A CHARACTERIZATION OF EXTREMALS FOR GENERAL MULTIPLE INTEGRAL PROBLEMS

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1. Introduction. The theory of quadratic functionals on a Hilbert space is applied here to variational problems for multiple integrals, involving several dependent variables and their partial derivatives up to any finite order, in nonparametric form. The treatment is confined to integration over open sets with fixed boundary, and to weak neighborhoods of an extremal. The basic result is the establishment of general sufficiency theorems for a weak relative minimum with isoperimetric or differential side conditions. A specific application is made of the sufficiency theorem in the case of isoperimetric side conditions to the extension of a well known characterization of of extremals. This characterization has been treated by Poincaré [16], Birkhoff and Hestenes [1], Karush [9] and others. It is analogous to the characterization of saddle points on two-dimensional surfaces as constrained extrema.

2. Quadratic forms. The theory of quadratic functionals (forms) has been developed explicitly by Hestenes [6, 7], and implicitly by Van Hove [19] and writers on elliptic partial differential equations.

If $Q(x)$ is a quadratic form on a Hilbert space $\mathfrak{H}$ with real scalars then there exist unique subspaces $\mathfrak{H}_+, \mathfrak{H}_0, \mathfrak{H}_-$ of $\mathfrak{H}$, having the null vector as their only common element, that are mutually orthogonal and Q-orthogonal, are such that Q is positive on $\mathfrak{H}_+$, negative on $\mathfrak{H}_-$, and zero on $\mathfrak{H}_0$, and are such that $\mathfrak{H} = \mathfrak{H}_+ + \mathfrak{H}_0 + \mathfrak{H}_-$ [6, p. 543]. The sum of the dimensions of the subspaces $\mathfrak{H}_-$ and $\mathfrak{H}_0$ will be called the *isoperimetric index* of Q on $\mathfrak{H}$. A quadratic form that is representable on $\mathfrak{H}$ as the sum of a positive definite quadratic form and a w-continuous quadratic form has been called a Legendre form by M. R. Hestenes. The fact that the isoperimetric index of a Legendre form is finite is significant for the characterization of extremals given below.

If $Q(x)$ and $K(x)$ are quadratic forms on $\mathfrak{H}$ such that $J(b; x) = Q(x) + bK(x)$ is a Legendre form for every positive number b, $K(x) \leqq 0$, and $Q(x) > 0$ whenever $K(x) = 0$ and $x \neq 0$, then there is a positive number c such that $J(c; x)$ is positive definite on $\mathfrak{H}$. A corollary to this is: *If $Q(x)$ and $K(x)$ are quadratic forms on $\mathfrak{H}$, $J(b_0; x) = Q(x) + b_0K(x)$ is a Legendre form for some number b_0, $K(x) \geqq 0$, and*

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$Q(x) > 0$ whenever $K(x) = 0$ and $x \neq 0$, then there is a positive number $c \geqq b_0$ such that $J(c; x)$ is positive definite on $\mathfrak{H}$.

Let Λ be a metric space. Let $M(\lambda; x)$ be a real-valued function of λ and x on the product space $\Lambda \times \mathfrak{H}$ which has the properties:

(1) for each element λ in Λ, $M(\lambda; x)$ is continuous in x on $\mathfrak{H}$;

(2) for every λ_0 in Λ, $M(\lambda; x)$ is continuous in λ at λ_0 uniformly for all x in $\mathfrak{H}$ for which $\|x\| = 1$. Then $M(\lambda; x)$ will be called a *λ-form on $\mathfrak{H}$.*

The proofs of the following three theorems are immediate consequences of the above definition:

THEOREM 2.1. *If $Q(\lambda; x)$ is a quadratic form in x, then $Q(\lambda; x)$ is a λ-form (a quadratic λ-form) if and only if for every $\varepsilon > 0$ and every λ_0 in Λ there is a neighborhood Λ_0 of λ_0 such that for λ in Λ_0, $|Q(\lambda; x) - Q(\lambda_0; x)| \leqq \varepsilon \|x\|^2$.*

THEOREM 2.2. *If $L(\lambda; x)$ is a linear form in x, then $L(\lambda; x)$ is a λ-form (a linear λ-form) if and only if for every $\varepsilon > 0$ and every λ_0 in Λ there is a neighborhood Λ_0 of λ_0 such that for λ in Λ_0, $|L(\lambda; x) - L(\lambda_0; x)| \leqq \varepsilon \|x\|$.*

THEOREM 2.3. *If the forms $L_\sigma(\lambda; x)$ $(\sigma = 1, \cdots, p)$ are linear λ-forms, then the form $K(\lambda; x) = L_\sigma(\lambda; x)L_\sigma(\lambda; x)$ is a quadratic λ-form.*

The sufficiency theorem of § 4 will be based on

THEOREM 2.4. *Suppose that $Q(\lambda; x)$ is a quadratic λ-form on $\mathfrak{H}$, that λ_0 is in Λ, and that $\mathscr{K}$ is a subspace of $\mathfrak{H}$ on which $Q(\lambda_0; x)$ is positive definite. Then there is a neighborhood Λ_0 of λ_0 and a positive number h such that for every λ in Λ_0, the inequality $Q(\lambda; x) \geqq h\|x\|^2$ holds for all x in $\mathscr{K}$.*

Proof. By hypothesis there exists $h > 0$ such that for each x in $\mathscr{K}$, $Q(\lambda_0; x) \geqq h\|x\|^2$. In Theorem 2.1, choose $\varepsilon = h/2$; then there is a neighborhood Λ_0 of λ_0 such that

$$-\frac{h}{2}\|x\|^2 \leqq Q(\lambda; x) - Q(\lambda_0; x) \leqq \frac{h}{2}\|x\|^2 ;$$

hence

$$Q(\lambda; x) \geqq Q(\lambda_0; x) - \frac{h}{2}\|x\|^2 \geqq h\|x\|^2 - \frac{h}{2}\|x\|^2 = \frac{h}{2}\|x\|^2 .$$

The sufficiency theorem of § 5 will be based on the following theorem. Let $Q(\lambda, b; x) = P(\lambda; x) + bK(\lambda; x)$ where $P(\lambda; x)$ and $K(\lambda; x)$ are quadratic forms in x defined on $\Lambda \times \mathfrak{H}$ and b is a real number. We will assume in the following theorem that for each λ in Λ, $K(\lambda; x)$ is a nonnegative quadratic form in x on $\mathfrak{H}$, and that for $\lambda = \lambda_0$, $P(\lambda_0; x) > 0$ whenever $K(\lambda_0; x) = 0$ and $x \neq 0$.

THEOREM 2.5. *If $Q(\lambda_0, b; x)$ has the property described above and is a Legendre form on $\mathfrak{H}$ for some value of b, then there exists a positive number c such that $Q(\lambda, c; x)$ is positive definite on $\mathfrak{H}$.*

Proof. The conclusion is a direct consequence of the corollary given earlier in this section.

3. General multiple integrals. Let T be a bounded open set in an m-dimensional euclidean space, and let t be a variable point in T. Let $z^1, z^2, \cdots, z^n$ be real-valued functions defined on T and let z represent the set $z^1, \cdots, z^n$ plus all the partial derivatives, when they exist, of these functions of orders less than or equal to some positive integer k. The boundary values of z and the first $k-1$ derivatives are prescribed. Finally, let f denote a real-valued function of t and z. With suitable restrictions on the function f and the variable z we can interpret meaningfully the *general multiple integral*

$$\int_T f[t, z(t)]dt \tag{3.1}$$

in the sense of m-dimensional Lebesgue integration. Further, let s be another variable point of T, let F denote a real-valued function of t and z, and let G denote a real-valued function of s, t, u, and v. The variables u and v are of the same type as z. Then the general multiple integral (of Fubini-Tonelli type)

$$\int_T\int_T G[s, u(s), t, v(t)]dsdt \tag{3.2}$$

is well-defined on the product space $T \times T$ in the same sense as (3.1), as is

$$J(z) = \int_T F[t, z(t)]dt + \int_T\int_T G[s, z(s), t, z(t)]dsdt\,. \tag{3.3}$$

The treatment of integrals of this type has for one purpose a uniform treatment of variational problems with side conditions.

Partial derivatives will be indicated in the accepted fashion: Given nonnegative integers $\alpha_1, \alpha_2, \cdots, \alpha_m$, let $\alpha = (\alpha_1, \cdots, \alpha_m)$ and let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$; then

$$z^i_\alpha = \frac{\partial^{|\alpha|} z^i}{\partial t_1^{\alpha_1} \cdots \partial t_m^{\alpha_m}}$$

denotes a partial derivative of z^i and $j = |\alpha|$ is the order of the derivative. In summary, we have $z = [z^{(0)}, z^{(1)}, \cdots, z^{(k)}]$ with $z^{(0)} = (z^1, z^2, \cdots, z^n)$ and $z^{(j)} = (z^i_\alpha)$, $|\alpha| = j \geqq 1$.

4. Sufficient conditions for a proper weak relative minimum.

Assume as given:

(1) a set T in t-space that is a union of a finite number of sets each of which is the image of an m-dimensional bounded open interval under a one-to-one continuous transformation which together with its inverse satisfies a uniform Lipschitz condition [7, p. 319];

(2) an open set $\mathscr{R}$ in tz-space; let $\mathscr{S} = \mathscr{R} \times \mathscr{R}$;

(3) integrand functions $F(t, z)$, $G(s, u, t, v)$ such that F is continuous on $\mathscr{R}$, G is continuous on $\mathscr{S}$, and F and G have continuous derivatives on $\mathscr{R}$ and $\mathscr{S}$, respectively, of the first and second orders with respect to the components of z, u, and v;

(4) a particular function e that is of class C^k on the closure of T,[1] and whose elements $[t, e(t)]$ are in $\mathscr{R}$ for all t in T.

A function z belongs to the *class* $\mathscr{A}$ *of comparison functions* if

(1) z is of class C^k on the closure of T;

(2) the difference $z^{(0)} - e^{(0)}$ together with its derivatives of orders $\leqq k - 1$ vanish on the boundary of T, that is, they have limit functions on the boundary that are identically zero;

(3) the elements $[t, z(t)]$ are in $\mathscr{R}$ for all t in T.

The *class* $\mathscr{B}$ *of variations* is the closure in the Hilbert space $\mathfrak{H}_k$ ([7], p. 319, restricted to real scalars) of the class of $x^{(0)}$ in $\mathfrak{H}_k$ whose kth derivatives are bounded on T, and which, together with their first $(k-1)$-derivatives, vanish on the boundary of T. The boundedness of T ensures that $z - e$ is a variation in the class $\mathscr{B}$ and that the integral $J(z)$ as defined by (3.3) has well-defined values. In what follows the derivatives of F with respect to z and of G with respect to u and v will appear and then their evaluations for the case $z = z(t)$, $u = z(s)$, $v = z(t)$. For all of this it will be convenient to use the abbreviations

$$F(t, z) = F[t, z(t)]\,, \quad G(s, t, z) = G[s, z(s), t, z(t)]\,,$$

$$G(t, s, z) = G[t, z(t), s, z(s)]\,, \quad F_{z^i_\alpha}(t, z) = \left[\frac{\partial F(t, z)}{\partial z^i_\alpha}\right]_{z=z(t)}\,,$$

$$G_{u^i_\alpha}(s, t, z) = \left[\frac{\partial G(s, t, z)}{\partial u^i_\alpha}\right]_{\substack{u=z(s)\\ v=z(t)}}\,, \quad G_{v^i_\alpha}(s, t, z) = \left[\frac{\partial G(s, t, z)}{\partial v^i_\alpha}\right]_{\substack{u=z(s)\\ v=z(t)}}\,,$$

[1] The function z belongs to class C^k if each component of $z^{(0)}$ is in the class. Similar remarks apply to the functions e, x, and y.

and similarly for higher derivatives of F and G.

The sufficient conditions will be stated in terms of the first and second variations of $J(z)$: The *first variation of* $J(z)$ *on* e is given by

$$(4.1)\quad J_1(e; x) = \int_T \left\{ F_{z^i_\alpha}(t, e) + \int_T [G_{u^i_\alpha}(t, s, e) + G_{v^i_\alpha}(s, t, e)] ds \right\} x^i_\alpha(t) dt$$

where i and α are summed for $i \leqq n$ and $|\alpha| \leqq k$. The *second variation of* $J(z)$ *on* e is given by

$$(4.2)\quad J_2(e; x) = \int_T r^{ij}_{\alpha\beta}(t, e) x^i_\alpha(t) x^j_\beta(t) dt + \int_T \int_T w^{ij}_{\alpha\beta}(s, t, e) x^i_\alpha(s) x^j_\beta(t) ds dt$$

(summed on i, j, α, β with $i \leqq n, j \leqq n, |\alpha| \leqq k, |\beta| \leqq k$) where

$$(4.3)\quad r^{ij}_{\alpha\beta}(t, e) = F_{z^i_\alpha z^j_\beta}(t, e) + \int_T [G_{u^i_\alpha u^j_\beta}(t, s, e) + G_{v^i_\alpha v^j_\beta}(s, t, e)] ds$$

and

$$(4.4)\quad w^{ij}_{\alpha\beta}(s, t, e) = G_{u^i_\alpha v^j_\beta}(s, t, e) + G_{v^i_\alpha u^j_\beta}(s, t, e) .$$

The function e satisfies *Condition* I if

$$(4.5)\quad J_1(e; x) = 0$$

for every variation in $\mathscr{B}$.

A function z of class C^k on the closure of T satisfying I will be called an *extremal of* $J(z)$.

The function e satisfies *Condition* III′ if there is a positive constant ε such that

$$(4.6)\quad r^{ij}_{\gamma\delta}(t, e) \xi^\gamma \xi^\delta \zeta^i \zeta^j \geqq \varepsilon |\xi|^{2k} |\zeta|^2$$

(summed on i, j, γ, δ, for $i \leqq n, j \leqq n, |\gamma| = |\delta| = k$) for all t in T and all sets of real numbers $(\neq 0)$ $\xi = (\xi_1, \cdots, \xi_m)$, $\zeta = (\zeta^1, \cdots, \zeta^n)$. The notation ξ^γ means $(\xi_1)^{\gamma_1}(\xi_2)^{\gamma_2} \cdots (\xi_m)^{\gamma_m}$; also, $|\xi|^2 = (\xi_1)^2 + \cdots + (\xi_m)^2$ and $|\zeta|^2 = (\zeta^1)^2 + \cdots + (\zeta^n)^2$. The functions $r^{ij}_{\gamma\delta}(t, z)$ are those given in (4.3). Condition III′ has been referred to as the *strengthened condition of Legendre*.

The function e satisfies *Condition* IV′ if

$$(4.7)\quad J_2(e; x) > 0$$

holds for every nonnull variation in $\mathscr{B}$. By a nonnull variation is meant one that does not vanish identically on T.

Sufficient conditions similar to these have been established by Van Hove [18, 19] for the case of derivatives of the first order only appearing in the integrand, and by Klotzler [11] for higher derivatives but with the assumption that J_2 is positive definite.

The sufficiency theorem here uses, in addition to Theorem 2.4, the following three theorems which were proved in [7] as Theorems 6: 1 and 8: 1:

(1) The quadratic form $K(x) = \int_T b^{ij}_{\mu\nu} x^i_\mu x^j_\nu dt$ (summed on i, j, μ, ν for $i \leqq n, j \leqq n, |\mu| \leqq k, |\nu| \leqq k, |\mu| + |\nu| < 2k$), where $b^{ij}_{\mu\nu}(t) = b^{ji}_{\nu\mu}(t)$ are bounded integrable functions of t on T, is w-continuous on $\mathfrak{H}_k$.

(2) The quadratic form $H(x) = \int_T \int_T c^{ij}_{\alpha\beta}(s, t) x^i_\alpha(s) x^j_\beta(t) ds dt$ (summed on i, j, α, β for $i \leqq n, j \leqq n, |\alpha| \leqq k, |\beta| \leqq k$) where $c^{ij}_{\alpha\beta}(s, t) = c^{ji}_{\beta\alpha}(t, s)$ are square integrable functions of s and t on $T \times T$, is w-continuous on $\mathfrak{H}_k$.

(3) Let J be a quadratic form of the type $J(x) = D(x) + K(x)$ where $K(x)$ is w-continuous on $\mathscr{B}$ and

$$D(x) = \int_T d^{ij}_{\gamma\delta} x^i_\gamma x^j_\delta dt \tag{4.8}$$

(summed over i, j, γ, δ for $|\gamma| = |\delta| = k, i \leqq n, j \leqq n$) where $d^{ij}_{\gamma\delta}(t) = d^{ji}_{\delta\gamma}(t)$ are continuous functions of t on the closure of T. Then the quadratic form J as described above is a Legendre form on $\mathscr{B}$ if and only if it satisfies on T the inequality

$$d^{ij}_{\gamma\delta} \xi^\gamma \xi^\delta \zeta^i \zeta^j \geqq \varepsilon |\xi|^{2k} |\zeta|^2 \tag{4.9}$$

(summed on i, j, γ, δ for $i \leqq n, j \leqq n, |\gamma| = |\delta| = k$) for some positive ε.

LEMMA 4.1. *For the function e, $J_2(e; x)$ is a Legendre form on $\mathscr{B}$ if and only if the Condition* III′ *is satisfied.*

Proof. Write, using (4.4), $J_2(e; x) = D(x) + K(x)$ where $D(x)$ is given by $D(x) = \int_T r^{ij}_{\gamma\delta} x^i_\gamma x^j_\delta dt$ (summed on γ and δ for $|\gamma| = |\delta| = k$). The immediately preceding theorems apply.

LEMMA 4.2. *For the function e, $J_2(e; x)$ is positive definite on $\mathscr{B}$ if and only if the Conditions* III′ *and* IV′ *are satisfied.*

Proof. If J_2 is positive definite on $\mathscr{B}$, then $J_2(e; x) > 0$ holds on $\mathscr{B}$ and also J_2 is a Legendre form; hence it follows that the inequality (4.6) is satisfied on T. On the other hand suppose J_2 is positive on $\mathscr{B}$ and (4.6) holds. Then J_2 is a Legendre form on $\mathscr{B}$ by the preceding lemma, and a positive Legendre form is positive definite.

To apply the theorems of §2 concerning quadratic forms, it is necessary to specify the metric space Λ. For this we will use the weak neighborhoods of the calculus of variations: Let $|z^i_\alpha - e^i_\alpha| = \max_t |z^i_\alpha(t) - e^i_\alpha(t)|$, $|z^{(j)} - e^{(j)}| = \max_{i,\alpha} |z^i_\alpha - e^i_\alpha|$ ($|\alpha| = j$); the metric space $\varLambda$ is the class $\mathscr{A}$ with the metric

$$|z - e| = |z^{(0)} - e^{(0)}| + |z^{(1)} - e^{(1)}| + \cdots + |z^{(k)} - e^{(k)}| . \tag{4.10}$$

LEMMA 4.3. *The second variation $J_2(z; x)$ is a quadratic z-form.*

Proof. Apply Theorem 2.1, recalling that F and G are assumed to be of class C^2.

THEOREM 4.1. *If the function e as given satisfies the Conditions* I, III′, *and* IV′ *then there is a (weak) neighborhood N of e and positive constant ρ such that for every z of class $\mathscr{A}$ in N*

$$J(z) - J(e) \geqq \rho \,\|z - e\|^2 . \tag{4.11}$$

Proof. For any z in $\mathscr{A}$, form $x = z - e$; then x is a variation in the class $\mathscr{B}$. The continuity properties of F and G make possible the expansion

$$J(z) - J(e) = \int_0^1 (1 - \theta) J_2(e + \theta x; x) d\theta$$

where we have used the fact that I implies $J_1(e; x) = 0$. To apply Theorem 2.4, we note that from Lemma 4.2 J_2 is positive definite on e, and from Lemma 4.3 J_2 is a quadratic z-form. Consequently there is a neighborhood N of e, which can be chosen to be convex, and a constant $\rho > 0$ such that when z is restricted to N and $x = z - e$, $J_2(e + \theta x; x) \geqq 2\rho \,\|x\|^2$. Hence $J(z) - J(e) \geqq \rho \,\|z - e\|^2$.

5. Isoperimetric side conditions.

The preceding sufficiency theorem will be applied to the problem of establishing sufficient conditions that a function e will minimize

$$I(z) = \int_T f[t, z(t)] dt + \int_T \int_T g[s, z(s), t, z(t)] ds dt \tag{5.1}$$

subject to a finite number of isoperimetric side conditions

$$I^\sigma(z) = \int_T f^\sigma[t, z(t)] dt = 0 \qquad (\sigma = 1, \cdots, p) . \tag{5.2}$$

For the complete formulation of the problem, it is expected that the assumptions of § 4 are met. In particular, the integrand functions $f(t, z), f^\sigma(t, z)$ and their first and second derivatives are specified to be continuous on $\mathscr{R}$ and similarly for $g(s, t, z)$ on $\mathscr{S}$.

Sufficient conditions for weak and strong minima with isoperimetric side conditions, in the case of one dependent variable and derivatives of first order, have been treated by Hestenes [5]. The extension of this work to the general case is still unknown. The following conditions[2] will prove to be sufficient that e provides a weak relative minimum for the system (5.1) and (5.2).

I_s. There exists a set of multipliers a_σ (possibly all zero) such that

$$I_1(e; x) + a_\sigma I_1^\sigma(e; x) = 0 \tag{5.3}$$

for every x in $\mathscr{B}$.

III_s'. The multipliers in I_s can be chosen so that there is a positive constant ε such that

$$R_{\gamma\delta}^{ij}(t, e)\xi^\gamma\xi^\delta\zeta^i\zeta^j \geqq \varepsilon\,|\,\xi\,|^{2k}\,|\,\zeta\,|^2 \tag{5.4}$$

(summed on i, j, γ, δ for $i \leqq n$, $|\gamma| = |\delta| = k$, $j \leqq n$) for all t in T. The coefficients $R_{\gamma\delta}^{ij}$ are given by

$$R_{\gamma\delta}^{ij}(t, z) = f_{z_\gamma^i z_\delta^j}(t, z) + a_\sigma f^\sigma_{z_\gamma^i z_\delta^j}(t, z) + \int_T [g_{u_\gamma^i u_\delta^j}(t, s, z) + g_{v_\gamma^i v_\delta^j}(s, t, z)]ds\,.$$

IV_s'. The multipliers in III_s' can be chosen so that

$$I_2(e; x) + a_\sigma I_2^\sigma(e; x) > 0 \tag{5.5}$$

for every nonnull variation in the subspace of $\mathscr{B}$ for which

$$I_1^\sigma(e; x) = 0 \qquad (\sigma = 1, \cdots, p)\,. \tag{5.6}$$

To reduce this problem to that of § 4, take for the integrands in (3.3): $F = f + a_\sigma f^\sigma$ and $G = g + bf^\rho f^\sigma$ summed on σ from 1 to p, with b a real number yet to be determined. Then, by Fubini's theorem, $J(z) = I(z) + a_\sigma I^\sigma(z) + bI^\sigma(z)I^\sigma(z)$; it is easily verified that

$$J_1(z; x) = I_1(z; x) + a_\sigma I_1^\sigma(z; x) + 2bI^\sigma(z)I_1^\sigma(z; x)$$

and

$$J_2(z; x) = I_2(z; x) + a_\sigma I_2^\sigma(z; x) + 2bI_1^\sigma(z; x)I_1^\sigma(z; x) + 2bI^\sigma(z)I_2^\sigma(z; x)\,.$$

Condition I_s implies the condition I directly, using the facts that $I_1^\sigma(z; x)$ is bounded for each value of σ, and $I^\sigma(e) = 0$: We have on e, $J_1(e; x) = I_1(e; x) + a_\sigma I_1^\sigma(e; x) = 0$.

Similarly, we can evaluate the expression in III′ from its definition by (4.3), and using again that $I^\sigma(e) = 0$ it follows that III_s' implies III′.

[2] In this and the following sections the subscript s will be used to distinguish sufficient conditions involving side conditions.

LEMMA 5.1. *If* III_s' *and* IV_s' *hold, then there is a positive value of* b *such that* $J_2(e;x)$ *is a positive definite form on* $\mathscr{B}$; *hence* IV′ *holds.*

Proof. Since $I^\sigma(e)=0$ and $I_2^\sigma(e;x)$ is bounded for each value of σ, $J_2(e;x)=I_2(e;x)+a_\sigma I_2^\sigma(e;x)+2bI_1^\sigma(e;x)I_1^\sigma(e;x)$. $J_2(e;x)$ is a Legendre form on $\mathscr{B}$ by Lemma 4.1. Let $P(x)=I_2(e;x)+a_\sigma I_2^\sigma(e;x)$ and $K(x)=2I_1^\sigma(e;x)I_1^\sigma(e;x)$. Then $K(x)$ is a nonnegative quadratic form in x. Also, $K(x)=0$ only if $I_1^\sigma(e;x)=0$ for $\sigma=1,\cdots,p$. Condition IV_s' implies that $P(x)>0$ whenever $K(x)=0$ and $x\neq 0$. Consequently the hypotheses of Theorem 2.5 are satisfied, and a positive value for b exists such that $J_2(e;x)$ is positive definite on $\mathscr{B}$. Since a positive definite form is positive, IV′ is satisfied.

In view of this last result, Theorem 4.1 has as its interpretation in the case of a finite number of isoperimetric side conditions,

THEOREM 5.1. *If the function* e *as prescribed in the formulation of* § 4 *satisfies the conditions* I_s, III_s' *and* IV_s', *then there is a (weak) neighborhood* N *of* e *and positive constants* b *and* ρ *such that for every* z *of class* $\mathscr{A}$ *in* N

$$I(z)-I(e)+a_\sigma I^\sigma(z)+bI^\sigma(z)I^\sigma(z)\geqq \rho\,||\,z-e\,||^2 \tag{5.7}$$

(*summed on* σ *for* $\sigma\leqq p$).

COROLLARY 5.1. *Under the hypotheses of Theorem* 5.1, *there is a (weak) neighborhood* N *of* e *and a positive constant* ρ *such that for every* z *of class* $\mathscr{A}$ *in* N

$$I(z)-I(e)\geqq \rho\,||\,z-e\,||^2 \tag{5.8}$$

holds whenever z *also satisfies the conditions*

$$I^\sigma(z)=0 \qquad (\sigma=1,\cdots,p)\,. \tag{5.9}$$

6. Differential side conditions. In this section the application of Theorem 4.3 is continued to the problem of establishing sufficient conditions that a function e will minimize the functional $I(z)$ given by (5.1) subject to a finite number of partial differential equations as side conditions of the form

$$\phi^\sigma[t,z(t)]=0 \qquad (\sigma=1,\cdots,p)\,. \tag{6.1}$$

In addition to the assumptions of § 4, it is required that, for $\sigma=1,2,\cdots,p$, $\phi^\sigma(t,z)$ is continuous and has continuous first and second derivatives on $\mathscr{R}$ with respect to the components of z. Further, to avoid an empty class of comparison functions, it will be assumed that the set (6.1) satisfies suitable integrability conditions. For the statement of sufficient conditions for a minimum it is convenient to form

$$H(z) = I(z) + \int_T a_\sigma(t)\phi^\sigma(t, z)dt \tag{6.2}$$

and

$$P(z) = \int_T \phi^\sigma(t, z)\phi^\sigma(t, z)dt \,. \tag{6.3}$$

Then for the first and second variations of H and P we take (with summations on $\sigma, i, j, \alpha, \beta$, for $\sigma \leqq p, i \leqq n, j \leqq n, |\alpha| \leqq k, |\beta| \leqq k$)

$$\begin{aligned} H_1(z; x) &= I_1(z; x) + \int_T a_\sigma(t)\phi^\sigma_{z^i_\alpha}(t, z)x^i_\alpha(t)dt, \\ P_1(z; x) &= 2\int_T \phi^\sigma(t, z)\phi^\sigma_{z^i_\alpha}(t, z)x^i_\alpha(t)dt \,, \\ H_2(z; x) &= I_2(z; x) + \int_T a_\sigma(t)\phi^\sigma_{z^i_\alpha z^j_\beta}(t, z)x^i_\alpha(t)x^j_\beta(t)dt \,, \\ P_2(z; x) &= 2\int_T \phi^\sigma_{z^i_\alpha}(t, z)\phi^\sigma_{z^j_\beta}(t, z)x^i_\alpha(t)x^j_\beta(t)dt \\ &\quad + 2\int_T \phi^\sigma(t, z)\phi^\sigma_{z^i_\alpha z^j_\beta}(t, z)x^i_\alpha(t)x^j_\beta(t)dt \,. \end{aligned} \tag{6.4}$$

As sufficient conditions for a proper weak relative minimum of $I(z)$ subject to $\phi^\sigma(t, z) = 0$, consider

I$_s$. There exist a set of multipliers $a_\sigma(t)(\sigma = 1, \cdots, p)$ defined and continuous on the closure of T such that

$$H_1(e; x) = 0 \tag{6.5}$$

for every x in $\mathscr{B}$.

III$'_s$. The multipliers $a_\sigma(t)$ in I$_s$ can be chosen so that there is a positive constant ε such that the inequality

$$R^{ij}_{\gamma\delta}(t, e)\xi^\gamma\xi^\delta\zeta^i\zeta^j \geqq \varepsilon\,|\,\xi\,|^{2k}\,|\,\zeta\,|^2 \tag{6.6}$$

(summed on i, j, γ, δ for $i \leqq n, j \leqq n, |\gamma| = |\delta| = k$) holds on T for all sets $(\xi, \zeta) \neq (0, 0)$ which satisfy $\phi^\sigma_{z^i_\gamma}(t, e)\xi^\gamma\zeta^i = 0\,(\sigma = 1, \cdots, p)$. The coefficients $R^{ij}_{\gamma\delta}$ are given by

$$R^{ij}_{\gamma\delta}(t, z) = f_{z^i_\gamma z^j_\delta}(t, z) + a_\sigma(t)\phi^\sigma_{z^i_\gamma z^j_\delta}(t, z) + \int_T [g_{u^i_\gamma u^j_\delta}(t, s, z) + g_{v^i_\gamma v^j_\delta}(s, t, z)]ds \,.$$

IV$'_s$. The multipliers $a_\sigma(t)$ in III$'_s$ can be chosen so that the inequality

$$H_2(e; x) > 0 \tag{6.7}$$

holds for every nonnull variation for which $\phi^\sigma_{z^i_\alpha}(t, e)x^i_\alpha(t) = 0$ $(\sigma = 1, \cdots, p)$ (summed on i, α for $i \leqq n$, $|\alpha| \leqq k$).

For this application take for the integrands in (3.3): $F = f + a_\sigma\phi^\sigma + b\phi^\sigma\phi^\sigma$ (summed on σ) with b an as yet undetermined real number, and $G = g$. Then $J(z) = H(z) + bP(z)$, and for the first and second variations of J we have $J_1(z; x) = H_1(z; x) + bP_1(z; x)$ and $J_2(z; x) = H_2(z; x) + bP_2(z; x)$.

Condition I_s implies the Condition I directly, using the fact that $\phi^\sigma(t, e) = 0$ in (6.4), so that $J_1(e; x) = H_1(e; x)$.

LEMMA 6.1. *If* III'_s *holds, then there is a positive value of* b *and a positive number* ε *such that for the arguments* $[t, e(t)]$, *the inequality*

$$(R^{ij}_{\gamma\delta} + b\phi^\sigma_{z^i_\gamma}\phi^\sigma_{z^j_\delta})\xi^\gamma\xi^\delta\zeta^i\zeta^j \geqq \varepsilon\,|\,\xi\,|^{2k}\,|\,\zeta\,|^2 \tag{6.8}$$

(*summed on* $i, j, \sigma, \gamma, \delta$ *for* $i \leqq n, j \leqq n, \sigma \leqq p, |\gamma| = |\delta| = k$) *holds for every set* (ξ, ζ). *Hence* III'_s *implies* III' *for this choice of* b.

Proof. Let $R = R^{ij}_{\gamma\delta}(t)\xi^\gamma\xi^\delta\zeta^i\zeta^j$ and $\phi = \phi^\sigma_{z^i_\gamma}(t)\phi^\sigma_{z^j_\delta}(t)\xi^\gamma\xi^\delta\zeta^i\zeta^j$, and without loss of generality consider the special case $|\xi| = |\zeta| = 1$. Then by III'_s, $R \geqq \varepsilon > 0$ whenever $\phi = 0$ and $(\xi, \zeta) \neq (0, 0)$; this means that both $R \leqq 0$ and $\phi = 0$ only if $(\xi, \zeta) = (0, 0)$. Now suppose no positive value of b exists such that III' holds. Then there is a sequence $\{t_q, \xi_q, \zeta_q\}$ with $|\xi_q| = |\zeta_q| = 1$, convergent to (t_0, ξ_0, ζ_0), such that in the limit (because of the continuity of R and ϕ in t, ξ, and ζ) $R \leqq 0$, $\phi = 0$, and $|\xi_0| = |\zeta_0| = 1$.

LEMMA 6.2. *If Conditions* III'_s *and* IV'_s *hold, then there is a positive value of* b *such that* $H_2(e; x) + bP_2(e; x)$ *is a positive definite form on* $\mathscr{B}$; *hence Condition* IV' *holds.*

Proof. Choose $b_0 > 0$ so that III' holds, as was shown possible in in Lemma 6.1; then $H_2(e; x) + b_0P_2(e; x)$ is a Legendre form on $\mathscr{B}$ (Lemma 4.1). Since $\phi^\sigma[t, e(t)] = 0$ $(\sigma = 1, \cdots, p)$, we see that $J_2(e; x) = H_2(e; x) + b_0K(x)$ where $K(x) = 2\int_T \phi^\sigma_{z^i_\alpha}\phi^\sigma_{z^j_\beta}x^i_\alpha x^j_\beta dt$. On $\mathscr{B}$, $K(x) \geqq 0$ and IV'_s ensures that $H_2(x) > 0$ whenever $K(x) = 0$ and $x \neq 0$. Hence by the corollary of § 2 there is a positive value of $b \geqq b_0$ such that $H_2(e; x) + bK(x)$ is positive definite, which implies IV'.

It has been shown that Conditions I_s, III'_s, and IV'_s imply Conditions I, III', and IV' of § 4. Consequently Theorem 4.1 is applicable and has the following theorem as its interpretation in the case of a finite number of partial differential equations as side conditions:

THEOREM 6.1. *If the function e as prescribed in the formulation of* § 4 *satisfies Conditions* I_s, III'_s, *and* IV'_s *of this section, then there is a (weak) neighborhood N of e and a positive constant* ρ *such that for every z of class* $\mathscr{A}$ *in N*

$$H(z) - H(e) + cP(z) \geqq \rho \,||\, z - e \,||^2 . \tag{6.9}$$

H(z) and P(z) are given by (6.2) *and* (6.3), *respectively.*

COROLLARY 6.1. *Under the hypotheses of Theorem* 6.1, *there is a (weak) neighborhood N of e and a positive constant* ρ *such that for every z of class* $\mathscr{A}$ *in N the inequality*

$$H(z) - H(e) \geqq \rho \,||\, z - e \,||^2 \tag{6.10}$$

holds whenever z also satisfies the conditions

$$\phi^\sigma[t, z(t)] = 0 \qquad (\sigma = 1, \cdots, p) . \tag{6.11}$$

7. A characterization of extremals. Let $J(z)$ be given by (3.3) and recall the assumptions and definitions of § 4.

THEOREM 7.1. *If the function e is an extremal of J(z) which satisfies the strengthened condition of Legendre, then there exists a finite set of functions* $F^\sigma(t, z)$ $(\sigma = 1, \cdots, p)$ *having the same continuity properties on* $\mathscr{R}$ *as* $F(t, z)$, *a (weak) neighborhood N of e, and a positive constant* ρ *such that for every z of class* $\mathscr{A}$ *in N the inequality* $J(z) - J(e) \geqq \rho \,||\, z - e \,||^2$ *holds whenever z further satisfies the p isoperimetric side conditions* $J^\sigma(z) = \int_T F^\sigma(t, z)dt = 0$. *Moreover, the number p can be taken equal to the isoperimetric index of* $J_2(e; x)$ *on T.*

Proof. We suppose given a function e satisfying the hypotheses. For the choice of the side conditions, recall from § 2 that there exist subspaces $\mathscr{B}_+$, $\mathscr{B}_0$, $\mathscr{B}_-$ of $\mathscr{B}$ that are mutually orthogonal. J_2 is positive on $\mathscr{B}_+$, negative on $\mathscr{B}_-$, zero on $\mathscr{B}_0$ and $\mathscr{B} = \mathscr{B}_- + \mathscr{B}_0 + \mathscr{B}_+$. J_2 is a Legendre form on $\mathscr{B}$ (Lemma 4.1), and so has a finite isoperimetric index. Therefore, let $y^\sigma (\sigma = 1, \cdots, p)$ be a basis for $\mathscr{B}_- + \mathscr{B}_0$, and set $F^\sigma(t, z) = y^{\sigma i}_\alpha (z^i_\alpha - e^i_\alpha)$ (summed on $i \leqq n$ and α, $|\,\alpha\,| \leqq k$). *Condition* I_s *is satisfied* by the choices $a_\sigma = 0$ $(\sigma = 1, \cdots, p)$, since $J_1(e; x) = 0$ by hypothesis; and the strengthened condition of Legendre ensures that III'_s *is satisfied.* Condition IV'_s reduces to considering $J_2(e; x)$. Observing that $J_1^\sigma(e; x) = (y^\sigma, x)$, $J_2(e; x)$ is positive for every x such that $J_1(e; x) = 0$; therefore IV_s' *is also satisfied.* The conclusion follows from Corollary 5.1.

The converse part of the characterization depends on proof of the necessity of the Condition I_s of § 5. This may be assumed as known from Mathis [13]. Other forms of isoperimetric conditions lead to the same result (cf. [1]).

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