we increase $R(\varphi)$, while $m(\varphi)$ is invariant. But if the segments are adjacent we have $R(\varphi) = m(\varphi)$. Therefore (42) is proved.

Evidently, the *MR* condition for $E(\varphi)$ is sufficient in order that $R(\varphi) = m(\varphi)$. Suppose now that $R(\varphi)$ is finite and that $E(\varphi)$ does not satisfy the *MR* condition. Then it is possible to find a subset F_1 of $E(\varphi)$ and a subset F_2 of the complement of $E(\varphi)$ on the ray $\arg(z - z_0) = \varphi$, such that the two subsets have equal, positive measures and F_2 separates F_1 from z_0. Replacing F_1 by F_2 we increase $R(\varphi)$, but not $m(\varphi)$. Therefore we must have $R(\varphi) < m(\varphi)$.

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