A SUFFICIENT CONDITION THAT AN ARC IN S^n BE CELLULAR

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An arc A in S^n, the n-sphere, is cellular if $S^n - A$ is topologically E^n, euclidean n-space. A sufficient condition for the cellularity of an arc in E^3 is given in [4] in terms of the property local peripheral unknottedness (L.P.U) [5]. We consider a weaker property and show that an arc in S^n with this property is cellular.

If A is an arc in S^n we say that A is p-shrinkable if A has an end point q and in each open set U containing q in S^n, there is a closed n-cell $C \subset U$ such that q lies in Int C (the interior of C), while BdC (the boundary of C) meets A in exactly one point. We note that A is p-shrinkable is precisely the condition that A be L.P.U. at an endpoint [5]. There is, however, a good geometric reason for using the p-shrinkable terminology here; the letter p denotes pseudo-isotopy.

LEMMA 0. *Let C^n be a closed n-cell and D^n a closed n-cell which lies in int C^n except for a single point q which lies on the boundary of each n-cell. If there is a homeomorphism h of C^n onto a geometric n-simplex such that $h(D^n)$ is also an n-simplex, then there is a pseudo-isotopy ρ_t of C^n onto C^n which is the identity on BdC^n, while $\rho_1(D^n)$, the terminal image of D^n, is the point q.*

The proof of this is omitted since it depends only on the same result when C^n and D^n are simplices.

LEMMA 1. *Let C^n be a closed n-cell and B an arc which lies in int C^n except for an endpoint b of B on BdC^n. Then there is a pseudo-isotopy of C^n onto C^n which is fixed on BdC^n and which carries B to b.*

Proof. Since $B \cap BdC^n = b$ we note that there is in C^n an n-cell D^n which contains B in its interior except for the point b, $D^n - b \subset \text{Int}\, C^n$, and D^n is embedded in C^n as in Lemma 0. Thus Lemma 0 can be applied to shrink B in the manner required by the Lemma.

THEOREM 1. *Let A be an arc in S^n such that for each subarc B of A, B is p-shrinkable. Then every arc in A is cellular.*

Proof. The proof is by contradiction. If A contains a non-cellular

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subarc there is no loss of generality in assuming this arc is A. Then $S^n - A \neq E^n$. By the characterization theorem of E^n in [1], there is a compact set C in $S^n - A$ and C lies in no open n-cell in $S^n - A$. By the Generalized Schoenflies Theorem [2], this is equivalent to the condition that no bicollared $(n-1)$-sphere in S^n separates C and A.

Let G be the set of all subarcs of A which cannot be separated from C by a bicollared sphere in S^n. We partially order G by set inclusion and select a maximal chain in G. Let B be the intersection of all arcs in this maximal chain. Evidently B cannot be separated from C by a bicollared sphere in S^n. Thus B is an arc and each proper subarc of B can be so separated from C in S^n.

By the hypothesis of the theorem, B is p-shrinkable. So let B be L.P.U. at an endpoint q. Let U be an open set containing q and $U \cap C = \square$. Then there is an n-cell $C^n \subset U$, $C^n \cap B = B^1$, an arc, while $B^1 \cap BdC^n = p$, a point. So by Lemma 1 there is a pseudo-isotopy ρ_t of S^n onto S^n, ρ_t is the identity in $S^n - C^n$, and $\rho_1(B^1) = p$. But $\rho_1(B)$ is a proper subarc of B which cannot be separated from C in S^n by a bicollared sphere. But this is a contradiction. Thus A is cellular as well as each subarc of A.

COROLLARY 1. *Let A be an arc in S^n which is the union of two p-shrinkable arcs, $A_1 \cup A_2$, which meet in a common endpoint p. Then A is cellular if A_1 is* L.P.U.

Proof. Each subarc of A is p-shrinkable.

COROLLARY 2. *Each non-cellular arc A in S^n contains a subarc which is not* L.P.U. *at either of its endpoints.*

Even in S^3 there is a difference between an arc being L.P.U. at each point and having the p-shrinkable property for each subarc. The simplest example is perhaps a mildly wild arc which is not a Wilder arc. [3].

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