# ON COVERINGS

J. SCHÖNHEIM

**1. Introduction.** Recently [2, 3, 4, 5] renewed interest has been aroused in the notion of covering and related problems, originally posed by Steiner [8] and later reformulated by Moore [6] as problems of the existence of *tactical configurations.*

A tactical configuration $C(k, l, \lambda, n)$ $(n \geqq k \geqq l)$ is a set of unordered $k$-tuples of $n$ different elements, such that each $l$-tuple of these elements appears exactly $\lambda$ times.

In view of the importance of the special cases $\lambda = 1$ and $l = 2$ the notions of *tactical systems* $S(k, l, n)$ for $C(k, l, 1, n)$ *and balanced incomplete block designs* (*BIBD*) $B(k, \lambda, n)$ for $C(k, 2, \lambda, n)$ have also been used.

A necessary condition [6] for the existence of a tactical configuration $C(k, l, \lambda, n)$ is known to be

$$(1) \qquad \lambda\binom{n-h}{l-h}\Big/\binom{k-h}{l-h} = \text{integer}\,, \qquad h = 0, 1, \cdots, l-1\,.$$

For $h = 0$ this integer, namely

$$(2) \qquad \lambda\binom{n}{l}\Big/\binom{k}{l}$$

is clearly the number of elements in $C(k, l, \lambda, n)$.

Condition (1) has been proved to be sufficient for $l = 2$, $k = 3$, $\lambda = 1$ by Moore [6] and Reiss [7], for $l = 2$, $k = 3$, $\lambda = 2$ by Bose [1], for $l = 2$, $k = 3$ and $k = 4$ and every $\lambda$, for $l = 2$, $k = 5$ $\lambda = 1, 4$ and 20, and for $l = 3$, $k = 4$ and every $\lambda$ by Hanani [3, 4, 5].

These results for $\lambda = 1$ show—and we note this here for future references—that necessary and sufficient conditions for the existence of tactical systems $S(4, 2, n)$, $S(5, 2, n)$ and $S(4, 3, n)$ are, respectively

$$(3) \qquad n \equiv 1 \text{ or } 4 \pmod{12}$$

$$(4) \qquad n \equiv 1 \text{ or } 5 \pmod{20}$$

$$(5) \qquad n \equiv 2 \text{ or } 4 \pmod{6}$$

More general *coverings* $R(k, l, \lambda, n)$ existing for every $n$ may be defined.

Received May 15, 1963. This paper will form part of the author's D. Sci. thesis in preparation at the Technion, Israel Institute of Technology, Haifa.

A covering $R(k, l, \lambda, n)$ $(n \geqq k \geqq l)$ is a set of unordered $k$-tuples of $n$ different elements, such that each $l$-tuple of these $n$ elements appears *at least* $\lambda$ times.

Coverings $R(3, 2, 1, n)$ have been studied by Fort Jr. and Hedlund [2]. These authors have proved that:

( i ) every covering $R(3, 2, 1, n)$ contains at least

$$\varphi(n) = \begin{cases} n^2/6 & \text{if } n \equiv 0 \\ n(n-1)/6 & \text{if } n \equiv 1 \text{ or } 3 \\ n^2 + 2/6 & \text{if } n \equiv 2 \text{ or } 4 \\ n^2 - n + 4/6 & \text{if } n \equiv 5 \end{cases} \pmod 6$$

triples;

(ii) for each $n$ there exists a covering $R(3, 2, 1, n)$ containing exactly $\varphi(n)$ triples.

In this paper we define the function

$$\psi(k, l, \lambda, n) = \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{k-1} \right\rceil \cdots \left\lceil \frac{n-l+2}{k-l+2} \left\lceil \frac{\lambda(n-l+1)}{k-l+1} \right\rceil \right\rceil \cdots \right\rceil$$

where $\lceil x \rceil$ denotes the smallest integer $y$, $y \geqq x$. This is a generalization of the function $\varphi(n)$. Indeed, $\varphi(n)$ equals $\psi(3, 2, 1, n)$.

We shall then prove (Theorem I) that every covering $R(k, l, \lambda, u)$ contains at least $\psi(k, l, \lambda, n)$ $k$-tuples.

Further, we denote coverings $R(k, l, \lambda, n)$ containing exactly $\psi(k, l, \lambda, n)$ $k$-tuples as *admissible coverings* $M(k, l, \lambda, n)$. Tactical configurations are such admissible coverings, because the number (2) of $k$-tuples in a tactical configuration $C(k, l, \lambda, n)$ equals $\psi(k, l, \lambda, n)$ as a consequence of conditions (1).

Finally, we shall prove (Theorem II) the existence of other admissible coverings, establishing that the existence of a tactical system $S(k, l, n)$ implies the existence of an admissible covering $M(k, l, 1, n+1)$. Thus, particularly (Corollaries 1, 2, 3) from conditions (3), (4), (5), derives the existence of admissible coverings $M(k, l, 1, n)$ for

$$\begin{aligned} k = 4,\ l = 2 \quad &\text{if} \quad n \equiv 2 \text{ or } 5 \pmod{12} \\ k = 5,\ l = 2 \quad &\text{if} \quad n \equiv 2 \text{ or } 6 \pmod{20} \\ k = 4,\ l = 3 \quad &\text{if} \quad n \equiv 3 \text{ or } 5 \pmod 6 . \end{aligned}$$

Our last result means in terms of *minimal coverings* (coverings containing the least possible number of $k$-tuples), that a minimal covering $R(k, l, \lambda, n)$ contains exactly $\psi(k, l, \lambda, n)$ $k$-tuples if a tactical system $S(k, l, n-1)$ exists.

## 2. The lower bound for the number of *k*-tuples in a covering.

THEOREM I. *Every covering $R(k, l, \lambda, n)$ contains at least*

$$\left\lceil \frac{n}{k}\left\lceil \frac{n-1}{k-1}\left\lceil \cdots \left\lceil \frac{n-l+2}{k-l+2}\left\lceil \frac{\lambda(n-l+1)}{k-l+1}\right\rceil\right\rceil \cdots \right\rceil\right\rceil\right\rceil \tag{6}$$

*k-tuples.*

*Proof.* We denote by $q(R, k, l, \lambda, n)$ the number of $k$-tuples contained in $R(k, l, \lambda, n)$ and by $\psi(k, l, \lambda, n)$ the expression (6). Under this notation, the statement of Theorem I is

$$q(R, k, l, \lambda, n) \geqq \psi(k, l, \lambda, n) . \tag{7}$$

We prove this inequality by induction on $l$. Let $l = 1$. Obviously $q(R, k, 1, \lambda, n) \geqq \lceil \lambda n/k \rceil = \psi(k, 1, \lambda, n)$. Suppose that inequality (7) is established for each $n \geqq k > l$ and $l \leqq l_0$. Now let $l = l_0 + 1$. Consider a $R(k, l_0 + 1, \lambda, n)$. It will contain $q(R, k, l_0 + 1, \lambda, n)$ $k$-tuples and therefore $k \cdot q(R, k, l_0 + 1, \lambda, n)$ elements. But each element must appear at least $q(R_1, k-1, l_0, \lambda, n-1)$ times, for otherwise $R(k, l_0 + 1, \lambda, n)$ could not contain $\lambda$ times the $l_0$-tuples of $n$ elements containing a given element. According to the hypothesis of the induction

$$q(R_1, k-1, l_0, \lambda, n-1) \geqq \psi(k-1, l_0, \lambda, n-1) .$$

It follows that

$$\begin{aligned} k \cdot q(R, k, l_0 + 1, \lambda, n) &\geqq nq(R_1, k-1, l_0, \lambda, n-1) \\ &\geqq n\psi(k-1, l_0, \lambda, n-1) \end{aligned}$$

and, since $q$ must be an integer, and as a consequence of the definition of $\psi(k-1, l_0, \lambda, n-1)$, we have

$$\begin{aligned} q(R, k, l_0 + 1, \lambda, n) &\geqq \left\lceil \frac{n}{k} \cdot \psi(k-1, l_0, \lambda, n-1) \right\rceil \\ &= \psi(k, l_0 + 1, \lambda, n) . \end{aligned}$$

This proves the validity of inequality (7) for each $l$, and the theorem is proved.

Theorem I justifies the following definition:

A covering $R(k, l, \lambda, n)$ may be called an admissible covering $M(k, l, \lambda, n)$ if it contains exactly $\psi(k, l, \lambda, n)$ $k$-tuples.

**3. The existence of admissible coverings which are not tactical configurations.** The fact that there exist admissible coverings which are not tactical configurations will be shown in Corollaries 1, 2 and 3 to Lemma 3, but for the purpose of obtaining the more general Theorem II, we shall prove the following four lemmas:

LEMMA 1. *If the expression*

$$\binom{n-h-1}{l-h}\Big/\binom{k-h}{l-h}$$

*is an integer for* $h = 0, 1, \cdots, l-1$, *and if we denote it by* $\alpha_{l-h}$ *and* 1 *by* $\alpha_0$, *we have, for* $i = 1, \cdots, l$

$$(8)\quad \sum_{j=0}^{i} \alpha_j = \left\lceil \frac{n-l+i}{k-l+i} \left\lceil \frac{n-l+i-1}{k-l+i-1} \left\lceil \cdots \left\lceil \frac{n-l+1}{k-l+1} \right\rceil \cdots \right\rceil \right\rceil \right\rceil .$$

*Proof.* We proceed by induction on $i$. Let $i = 1$. Then

$$\left\lceil \frac{n-l+1}{k-l+1} \right\rceil = \left\lceil \alpha_1 + \frac{1}{k-l+1} \right\rceil = \alpha_1 + 1 = \alpha_1 + \alpha_0 = \sum_{j=0}^{1} \alpha_j .$$

Let equality (8) be valid for $i = m < l$. This implies

$$\left\lceil \frac{n-l+m+1}{k-l+m+1} \left\lceil \frac{n-l+m}{k-l+m} \left\lceil \cdots \left\lceil \frac{n-l+1}{k-l+1} \right\rceil \right\rceil \cdots \right\rceil \right\rceil$$

$$= \left\lceil \frac{n-l+m+1}{k-l+m+1} \left( \sum_{j=0}^{m} \alpha_j \right) \right\rceil$$

$$= \left\lceil \frac{\sum_{j=0}^{m} (n-l+j)\alpha_j + \sum_{j=0}^{m} (m-j+1)\alpha_j}{k-l+m+1} \right\rceil$$

$$= \left\lceil \frac{\sum_{j=0}^{m} (k-l+j+1)\alpha_{j+1} + \sum_{j=0}^{m} (m-j+1)\alpha_j}{k-l+m+1} \right\rceil$$

$$= \left\lceil \frac{\sum_{j=1}^{m+1} (k-l+j)\alpha_j + \sum_{j=0}^{m} (m-j+1)\alpha_j}{k-l+m+1} \right\rceil$$

$$= \left\lceil \frac{(k-l+m+1)\alpha_{m+1} + m + 1 + \sum_{j=1}^{m} (k-l+m+1)\alpha_j}{k-l+m+1} \right\rceil$$

$$= \left\lceil \alpha_{m+1} + \sum_{j=1}^{m} \alpha_j + \frac{m+1}{k-l+m+1} \right\rceil = \sum_{j=0}^{m+1} \alpha_j .$$

And the lemma is proved.

LEMMA 2. *If the expression*

$$(9)\qquad \binom{n-h-1}{l-h}\Big/\binom{k-h}{l-h}$$

*is an integer for* $h = 0, 1, \cdots, l-1$ *we have*

$$(10)\qquad \frac{(n-1)(n-2)\cdots(n-l)}{k(k-1)\cdots(k-l+1)} + \left\lceil \frac{n-1}{k-1}\left\lceil \frac{n-2}{k-2}\left\lceil \cdots \left\lceil \frac{n-l+1}{k-l+1}\right\rceil \cdots \right\rceil\right\rceil\right\rceil = \left\lceil \frac{n}{k}\left\lceil \frac{n-1}{k-1}\left\lceil \cdots \left\lceil \frac{n-l+1}{k-l+1}\right\rceil \cdots \right\rceil\right\rceil\right\rceil .$$

*Proof.* Denote the integer (9) by $\alpha_{l-h}$ and $\alpha_0 = 1$. According to Lemma 1 and under this notation, the left hand side of equality (10) becomes

$$\alpha_l + \sum_{j=0}^{l-1}\alpha_j = \sum_{j=0}^{l}\alpha_j = \left\lceil \frac{n}{k}\left\lceil \frac{n-1}{k-1}\left\lceil \cdots \left\lceil \frac{n-l+1}{k-l+1}\right\rceil \cdots \right\rceil\right\rceil\right\rceil .$$

LEMMA 3. *If there exists a tactical system* $S(k, l, n-1)$ *and an admissible covering* $M(k-1, l-1, 1, n-1)$, *then there also exists an admissible covering* $M(k, l, 1, n)$.

*Proof.* Let $N$ be a fixed element. Let $V = \{(x, N) : x \in M(k-1, l-1, 1, n-1)\}$ and $T = S(k, l, n-1) \cup V$. It will then, be shown that $T$ is an admissible covering.

Indeed, it is a covering $R(k, l, 1, n)$, as all the $l$-tuples of $n$ elements not containing the element $N$ appear in one of the $k$-tuples in $S(k, l, n-1)$, while the $l$-tuples containing the element $N$ appear in at least one of the $k$-tuples in $V$. Moreover, the covering $R(k, l, 1, n)$ is an admissible covering $M(k, l, 1, n)$. In fact, it contains

$$(11)\qquad \frac{(n-1)(n-2)\cdots(n-l)}{k(k-1)\cdots(k-l+1)} + \left\lceil \frac{n-1}{k-1}\left\lceil \frac{n-2}{k-2}\left\lceil \cdots \left\lceil \frac{n-l+1}{k-l+1}\right\rceil \cdots \right\rceil\right\rceil\right\rceil$$

$k$-tuples, which is the sum of the number of $k$-tuples in $S(k, l, n-1)$ and of $(k-1)$-tuples in $M(k-1, l-1, 1, n-1)$.

The conditions of Lemma 2 are satisfied, and accordingly, (11) equals $\psi(k, l, 1, n)$, which proves the lemma.

COROLLARY 1. *If*

$$(12)\qquad n \equiv 3 \text{ or } 5 \pmod 6$$

*then there exists an admissible covering* $M(4, 3, 1, n)$.

*Proof.* For $M$ satisfying (12), according to (5), there exists a tactical system $S(k, 3, n-1)$, and according to (ii) there also exists an admissible covering $M(3, 2, 1, n-1)$. Lemma 3 then implies the existence of an admissible covering $M(4, 3, 1, n)$

COROLLARY 2. *If*

$$n \equiv 2 \text{ or } 5 \pmod{12} \tag{13}$$

*then there exists an admissible covering* $M(4, 2, 1, n)$.

*Proof.* For $n$ satisfying (13), according to (3), there exists a *BIBD* $B(4, 1, n-1)$. The existence of an admissible covering $M(3, 1, 1, n-1)$ being obvious, Lemma 3 implies the existence of an admissible covering $M(4, 2, 1, n)$.

COROLLARY 3. *If*

$$n \equiv 2 \text{ or } 6 \pmod{20}$$

*then there exists an admissible covering* $M(5, 2, 1, n)$.

*Proof.* Similar to that of the preceding corollary, but using (1) instead of (3).

LEMMA 4. *The existence of a tactical system* $S(k, l, n)$ *implies that of an admissible covering* $M(k-1, l-1, 1, n)$.

*Proof.* By induction on $l$. Let $l = 2$. The existence of an admissible covering $M(k-1, 1, 1, n)$ is obvious. Suppose now that the lemma is proved for $l = l_0$ and let $l = l_0 + 1$. The existence of a $S(k, l_0 + 1, n)$ implies that of a $S(k-1, l_0, n-1)$ which, according to the hypothesis of the induction, implies the existence of a $M(k-2, l_0 - 1, 1, n-1)$. The existence of a $S(k-1, l_0, n-1)$ and of a $M(k-2, l_0 - 1, 1, n-1)$ implies, according to Lemma 3, that of a $M(k-1, l_0, 1, n)$.

THEOREM II. *If a tactical system* $S(k, l, n)$ *exists, then there also exists an admissible covering* $M(k, l, 1, n+1)$.

*Proof.* According to Lemma 4, the second hypothesis of Lemma 3 is automatically satisfied if the first hypothesis holds.

## References

1. R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugenics, **9** (1939), 353-399.
2. M. K. Fort Jr. and G. A. Hedlund, *Minimal coverings of pairs by triples*, Pacific J. Math., **8** (1958).
3. H. Hanani, *On quadruple systems*, Canadian J. Math., **12** (1960).
4. ———, *The existence and construction of balanced incomplete block designs*, Annals of Math. Statistics VI (1961).
5. ———, *On some tactical configurations* Canadian J. Math., in print, 1963.
6. E. H. Moore, *Tactical Memoranda*, Amer. J. Math., **18** (1896), 264-303.
7. (M.) Reiss, *Üeber eine Steinersche combinatorische Aufgabe*, J. für die reine angewandte Mathematik, **56** (1859), 326-344.
8. J. Steiner, *Combinatorische Aufgabe*, J. für die reine angewandte Mathematik **45** (1853), 273-280.