FUNCTIONS WITH CONVEX MEANS

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1. Introduction. Let f be Lebesgue summable on $[0, a]$, $a > 0$. The function F_1 defined on $[0, a]$ by $F_1(x) = 1/x\int_0^x f(t)dt$, $F(0) = f(0)$ is called the mean of the function f. Inductively, we may define the Nth mean of f on $[0, a]$ by $F_N(x) = 1/x\int_0^x F_{N-1}(t)dt$, $F_N(0) = f(0)$, provided, of course, that F_{N-1} is summable on $[0, a]$. Some questions involving the mean of certain classes of functions have been examined in Beckenbach [1] and Bruckner and Ostrow [2].

The primary purpose of this paper is to consider the problem of determining when a function f has its Nth mean convex for N sufficiently large. To develop the necessary machinery, we devote § 2 to obtaining some properties of the means which we shall need in the sequel, and we devote § 3 to obtaining representations for functions f possessing means of all orders. In particular, Theorem 3 shows how a wide class of functions f admit representations as sums of infinite series whose Nth term involves the Nth mean of f. Then in § 4 we examine the question posed at the beginning of this paragraph. (Lemma 6 together with Theorem 5 yields a condition which is necessary and sufficient for a sufficiently well behaved function to have its Nth mean convex. We then use this theorem to obtain two sufficient conditions for a starshaped function to have its Nth mean convex for N sufficiently large. This is done in Theorems 6 and 7. Finally, in Theorem 8, we use the Baire Category Theorem to show that not every continuous starshaped function has one of its means convex.

2. Preliminaries. We begin with some remarks and simple observations. In the sequel, we shall denote the Nth mean of f by $F_N(f : x)$. As circumstances warrant, this notation will be shortened to $F_N(f)$, $F_N(x)$, or simply F_N.

The fact that f is summable does not insure that means of all orders exist. Thus, the mean of the function f given by $f(x) = x^{-1}(\log x)^{-2}, f(0) = 0$ is $F_1(x) = -x^{-1}(\log x)^{-1}$, $F_1(0) = 0$, but F_1 is not summable on any neighborhood of the origin.

DEFINITION. The function f is in the class $M(a)$ provided f possesses means of all orders on $[0, a]$.

It is easily seen from Lemma 1 below that $f \in M(a)$ if the functions

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$f(t)(\log t)^N$ are summable for all $N = 1, 2, 3, \cdots$.

LEMMA 1. *Let $f \in M(a)$. Then the following representations for $F_N(f:x)$ are valid for $x > 0$ and $N = 1, 2, 3, \cdots$:*

$$F_N(f:x) = \frac{1}{(N-1)!\,x}\int_0^x f(u)\left[\log\left(\frac{x}{u}\right)\right]^{N-1} du \tag{1}$$

$$F_N(f;x) = \frac{(-1)^{N-1}}{(N-1)!}\int_0^1 f(xu)(\log u)^{N-1} du\ . \tag{2}$$

Proof. Equation (1) is obtained by interchanging the order of integration on the N times iterated integral. Equation (2) is then obtained from (1) by the change of variable $v = xu$.

Now write $k_N(u) = [(-1)^{N-1}/(N-1)!](\log u)^{N-1}$, $N = 1, 2, 3, \cdots$. Then equation (2) becomes

$$F_N(f:x) = \int_0^1 f(xu)k_N(u)du\ . \tag{3}$$

The following lemma shows that k_N is an approximation to the Dirac δ function.

LEMMA 2. *The kernel $k_N(u)$ has the following properties:*

(i) $k_N(u) \geqq 0$ $\qquad N = 1, 2, 3, \cdots$

(ii) $\int_0^1 k_N(u)du = 1$ $\qquad N = 1, 2, 3, \cdots$

(iii) $\lim_{N\to\infty} k_N(u) = 0$. *The convergence is uniform on every interval* $[\varepsilon, 1]$, $\varepsilon > 0$.

(iv) $k_N(u)$ *is nonincreasing on* $(0, 1)$ *for each* N.

Proof. Parts (i), (iii) and (iv) follow from inspection of the kernel k_N, and part (ii) can be obtained by setting $f \equiv 1$ in (3).

THEOREM 1. *Let $f \in M(a)$. If $f(0^+)$ exists and is finite, then the sequence $\{F_N(f:x)\}$ converges uniformly to $f(0^+)$ on $[0, a]$.*

Proof. Let $\varepsilon > 0$. Choose $\delta > 0$ so that $|f(x) - f(0^+)| < \varepsilon$ when $0 < x \leqq \delta$. Noting Lemma 2, we have

$$\begin{aligned}
|F_N(x) - f(0^+)| &\leqq \int_0^1 |f(xu) - f(0^+)|\, k_N(u)du \\
&\leqq \varepsilon\int_0^\delta k_N(u)du + k_N(\delta)\int_\delta^1 |f(xu) - f(0^+)|\, du \\
&\leqq \varepsilon + k_N(\delta)\int_\delta^1 |f(xu) - f(0^+)|\, du\ .
\end{aligned}$$

Since $\lim_{N\to\infty} k_N(\delta) = 0$ and the last integral is bounded (as a function of x), we infer that for N sufficiently large,

$$| F_N(x) - f(0^+) | < 2\varepsilon \quad \text{for all } x \in [0, a] .$$

As ε was arbitrary, the theorem follows.

It is easy to see from the definition of the means that if $f \in M(a)$, then $F_N(f)$ is differentiable on $(0, a]$ for $N > 1$ and $F_1(f)$ is differentiable a.e. on $(0, a]$. Theorem 2 considers the differentiability of F_N at the origin.

THEOREM 2. *Let $f \in M(a)$. If f is differentiable at the origin then so too is $F_N(f)$ for $N = 1, 2, 3, \cdots$, and $F'_N(f:0) = 2^{-N}f'(0)$. Furthermore, for $N > 1$, $F'_N(f)$ is continuous, and if $F_1(f)$ is differentiable in a neighborhood of the origin, then $F'_1(f)$ is continuous at the origin.*

Proof. We prove the theorem for $N = 1$. The general result follows by induction and by observing that for $N > 1$, $F_N(f)$ is differentiable on $(0, a]$.

We have

$$\begin{aligned}\frac{F_1(x) - F_1(0)}{x} &= \frac{F_1(x) - f(0)}{x} \\ &= \frac{1}{x^2}\int_0^x \Big[\frac{f(u) - f(0)}{u}\Big] u\,du \\ &= \frac{1}{x^2}\int_0^x [f'(0) + o(1)]u\,du \\ &= \frac{f'(0)}{2} + \frac{1}{x^2}\int_0^x o(u)du = \frac{f'(0)}{2} + o(1)\end{aligned}$$

as $x \to 0$. This proves that $2F'_1(0) = f'(0)$.

Now, if F_1 is differentiable on a neighborhood U of the origin, then $F'_1(x) = [f(x)/x] - [F_1(x)/x] = \{[f(x) - f(0)]/x\} - \{[F_1(x) - F_1(0)]/x\}$ on U. Since f and F_1 are assumed differentiable at the origin, $\lim_{x\to 0} F'_1(x) = f'(0) - F'_1(0) = F'_1(0)$.

3. Representation theorems. In this section we show how a function $f \in M(a)$ for which $f(0^+)$ exists can be represented as the sum of an infinite series involving its means.

Let $f \in M(a)$ and let x be a point at which f is the derivative of its integral. Then from the equation $F_1(x) = 1/x\int_0^x f(t)dt$ we obtain $f(x)/x = F'_1(x) + [F_1(x)/x]$. Inductively we obtain the finite representation

$$\frac{f(x)}{x} = \sum_{n=1}^{N} F_n'(x) + \frac{F_N(x)}{x}, \tag{4}$$

which is valid for any x at which f is the derivative of its integral. In particular, (4) is valid at every Lebesgue point of f.

THEOREM 3. *Let $f \in M(a)$. If $f(0^+)$ is finite then*

$$\frac{f(x) - f(0^+)}{x} = \sum_{n=1}^{\infty} F_n'(x)$$

a.e. on any interval $[\varepsilon, a]$, $0 < \varepsilon \leqq a$. In particular, the convergence is uniform on the set of points in $[\varepsilon, a]$ at which f is the derivative of its integral.

Proof. By noting (4) we see that it suffices to show that $\lim_{N\to\infty} [F_N(f:x) - f(0^+)]/x = 0$, uniformly on $[\varepsilon, a]$. Since $F_N(f:x) - f(0^+) = F_N(f - f(0^+):x)$ is the mean of a function which has zero for the limiting value at the origin, it tends uniformly to zero on $[\varepsilon, a]$ by Theorem 1.

LEMMA 3. *Let $f(0^+) = 0$. If $f(x)/x$ is summable on $[0, a]$, then*

$$\int_0^x \frac{F_N(f:u)}{u} du = F_N\left(\int_0^x \frac{f(u)}{u} du : x\right) \quad \text{for} \quad 0 < x \leqq a. \tag{5}$$

Proof. We have

$$\begin{aligned}\int_0^x \frac{F_N(f:u)}{u} du &= \int_0^x \frac{1}{v} \int_0^1 f(vu) k_N(u) du\, dv \\ &= \int_0^1 \left[\int_0^{xu} \frac{f(v)}{v} dv\right] k_N(u) du \\ &= F_N\left(\int_0^x \frac{f(u)}{u} du : x\right) \quad \text{for} \quad 0 < x \leqq a.\end{aligned}$$

The first and third equalities follow from (3) and the interchange of order of integration in the second equality is justified by the summability of $f(x)/x$.

THEOREM 4. *If $f(0^+)$ is finite and $[f(x) - f(0^+)]/x$ is summable on $[0, a]$, then*

$$\int_0^x \frac{f(u) - f(0^+)}{u} du = \sum_{N=1}^{\infty} (F_N(x) - F_N(0^+)).$$

The convergence is uniform on $[0, a]$.

Proof. Without loss of generality, assume $f(0^+) = 0$. Using (4) and Lemma 3, we have

$$\int_0^x \frac{f(u)}{u} du = \int_0^x \sum_{n=1}^N F_n'(f:u)du + \int_0^x \frac{F_N(f:u)}{u} du$$
$$= \sum_{n=1}^N F_n(f:x) + F_N\left(\int_0^x \frac{f(u)}{u} du : x\right) \qquad \text{for} \quad 0 \leqq x \leqq a .$$

We see from Theorem 1 that $\lim_{N\to\infty} F_N\left(\int_0^x [f(u)/u]du : x\right) = 0$, uniformly on $[0, a]$ and Theorem 4 follows.

4. Convexity of the means. We now turn to the problem of determining when a function f has convex means $F_N(f)$ for sufficiently large N. Theorem 5 together with Lemma 6 gives a necessary and sufficient condition for the Nth mean $F_N(f)$ to be convex. This condition is in terms of a kernel integral which is not practical to use for specific functions. We use this condition to obtain two sufficient conditions in Theorems 6 and 7. These theorems state that a starshaped function f (that is, one for which the function $[f(x) - f(0)]/x$ is increasing) has one of its means convex provided $f(x)$ behaves sufficiently well for small x.

To motivate these results, we first observe that it was shown in [2] that any function f for which $F_1(f)$ is convex, must be starshaped. It is easy to verify that the converse is not valid. However, if f is starshaped, then it follows immediately from Theorem 4 that $\sum_{n=1}^{\infty} F_n(f:x)$ is convex. Since the operation of taking a mean is a "smoothing" operation, this suggests that $F_N(f)$ is convex for N sufficiently large. This is false as Theorem 8 shows, but is true, as stated before, if f is sufficiently well behaved for small x. We note in passing that the condition of starshapedness cannot be entirely removed from the hypothesis of Theorems 6 and 7. Thus, the function which is equal to x on $[0, 1]$ and one on $[1, \infty)$ satisfies the hypotheses of these theorems (except for starshapedness), but all of its means are concave. To simplify some of the calculations, we shall henceforth assume that all functions vanish at the origin.

It was shown in [2] that the means of a convex function are convex, and the means of a continuous starshaped function are starshaped. Lemma 4 below does not require f to be continuous.

LEMMA 4. *If f is convex* (starshaped) *on* $[0, a]$, *then* $F_N(f)$ *is convex* (starshaped) *on* $[0, a]$, $N = 1, 2, 3, \cdots$.

Proof. A function g is convex on $[0, a]$ if and only if

$g(\alpha x + (1-\alpha)y) \leqq \alpha g(x) + (1-\alpha)g(y)$ for all α, $0 \leqq \alpha \leqq 1$ and all x and y in $[0, a]$. If f satisfies this inequality, then equation (3) shows $F_N(f)$ does also.

Now let f be starshaped on $[0, a]$. Then $f(x)/x$ is an increasing function of x, so that $\int_0^x [f(u)/u]du$ is convex. From Lemma 3, we have $\int_0^x [F_N(f:u)/u]du = F_N\left(\int_0^x [f(u)/u]du : x\right)$. The right side of this equality is the Nth mean of a convex function, hence convex. Thus the left side is also convex, so that $F_N(f:u)/u$ must be increasing, so that $F_N(f:u)$ is starshaped.

The following lemma was proved in [2].

LEMMA 5. *The function $F_N(f)$ is starshaped on $[0, a]$ if and only if $2F_N(x) \leqq F_{N-1}(x)$ on $[0, a]$. It is convex on $[0, a]$ if and only if $2F'_N(x) \leqq F'_{N-1}(x)$ on $[0, a]$.* (If $N = 1$, we must replace these derivatives by lower derivatives at points where the derivative fails to exist).

Since $F'_N(x) = [F_{N-1}(x) - F_N(x)]/x$, the condition for convexity can be written in the form

$$2[F_{N-1}(x) - F_N(x)] \leqq F_{N-2}(x) - F_{N-1}(x) , \qquad 0 \leqq x \leqq a$$

or

$$(5) \qquad Q_N(x) \equiv 2F_N(x) - 3F_{N-1}(x) + F_{N-2}(x) \geqq 0 , \qquad 0 \leqq x \leqq a .$$

LEMMA 6. *If $N \geqq 3$, then F_N is convex on $[0, a]$ if and only if for all $x \in [0, a]$*

$$(6) \qquad Q_N(x) = \int_0^1 f(xu)k_{N-2}(u)P_N(u)du \geqq 0 \qquad (N \geqq 3)$$

where

$$P_N(u) = \frac{2(\log u)^2}{(N-1)(N-2)} + \frac{3(\log u)}{N-2} + 1 .$$

Proof. Utilizing the integral representation (3) to express (5) yields (6).

We are especially interested in the case where f is starshaped on $[0, a]$ and f is convex on $[0, \varepsilon]$, $\varepsilon < a$. Thus we need the following theorem.

THEOREM 5. *If $f(x) = xg(x)$ on $[\varepsilon, a]$ where g is absolutely continuous, and if f and f' are absolutely continuous on $[0, \varepsilon]$, then for $N \geqq 3$*

$$Q_N(x) = x^2\int_0^\delta f''(xu)r_N(u)du + x^2\int_\delta^1 g'(xu)s_N(u)du \tag{7}$$

where δ is any number such that $0 \leqq \delta A \leqq \varepsilon$ and

$$r_N(u) = u^2k_N(u) - \delta uk_N(\delta)$$

and

$$s_N(u) = u^2k_{N-1}(u)\left[1 + \frac{\log u}{N-1}\right].$$

Proof. We will write equation (6) in two parts as $Q_N(x) = \int_0^\delta + \int_\delta^1 = I_1 + I_2$ where I_1 and I_2 are the integrals from $[0, \delta]$ and $[\delta, 1]$ respectively.

Two integrations by parts gives

$$I_1 = f(x\delta)\int_0^\delta k_{N-2}(u)P_N(u)du - xf'(x\delta)\int_0^\delta\int_0^u k_{N-2}(u)P_N(u)dtdu$$
$$+ x^2\int_0^\delta f''(xu)\left(\int_0^u\int_0^t k_{N-2}(\xi)P_N(\xi)d\xi dt\right)du .$$

Using the fact that the second derivative of $u^2k_N(u)$ is

$$(u^2k_N(u))'' = (2uk_N(u) - uk_{N-1}(u))' = k_{N-2}(u)P_N(u)$$

we have

$$I_1 = f(x\delta)\delta[2K_N(\delta) - k_{N-1}(\delta)] - x\delta^2 f'(x\delta)k_N(\delta)$$
$$+ x^2\int_0^\delta f''(xu)u^2k_N(u)du .$$

Using the fact that $\int_0^u tk_{N-2}(t)P_N(t)dt = u^2(k_N(u) - k_{N-1}(u))$ (in particular, this is zero when $u = 1$, $N \geqq 2$) integration by parts shows

$$I_2 = xg(xu)\int_0^u tk_{N-2}(t)P_N(t)dt\,\Big|_\delta^1 - x^2\int_\delta^1 g'(xu)u^2[k_N(u) - k_{N-1}(u)]du$$
$$= -f(x\delta)\delta[k_N(\delta) - k_{N-1}(\delta)] + x^2\int_\delta^1 g'(xu)u^2[k_{N-1}(u) - k_N(u)]du .$$

Thus

$$\begin{aligned} Q_N(x) &= I_1 + I_2 = [f(x\delta) - (x\delta)f'(x\delta)]\delta k_N(\delta) \\ &\quad + x^2\int_0^\delta f''(xu)u^2k_N(u)du + x^2\int_\delta^1 g'(xu)u^2[k_N(u) - k_{N-1}(u)]du . \end{aligned} \tag{8}$$

Since $f(0) = 0$ we can use the expression

$$-x^2\int_0^\delta f''(xu)udu = -\int_0^{x\delta} f''(u)udu = -f(0) + f(x\delta) - (x\delta)f'(x\delta)$$

to rewrite the first term on the right in equation (8); thus yielding equation (7).

In order to utilize Theorem 5 we need to know some of the properties of the kernel functions s_N and r_N. These are given in the following lemma.

LEMMA 7. *The kernel* s_N *has the properties:*

(i) $s_N(u) < 0 \qquad u \in (0, e^{-(N-1)})$

$s_N(u) > 0 \qquad u \in (e^{-(N-1)}, 1)$

(ii) $3^N \int_0^1 s_N(u)du = 2 \qquad N = 3, 4, \cdots$

(iii) $3^N \int_0^{e^{-(N-1)}} s_N(u)du \to 0 \qquad as\ N \to \infty$.

The kernel r_N *has the following properties on* $[0, \delta]$ *for sufficiently large N:*

(iv) *there is exactly one point,* δ_N, *such that* $0 < \delta_N < \delta$ *and* $r_N(\delta_N) = 0$. *We have* $\delta_N \to 0$ *as* $N \to \infty$.

(v) r_N *has exactly two inflection points on* $[0, \delta]$. *If they occur at* $u_{1,N}$ *and* $u_{2,N}$, $0 < u_{1,N} < u_{2,N} < \delta$ *then* $u_{2,N} \to 0$ *as* $N \to \infty$.

(vi) $r_N(u) \geqq r'_N(\delta)(u - \delta)$ *on* $[u_{2,N}, \delta]$.

(vii) *If* $\min_{[0,\delta_N]} r_N(u) = -\varepsilon_N$, *then* $\varepsilon_N = o(|r'_N(\delta)|)$ *as* $N \to \infty$.

Proof. (i) Is easily established by inspecting s_N.

(ii) The integral from 0 to 1 of s_N is just $F_{N-1}(x^2 : 1) - F_N(x^2 : 1) = 3^{-(N-1)} - 3^{-N} = 2 \cdot 3^{-N}$.

(iii) It is clear that

$$| s_N(u) | \leqq u^2 k_N(u)$$

and since $u^2 k_{N+1}(u)$ is monotone increasing on $[0, e^{-N}]$ we have $3^{N+1} | s_{N+1}(u) | \leqq 3^{N+1} e^{-2N} N^N / N!$. By Stirling's formula the integral in (iii) is $0((3e^{-2})^{N-1}) = o(1)$.

(iv) r_N is negative near zero and positive near δ. Thus, there is at least one zero of r_N between zero and δ. We will utilize property (v) to complete (iv). Since there are exactly two inflection points on $[0, \delta]$, and r_N is concave up near δ, there is at most one zero, δ_N, and this lies to the left of $U_{2,N}$. Thus $\delta_N \to 0$.

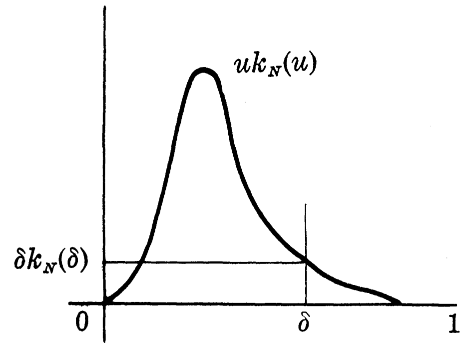
(v) $r''_N(u) = k_{N-2}(u) P_N(u)$ and $k_{N-2}(u) > 0$ on $(0, 1)$. P_N has exactly two zeros $u_{1,N}$ and $u_{2,N}$ on $(0, 1)$. If N is sufficiently large these lie on $[0, \delta]$. As $N \to \infty$, $u_{2,N} \to 0$.

(vi) r_N is concave upwards on $[u_{2,N}, \delta]$ and $r_N(\delta) = 0$.

(vii) Since $-\delta k_N(\delta) \leqq u k_N(u) - \delta k_N(\delta) \leqq 0$ on $[0, \delta_N]$, we have $-\delta_N \delta k_N(u) \leqq r_N(u) \leqq 0$ on that interval. Thus $\varepsilon_N \leqq \delta_N \delta k_N(\delta)$. Now

$$r'_N(\delta) = \delta(k_N(\delta) - k_{N-1}(\delta)) = -\delta k_{N-1}(\delta)[1 + (1/N - 1) \log \delta] ,$$

so that $|r'_N(\delta)| \geqq (\delta/2)k_{N-1}(\delta)$ when N is sufficiently large; thus $\varepsilon_N \leqq [2\delta_N \log(1/\delta)/(N-1)]\,|r'_N(\delta)|$ which completes the proof of the lemma.



THEOREM 6. *Let f satisfy the continuity and differentiability conditions of Theorem 5. If f is starshaped on $[0, a]$ and convex on $[0, \varepsilon]$ then F_N is convex on $[0, a]$ for sufficiently large N.*

Proof. We will show that $Q_N(x)$ is nonnegative on $[0, a]$ for sufficiently large N. Since $g'(xu) \geqq 0$ a.e. for $u \in [0, 1]$, $x \in [0, a]$, it follows from property (i) of Lemma 7 that the second integral on the right in (7) is nonnegative for sufficiently large N. We will show that there is an N such that

$$J_N(x) = \int_0^\delta f''(xu)r_N(u)du \geqq 0$$

for all $x \in [\varepsilon, a]$ by using the fact that $f''(xu) \geqq 0$ for $u \in [0, \delta]$ and $x \in [0, a]$. We need only consider the case where f'' vanishes on no neighborhood $[0, \varepsilon']$, since otherwise by providing that $\delta a < \varepsilon'$ we have $J_N \equiv 0$ and are done. If f'' vanishes on no neighborhood of the origin, using the notation of Lemma 7, we have

$$J_N(x) \geqq r'_N(\delta)\int_{u_{2,N}}^\delta f''(xu)(u-\delta)du - \varepsilon_N\int_0^{\delta_N} f''(xu)du$$

and since $r'_N(\delta) < 0$,

$$(9) \qquad \frac{J_N(x)}{|r'_N(\delta)|} \geqq \int_{u_{2,N}}^\delta f''(xu)(\delta-u)du - \frac{\varepsilon_N}{x\,|r'_N(\delta)|}\int_0^{x\delta_N} f''(u)du\,.$$

since $\int_0^{x\delta_N} f''(u)du \leqq \int_0^{a\delta_N} f''(u)du \to 0$ as $N \to \infty$ (uniformly in x on $[0, a]$), property (vii) of Lemma 7 shows that the last term on the right in (9) is $o(1)$ (uniformly in x on $[\varepsilon, a]$). But, the continuous functions

$$K_N(x) = \int_{u_{2,N}}^{\delta} f''(xu)(\delta - u)du \to \int_0^{\delta} f''(xu)(\delta - u)du$$

uniformly on $[\varepsilon, a]$ as $N \to \infty$, and since f'' vanishes on no neighborhood of the origin the continuous function

$$K(x) = \int_0^{\delta} f''(xu)(\delta - u)du$$

is bounded away from zero on the compact interval $[\varepsilon, a]$. It follows that past some sufficiently large N the functions $K_N(x)$ are uniformly bounded away from zero and thus the $J_N(x)$ are strictly positive on $[\varepsilon, a]$.

This shows $Q_N(x) > 0$ on $[\varepsilon, a]$ for N sufficiently large. Since $Q_N(x) \geqq 0$ on $[0, \varepsilon]$ as a result of the convexity of f on $[0, \varepsilon]$, it follows that $Q_N(x) \geqq 0$ on $[0, a]$ for sufficiently large N.

COROLLARY. *Let f be a function defined on $[0, a]$. If there is a positive integer k such that $F_k(f)$ satisfies the conditions of Theorem 6, then $F_N(f)$ is convex on $[0, a]$ for sufficiently large N. In particular, if f is superadditive on $[0, a]$ and satisfies the continuity and differentiability conditions of Theorem 5, then the conclusion follows.*

Proof. The first statement is obvious. The second follows by noting that if f is superadditive, then $F_1(f)$ is starshaped [2].

We are now ready to prove a theorem which shows that for a wide class of starshaped functions, the means are eventually convex.

THEOREM 7. *Let f be starshaped on $[0, a], f(0) = 0$. Let $g(x) = f(x)/x, g(0) = f'(0)$. If g is absolutely continuous on $[0, a]$ and if there exists a nondecreasing function p and a positive constant α such that $\alpha p(x) \leqq g'(x) \leqq p(x)$ for all x in $[0, \varepsilon]$, then $F_N(f)$ is convex on $[0, a]$ for sufficiently large N.*

Proof. We take $\delta = 0$ in equation (7). Then

$$\begin{aligned} 3^{N+1}Q_{N+1}(x) &= x^2\int_0^1 g'(xu)s_{N+1}(u)du \\ &\geqq x^2\left[\alpha p(xe^{-N})\int_{e^{-N}}^1 s_{N+1}(u)du - p(xe^{-N})\int_0^{e^{-N}} s_{N+1}(u)du\right] \\ &= x^2 p(xe^{-N})[2\alpha + o(1)] \end{aligned}$$

as $N \to \infty$ by properties (ii) and (iii) of Lemma 7. Thus $Q_N(x) \geqq 0$ on $[0, \varepsilon]$ for sufficiently large N. By Theorem 6 $F_N(f)$ is convex on $[0, a]$ for sufficiently large N.

We note that if the function g of Theorem 7 is differentiable, then g must also be absolutely continuous. This follows from the fact that if f is starshaped, g is increasing, and any differentiable monotonic function must be absolutely continuous.

COROLLARY. *Let f be starshaped on $[0, a], f(0) = 0$, and let g be defined as in Theorem 6. If g is absolutely continuous and*

either (1) *f is a polynomial*
or (2) $g'(0) > 0$

then $F_N(f)$ is convex for sufficiently large N.

The conditions of Theorem 7 are not necessary for $F_N(f)$ to be convex for sufficiently large N.

EXAMPLE. Let $f(x) = x^2 + x\int_0^x \sin(1/u)du, f(0) = 0$. Then $g(x) = x + \int_0^x \sin(1/u)du$ and $g'(x) = 1 + \sin(1/x)$. It is clear that there exist no p and α satisfying the conditions of Theorem 7. However, equation (7) (with $\delta = 0$) gives

$$\begin{aligned} 3^N Q_N(x) &= x^2\left[\int_0^1 s_N(u)du + \int_0^1 \sin\left(\frac{1}{x}\right)s_N(u)du\right] \\ &= x^2\left[2 + \int_1^\infty \frac{s_N(1/v)}{v^2} \sin\left(\frac{v}{x}\right)dv\right]. \end{aligned}$$

From the Riemann-Lebesgue lemma it is easy to see that for fixed N, there is a $\varepsilon > 0$ such that when $0 < x < \varepsilon$ the last integral above is less than 2 in absolute value and thus $Q_N(x) \geqq 0$ on $[0, \varepsilon]$. It follows then from Theorem 6 that for any $a > 0$ there is an N for which $F_N(f)$ is convex on $[0, a]$.

We now show that not every starshaped continuous function which vanishes at the origin must have its means $F_N(f)$ be convex for N sufficiently large. We first prove a lemma.

LEMMA 8. *Let α and δ be positive numbers and let $c > 2\delta$. Define a function $f(\alpha, \delta : x)$ by*

$$f(\alpha, \delta : x) = \begin{cases} 0 & 0 \leqq x \leqq \delta \\ 2\alpha(x - \delta) & \delta < x \leqq 2\delta \\ \alpha x & 2\delta < x \leqq c\,. \end{cases}$$

For every positive integer N and every $\alpha > 0$, there exists $\delta > 0$ such that $F_N(f(\alpha, \delta : x))$ is not convex on $[0, c]$.

Proof. Let $g(\alpha, \delta : x) = f(\alpha, \delta : x)/x$. The function $g(\alpha, \delta : x)$ is absolutely continuous on $[0, c]$ and differentiable except at $x = \delta$ and $x = 2\delta$. We have

$$g'(\alpha, \delta : x) = \begin{cases} 0 & 0 \leqq x < \delta \\ 2\alpha\delta/x^2 & \delta < x < 2\delta \\ 0 & 2\delta < x \leqq c\,. \end{cases}$$

Now choose δ so that $2\delta c < e^{-N}$. Since the function s_{N+1} is negative on $(0, e^{-N})$, we have, $\int_0^1 g'(\alpha, \delta : cu)s_{N+1}(u)du < 0$. It now follows from Theorem 5, that $F_n(f(\alpha, \delta : x))$ is not convex on $[0, c]$.

THEOREM 8. *There is a continuous starshaped function defined on* $[0, a]$ *such that none of its means is convex on* $[0, a]$.

Proof. Let S be the class of functions which are continuous, nonnegative, and starshaped on $[0, a]$ and vanish at the origin. This class may be considered as a subspace of $C[0, a]$. Since S is closed in $C[0, a]$, S is a complete metric space. For each positive integer N, let S_N denote that class of functions in S whose Nth means are convex. It suffices to show that each S_N is closed and has a dense complement in S. For then each S_N is nowhere dense in S and Theorem 8 follows from the Baire Category Theorem.

Let $\{f_k\}$ be a sequence of functions in S_N converging uniformly to a function f. It is clear that $f \in S$, and it follows from [2; Theorem 2] that $F_N(f_k) \to F_N(f)$. Since $F_N(f_k)$ is convex on $[0, a]$ by hypothesis, and the limit of a sequence of convex functions is convex, the function $F_N(f)$ is also convex on $[0, a]$. Thus $f \in S_N$ and S_N is closed.

Now let $\bar{f} \in S$ and let $\varepsilon > 0$. Choose $c < a$ so that $[\bar{f}(c)/c]x < \varepsilon$ for all x, $0 \leqq x \leqq c$. Let $\alpha = \bar{f}(c)/c$. Now choose δ so that the function $f(\alpha, \delta : x)$ satisfies the conditions of Lemma 8. Define a function f^* by

$$f^*(x) = \begin{cases} f(\alpha, \delta : x) & \text{if } \ 0 \leqq x \leqq c \\ \bar{f}(x) & \text{if } \ c \leqq x \leqq a\,. \end{cases}$$

It is clear that $\sup_{0 \leqq t \leqq a} |f^*(x) - \bar{f}(x)| < \varepsilon$ and it follows from Lemma 8 that $F_N(f^*)$ is not convex on $[0, c]$, and *a fortiori*, f^* is not in S_N. Thus the complement of S_N is dense.

In view of Theorem 6, Theorem 8 can be restated as follows:

COROLLARY. *There exist a continuous, nonnegative, starshaped function* f *with the property that no mean* $F_N(f)$ *is convex on any interval* $[0, \varepsilon]$.

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