INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE TRANSFORM

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1. Introduction. In a series of recent papers I have discussed various properties and inversion theorems etc. for the transform

$$F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1}\int_0^\infty (xy)^\beta {}_1F_1(\beta+\eta+1; \alpha+\beta+\eta+1; -xy)f(y)dy \,. \tag{1.1}$$

where $f(y) \in L0, \infty)$, $\beta \geqq 0$, $\eta > 0$.

$$= A\int_0^\infty (xy)^\beta \psi(x, y)f(y)dy$$

where for convenience we denote $\Gamma(\beta+\eta+1)/\Gamma(\alpha+\beta+\eta+1)$ by A and ${}_1F_1(a; b; -xy)$ by $\psi(xy)$; a and b standing respectively for $\beta+\eta+1$ and $a+\alpha$. For $\alpha=\beta=0$ (1.1) reduces to the wellknown Laplace transform

$$F(x) = \int_0^\infty e^{-xy} f(y)dy \,. \tag{1.2}$$

The transform (1.1), which may be called a generalization of the Laplace transform, arises if we apply Kober's operators of fractional integration [2] to the function $x^\beta e^{-x}$[1].

The object of the present paper is to obtain an inversion and a representation theorem for the transform (1.1) by using properties of Kober's operators defined below.

2. Definition of operations. The operators given by Kober are defined as follows.

$$I^+_{\eta,\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} x^{-\eta-\alpha}\int_0^x (x-u)^{\alpha-1}u^\eta f(u)du$$

$$K^-_{\zeta\,\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} x^\zeta \int_n^\infty (u-x)^{\alpha-1}u^{-\zeta-\alpha}f(u)du$$

where $f(x) \in L_p(0, \infty)$, $1/p + 1/q = 1$, if $1 < p < \infty$ and $1/p$ or $1/q$ 0 if p or $q=1$, $\alpha > 0$, $\zeta > -(1/p)$, $\eta > -(1/q)$.

The Mellin transform $\bar{M}f(x)$ of a function $f(x) \in L_p(0, \infty)$ is defined as

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$$\bar{M}f(x) = \int_0^\infty f(x)x^{it}du \qquad (p = 1)$$

and

$$= \underset{x\to\infty}{\overset{\text{index}V}{\lim}} \int_{1/x}^{x} f(x)^{it-1/q}dn \qquad (p > 1) .$$

The inverse Mellin transform $M^{-1}\phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$\text{(2.1)} \qquad M^{-1}\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)x^{-it}dt \qquad (q = 1)$$

and

$$= \frac{1}{2\pi} \underset{T\to\infty}{\overset{\text{index}p}{\lim}} \int_{-T}^{T} \phi(t)x^{-it-1/p}dt \qquad (q > 1) .$$

If Mellin transform is applied to Kober's operators and the orders of integrations are interchanged we obtain, under certain conditions

$$\bar{M}\{I_{\eta}^{+}{}_{\alpha}f(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left(\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right]}\bar{M}f(x)$$

and

$$\bar{M}\{K_{\zeta}^{-}{}_{\alpha}f(x)\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]}\bar{M}f(x) .$$

But

$$\bar{M}(e^{-x} \cdot x^{\beta}) = \int_0^\infty e^{-x}x^{\beta+it-1/q}dx = \Gamma\left(\beta + it + \frac{1}{p}\right), \quad \text{if } Re\left(\beta + \frac{1}{p}\right) > 0 .$$

Therefore

$$\bar{M}\{I_{\eta,\alpha}^{+}(x^{\beta}e^{-x})\} = \frac{\Gamma\left[\left(\eta + \frac{1}{q} - it\right)\right]\Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right]}$$

and

$$\bar{M}\{K^-_{\zeta,\alpha}(x^\beta e^{-x})\} = \frac{\Gamma\left(\beta + it + \frac{1}{p}\right)P\left(\zeta + it + \frac{1}{p}\right)}{\Gamma\left[\alpha + \left\{\zeta + \frac{1}{p} + it\right\}\right]} .$$

By (2.1) we then have

$$(2.2) \qquad I^+_{\eta,\alpha}(x^\beta e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]} x^{-it-1/p}dt$$

and

$$K^-_{\zeta,\alpha}(x^\beta e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]} x^{-it-1/p}dt ,$$

provided that $1/p > 0$, $\eta + 1/q > 0$ and $\zeta + 1/p > 0$.

3. **Inversion theorem.** We now define an inversion operator which will serve to invert (1.1).

An operator is defined for integral values of n by the relations

$$W_0[G(x)] = G(x) ,$$

$$W_n[G(x)] = (-)^n n^{\beta+n+1}\left(\frac{d}{dx}\right)^n[x^{-\beta}G(x)], \ (n = 1, 2, \cdots)$$

$$Q_{n,t}[G(x)] = \frac{1}{\Gamma(n + 1 + \beta - \alpha)}[W_n[G(x)]]_{n=n/t}(n = 1, 2, \cdots) .$$

THEOREM 3.1. *If $f(t)$ is bounded in $(0 < t < \infty)$ then, provided that the integral* (1.1) *converges, $\eta > 0$, $\beta \geqq 0$*

$$f(t) = \lim_{n\to\infty} Q_{n,t}[F(x)]$$

for almost all positive t.

Proof. Let x be any number greater than zero. Then, since the integral (1.1) converges, we can differentiate under the integral sign. Also (2.2) gives

$$(3.1) \qquad \left(\frac{d}{dx}\right)[x^{-\beta}I_{n,\alpha}(x^\beta e^{-x})] = -x^{-\beta}I_{\eta+1,\alpha}[x^\beta e^{-x}] .$$

Using this relation we get

$$W_n[F(n)] = (-)^n n^{\beta+n+1}\int_0^\infty x^{-\beta}y^n I_{\eta+n,\alpha}\{(xy)^\beta e^{-xy}\}f(y)dy$$
$$= \frac{\Gamma(\beta+\eta+n+1)}{\Gamma(\alpha+\beta+\eta+n+1)}\int_0^\infty y^{\beta+n}{}_1F_1(\beta+\eta+n+1;$$
$$\alpha+\beta+\eta+n+1-xy)f(y)dy\ .$$

Therefore

$$Q_{n,t}\{F(x)\}$$
$$= \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\left(\frac{n}{t}\right)^{\beta+n+1}\frac{1}{\Gamma(n+\beta+1-\alpha)}$$
$$\times\int_0^\infty y^{\beta+n}{}_1F_1(n+\beta+\eta+1;\alpha+\beta+\eta+1+n;\ -xy)f(y)dy$$
$$= \frac{1}{\Gamma(n+\beta+1-\alpha)}\left(\frac{n}{t}\right)^{\beta+n+1}\frac{\Gamma(a)}{\Gamma(b)}$$
$$\times\int_0^\infty y^{\beta+n}{}_1F_1(a+n;b+n;\ -xy)f(y)dy$$

in the notation of § 1.

$$= \frac{\Gamma(a+n)}{\Gamma(b+n)\Gamma(n+\beta+1-\alpha)}\left(\frac{n}{t}\right)^{n+\beta+1}$$
$$\times\int_0^\infty (tv)^{n+\beta}{}_1F_1(a+n;b+n;\ -nv)f(tv)dt$$
$$= \frac{\Gamma(a+n)}{\Gamma(b+n)\Gamma(n+\beta+1-\alpha)}\left(\frac{n}{t}\right)^{n+\beta+1}$$
$$\times\int_0^\infty v^{n+\beta}{}_1F_1(\beta+\eta+n+1;\alpha+\beta+\eta+n+1;\ -nv)f(tv)dt$$

by a simple change of variable. Now by using a result of Slater [4] we have

$$\frac{\Gamma(a+n)}{\Gamma(b+n)}{}_1F_1(a+n;b+n;\ -v)\sim(nv)^{a-b}e^{-nv}\qquad(n\to\infty).$$

Therefore

$$\lim_{n\to\infty} Q_{n,t}\{F(n)\} = \lim_{n\to\infty}\frac{n^{\beta+n+1-\alpha}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty v^{n+\beta-\alpha}e^{-nv}f(tv)dv\ .$$

But [3] we have for almost all positive t

$$\lim_{n\to\infty}\frac{n^{\beta+n+1-\alpha}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty y^{n+\beta-\alpha}e^{-ny}\{f(ty)-f(t)\}dy = 0$$

and so we have our theorem.

5. Representation theorem. In this section we propose to give a set of necessary and sufficient conditions for the representation of a function as an integral of the form (1.1). We shall need a lemma which we now prove.

LEMMA 4.1. *If n is a positive integer and x and t are positive variables then*

$$\left(\frac{\partial}{\partial t}\right)^n\left[t^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] = \frac{n^n}{t^{n+1-\beta}}I_{\eta+n,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}.$$

Proof. It is plain that

$$\left(\frac{t}{x}\right)^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}$$

is a homogeneous function of zero order. Therefore applying Euler's theorem we get

$$t\left(\frac{\partial}{\partial t}\right)\left[\left(\frac{t}{x}\right)^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] + n\left(\frac{\partial}{\partial x}\right)\left[\left(\frac{t}{x}\right)^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] = 0$$

or

$$\left(\frac{\partial}{\partial t}\right)\left[\frac{t^{\beta+n-1}}{x^{\beta+n}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] = -\left(\frac{\partial}{\partial x}\right)\left[\frac{t^{\beta+n-2}}{x^{\beta+n-1}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right]$$

or

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left[\frac{t^{\beta+n-1}}{x^{\beta+n}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] &= -\frac{\partial^2}{\partial t\partial x}\left[\frac{t^{\beta+n-2}}{x^{\beta+n-1}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] \\ &= -\left(\frac{\partial}{\partial x}\right)\left[\frac{\partial}{\partial t}\left\{\frac{t^{\beta+n-2}}{x^{\beta+n-1}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right\}\right] \\ &= (-)^2\frac{\partial^2}{\partial x^2}\left[\frac{t^{\beta+n-3}}{x^{\beta+n-2}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right].\end{aligned}$$

Proceeding in the same manner we have

$$\frac{\partial^n}{\partial t^n}\left[\frac{t^{\beta+n-1}}{x^{\beta+n}}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-n/t}\right\}\right] = \frac{t^{\beta-n-1}}{x^\beta}I_{\eta+n,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right]$$

using (3.1).

THEOREM 4.1. *The necessary and sufficient conditions that a given function $F(x)$ may have the representation* (1.1) *with $f(y)$ bounded and $Re\,\eta > 0$ $Re\,\beta \geqq 0$ are that*

(i) *$F(x)$ has derivatives of all orders in $0 < x < \infty$.*

(ii) *$F(x)$ tends to zero as x tends to infinity and*
(iii) $|Q_{n,t}\{F(x)\}| < M$ *for all integral* n $(0 < t < \infty)$.

Proof. First let us suppose that $F(x)$ has the representation (1.1). Under the conditions of the theorem it is obvious that all the derivatives of $F(x)$ exist. Also

$$\begin{aligned} F(x) &\leqq M' \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \\ &\quad \times \int_0^\infty (xy)^\beta {}_1F_1(\beta+\eta+1;\ \alpha+\beta+\eta+1;\ -xy)dy \\ &= \frac{M'\Gamma(\eta)\Gamma(\beta+1)}{x\Gamma(\alpha+\eta)} \end{aligned}$$

since $f(y)$ is bounded. So $F(x)$ tends to zero as x tends to infinity. To prove the necessity of (iii) we see, as in Theorem 3.1, that

$$|Q_{n,t}\{F(x)\}| \leqq \left\{\frac{n^{\beta+n+1-\alpha}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty v^{n+\beta-\alpha}e^{-nv}dv\right\}\left\{\operatorname*{lub}_{0\leqq t<\infty}|f(tv)|\right\} = M .$$

To prove the sufficiency let us suppose that the conditions are satisfied. If we now set

$$J_n = \int_0^\infty I_{\eta,\alpha}\{(xy)^\beta e^{-xy}\}Q_{n,y}\{F(x)\}dy$$

we have

$$\begin{aligned} J_n &= \frac{1}{\Gamma(n+1+\beta-\alpha)}\int_0^\infty \frac{n}{t^2} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\} W_n\{F(x)\}dn \\ &= (-)^n \int_0^\infty nt^{n+\beta-1} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\}\left(\frac{d}{dt}\right)^n \{t^{-\beta}F(t)\}dt . \end{aligned}$$

It will be seen in the course of the arguement that this integral exists. Integrating by parts we have

$$\begin{aligned} J_n = &\frac{(-)^n n}{\Gamma(n+\beta+1-\alpha)}\left[t^{n+\beta-1}I_{\eta,\alpha}\left\{\left(\frac{nn}{t}\right)^\beta e^{-nn/t}\right\}\left(\frac{d}{dt}\right)^{n-1}\{t^{-\beta}F(t)\}\right]_0^\infty \\ &+ \frac{(-)^{n-1}n}{\Gamma(n+1+\beta-\alpha)}\int_0^\infty \left(\frac{d}{dt}\right)^{n-1}\{t^{-\beta}F(t)\}\left(\frac{\partial}{\partial t}\right)\{t^{n+\beta-1}I_{\eta,\alpha}\phi\}dt \end{aligned}$$

where

$$\phi \equiv \left(\frac{nx}{t}\right)^\beta e^{-nx/t} .$$

Now

$$\begin{aligned} I_{\eta\alpha}\phi &= 0(t^{\eta+1}) \qquad (t \to 0) \\ &= 0(1) \quad \beta = 0(t \to \infty) \\ &= 0(1) \quad \beta > 0(t \to \infty) \end{aligned}$$

for [1]

$$I_{\eta,\alpha}(\phi) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\left(\frac{nx}{t}\right)^{\beta} {}_1F_1\left(\beta+\eta+1; \alpha+\beta+\eta+1; -\frac{nx}{t}\right).$$

Also the hypotheses of the theorem by implications mean that

$$F(x) = 0(x^{-1})$$

and in general

$$F^{(n)}(x) = 0(x^{-n-1})$$

and

$$\begin{aligned} &\left(\frac{d}{dt}\right)^{n-1}[t^{-\beta}F(t)] \\ &\quad = \{(-)^{n-1}\beta(\beta+1)\cdots(\beta+n-2)t^{-\beta-n+1}F(t) + \cdots t^{-\beta}F^{(n-1)}(t)\} . \end{aligned}$$

Therefore the integrated part

$$= 0[t^{\eta+1}\{A_1F(t) + \cdots t^{n-1}F^{(n-)}(t)\}] \to 0 \quad \text{as} \quad t \to 0 .$$

Also it is

$$= 0[A_1F(t) + \cdots tF^{(n-1)}(t)] \to 0 \qquad \text{as} \quad t \to \infty .$$

Therefore the integrated part is zero and integrating by parts again

$$\begin{aligned} J_n &= \frac{(-)^{n-1}n}{\Gamma(n+\beta+1-\alpha)}\left[\frac{\partial}{\partial t}(t^{n+\beta-1}I_{\eta\alpha}\phi)\left(\frac{d}{dt}\right)^{n-2}\{t^{-\beta}F(t)\}\right]_0^{\infty} \\ &\quad + \frac{(-)^{n-2}n}{\Gamma(n+\beta+1-\alpha)}\int_0^{\infty}\left(\frac{d}{dt}\right)^{n-2}\{t^{-\beta}F(t)\}\frac{\partial^2}{\partial t^2}(t^{n+\beta-1}I_{\eta,\alpha}\phi)dt . \end{aligned}$$

Now

$$\left(\frac{\partial}{\partial t}\right)\{t^{\beta+n-1}I_{\eta,\alpha}\phi\} = [(n-1)t^{\beta+n-2}I_{\eta,\alpha}\phi + \cdots + nnt^{\beta+n-3}I_{\eta+1,\alpha(\varphi)}]$$

and

$$\begin{aligned} &\left(\frac{d}{dt}\right)^{n-2}\{t^{-\beta}F(t)\} \\ &\quad = \{(-)^{n-2}\beta(\beta+1)\cdots(\beta+n-3)t^{-\beta-n+2}F(t) + \cdots t^{-\beta}F^{(n-2)}(t)\} . \end{aligned}$$

Therefore as before the integrated part again approaches zero when t tends to zero and t tends to infinity. Proceeding in the same manner we obtain

$$J_n = \frac{n}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty t^{-\beta}F(t)\frac{\partial^n}{\partial t^n}\{t^{\beta+n-1}I_{\eta,\alpha}\phi\}dt$$
$$= \frac{n}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty t^{-\beta}F(t)\frac{(nx)^n}{t^{n+1}}t^{\beta}I_{\eta+n,\alpha}(\phi)dt$$

by the Lemma 4.1. Hence

$$J_n = \frac{n^{n+\beta+1}n^{n+\beta}\Gamma(a)}{\Gamma(n+\beta+1-\alpha)\Gamma(b)}\int_0^\infty t^{-\beta-n-1}{}_1F_1\Big(a;\, b;\, -\frac{nx}{t}\Big)F(t)dt\ .$$

It is clear that this integral exists under the hypotheses of the theorem and therefore all the previous integrals exist. By a simple substitution this gives on using the asymptotic expansion of ${}_1F_1(a; b; x)$ [4]

$$J_n \sim \frac{n^{\beta+n+1}n^{n+\beta}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty u^{\beta+n-1}e^{-nxu}F\Big(\frac{1}{u}\Big)du\ .$$

Let

$$(1/u)F\Big(\frac{1}{u}\Big) \equiv \psi(u)\ .$$

Now

$$(1/u)F(1/u) = 0(1) \quad (u \to \infty) \quad \text{and} \quad F\Big(\frac{1}{u}\Big) = 0(1) \quad (u \to 0)\ .$$

Hence it is easily seen
(i) $\psi(u) \in L$ $(1/R \leqq t < R)$ for every $R > 1$.
(ii) $\int_1^\infty \psi(u)e^{-cu}du$ converges for any fixed $c > 0$, and
(iii) $\int_0^1 u\psi(u)du$ also converges. Therefore [3]

$$\lim_{n\to\infty} J_n = \frac{1}{u}\psi\Big(\frac{1}{u}\Big) = F(u)\ .$$

Now if

$$\chi(x, y) = \frac{\Gamma(a)}{\Gamma(b)}(xy)^{\beta}{}_1F_1(a;\, b;\, -xy)\ .$$

Then $\chi(xy) \in L$ in $0 \leqq y < \infty$ under the conditions assumed for the convergence of (1.1). Therefore by a theorem on weak compactness of a set of functions [5] the inequalities in the hypothesis (iii) of the theorem imply the existence of a subset $\{n_i\}$ of the positive integers and a bounded function $f(y)$ such that

$$\lim_{i\to\infty}\int_0^\infty [Q_{n_i,y}\{F(x)\}]\chi(x, y) = \int_0^\infty \chi(x, y)f(y)dy \ .$$

Hence

$$F(x) = \int_0^\infty \chi(x, y)f(y)dy$$

and the theorem is established.

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