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# THE ASYMPTOTIC NATURE OF THE SOLUTIONS OF CERTAIN LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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**Suppose $y'(t) = [A + V(t) + R(t)]y(t)$ is a system of differential equations defined on $[0, \infty)$, where $A$ is a constant matrix, $V(t) \to 0$ as $t \to \infty$ and the norms of the matrices $V'(t)$ and $R(t)$ are summable. If the roots of the characteristic polynomial of $A$ are simple, then under suitable conditions on the real parts of the roots of the characteristic polynomials of $A + V(t)$ a theorem of N. Levinson gives an asymptotic estimate of the behavior of the solutions of the differential system as $t \to \infty$. In this paper Levinson's theorem is improved by removing the condition that the characteristic roots of $A$ are simple. Under suitable conditions on $V(t)$ and $R(t)$ and the characteristic roots of $A + V(t)$, which reduce to Levinson's conditions when the characteristic roots of $A$ are simple, asymptotic estimates are obtained for the solutions of the given system.**

The proof given here, with essential modifications, will follow the proof given by Levinson [3] [2, p. 92]. One interest in the improved theorem is in its application to the problem of finding the deficiency index of an ordinary self-adjoint differential operator, which will appear in a subsequent paper. We shall establish the following.

THEOREM.[1] *Let $A$ be a constant $n \times n$ matrix whose minimal polynomial is of degree $n$ and is of the form*

$$\chi(\lambda) = \prod_{k=1}^{m} (\lambda - \lambda_k)^{n_k},\ \lambda_j \neq \lambda_k \ \textit{for}\ j \neq k,\ \sum_{k=1}^{m} n_k = n\ .$$

*Let $q + 1 = \max n_k$, $V(t)$ an $n \times n$ matrix with $(q + 1)$-times continuously differentiable elements satisfying $t^{2q} | v_{ij}^{(r)}(t) |^{1/r} \in L^1$ for $1 \leqq r \leqq q + 1$ and $V(t) \to 0$ as $t \to \infty$. Let the roots of $\det(A + V(t) - \lambda I) = 0$ be $\{\lambda_k(t)\}_1^m$ and for $t \geqq \tau_0$ we suppose the minimal polynomial of $A + V(t)$ is*

$$\chi(\lambda, t) = \prod_{k=1}^{m} (\lambda - \lambda_k(t))^{n_k}\ ,$$

*where $\lambda_k(t) \to \lambda_k$ as $t \to \infty$. For a given $k$, let*

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[1] If $A$ is an $n \times n$ matrix with entries $a_{ij}$ we shall write $| A | = \sum_{ij} | a_{ij} |$. If $x$ is a vector with entries $x_i$ we shall write $| x | = \sum_i | x_i |$.

$$d_{kj}(t) = Re(\lambda_k(t) - \lambda_j(t)) ,$$

*and suppose that all* $j, 1 \leqq j \leqq n$, *fall into one of two classes* $I_1$ *and* $I_2$, *where* $j \in I_1$, *if and only if* $t^{-q} \exp \int_0^t d_{kj} \to \infty$ *as* $t \to \infty$ *and*

$$(|t - \tau|^q + 1) \exp - \int_\tau^t d_{kj} < M < \infty \quad \textit{for} \quad t \geqq \tau \geqq 0 ,$$

$j \in I_2$ *if and only if* $\int_\tau^t d_{kj} < \log M$ *for* $t \geqq \tau \geqq 0$. *Let* $R(t)$ *be a matrix valued function with measurable elements such that* $t^{2q} | R(t) | \in L^1$. *Let* $\{q_{kj}; 1 \leqq j \leqq n_k\}$ *be a set of "principal vectors" for* $\lambda_k$; *i.e.,* $q_{kj} = (A - \lambda_k I)^{n_k - j} g_{kn_k}$, $(A - \lambda_k I)^{n_k - 1} g_{kn_k} \neq 0$ *and* $(A - \lambda_k I)^{n_k} g_{kn_k} = 0$. *Then, given the differential equation*

$$y'(t) = [A + V(t) + R(t)]y(t) \tag{1.1}$$

*there exists a* $t_0$ *and a fundamental system of solutions* $\{y_{kj}(t); 1 \leqq j \leqq n_k, 1 \leqq k \leqq m\}$ *such that*

$$\left[\frac{t^{j-1}}{(j-1)!} \exp \int_{t0}^t \lambda_k(\tau) d\tau\right]^{-1} y_{kj}(t) - q_{kj} \to 0, t \to \infty .$$

2. We begin the proof by first considering a differential system of the form

$$y'(t) = (A(t) + R(t))y(t) , \tag{2.1}$$

where $A(t)$ is a matrix with blocks $\{J_j(t)\}_1^m$ down the main diagonal and zeros elsewhere, $J_j(t)$ being an $n_j \times n_j$ matrix with the same number $\lambda_j(t)$ down the main diagonal, 1 down the superdiagonal and zeros elsewhere, and $R(t)$ has measurable entries with $t^{2q} | R(t) | \in L^1$, where $q + 1 = \max \{n_j, 1 \leqq j \leqq m\}$.

One fundamental matrix $\Psi$ for the system

$$y'(t) = A(t)y(t) \tag{2.2}$$

has blocks $\{P_j\}_1^m$ down the main diagonal and zeros elsewhere, where $P_j$ is an $n_j \times n_j$ matrix of the form

$$P_j(t) = \exp \int_{t_0}^t \lambda_j \begin{bmatrix} 1 & t & t^2/2! \cdots t^{n_j - 1}/(n_j - 1)! \\ 0 & 1 & t \quad \cdots t^{n_j - 2}/(n_j - 2)! \\ \vdots & \cdot & \cdot \\ \vdots & & \quad \cdot \\ 0 & \cdot \quad \cdot & \cdot \quad 0 \quad 1 \end{bmatrix} . \tag{2.3}$$

This may be checked by a direct computation. Again, it may be easily checked that

$$P_j^{-1}(t) = \exp - \int_{t_0}^{t} \lambda_j \begin{bmatrix} 1 & -t & t^2/2! & -t^3/3! \cdots (-1)^{n_j-1}t^{n_j-1}/(n_j-1)! \\ 0 & 1 & -t & t^2/2! \cdots (-1)^{n_j-2}t^{n_j-2}/(n_j-2)! \\ \vdots & \cdot & \cdot & \\ \vdots & & & \cdot \\ 0 & \cdot & \cdot & \cdot \quad 0 \quad 1 \end{bmatrix}$$

and

(2.4)
$$P_j(t)P_j^{-1}(\tau) = \exp \int_{\tau}^{t} \lambda_j \begin{bmatrix} 1 & (t-\tau) & (t-\tau)^2/2! \cdots (t-\tau)^{n_j-1}/(n_j-1)! \\ 0 & 1 & (t-\tau) \quad \cdots (t-\tau)^{n_j-2}/(n_j-2)! \\ \vdots & \cdot & \cdot \\ \vdots & & \quad \cdot \\ 0 & \cdot & \cdot \quad \cdot \quad 0 \quad 1 \end{bmatrix} .$$

Let us fix $k$ and let $\Psi_1$ be that matrix with zeros everywhere except for diagonal blocks $\{P_j; j \in I_1\}$, where each such $P_j$ has the same position as in the matrix $\Psi$. Let $\Psi_2$ be the corresponding type matrix with diagonal blocks $\{P_j; j \in I_2\}$. Clearly $\Psi = \Psi_1 + \Psi_2$.

Let $e_i$ be the vector with $j$th component equal to $\delta_{ij}$, $\delta_{ij}$ being the Kronecker symbol. Now set $i = l + \sum_{j=1}^{k-1} n_j$, where $1 \leqq l \leqq n_k$, and consider the equation

$$\phi(t) = \Psi(t)e_i + \int_{t_0}^{t} \Psi_1(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau - \int_{t}^{\infty} \Psi_2(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau . \tag{2.5}$$

It may be checked by a straightforward computation that, at least formally, $\phi$ is a solution to (2.1). Hence, if it can be shown that a solution to (2.5) exists, where the integrands are in $L^1$, then this solution will also be a solution to (2.1).

We proceed by successive approximations. Choose $\phi^0 = 0$ and hence $\phi^1 = \Psi(t)e_i$. It follows that

$$|\phi^1 - \phi^0| \leqq \left[\exp \int_{t_0}^{t} Re\lambda_k\right] \sum_{j=0}^{l-1} t^j/j! \tag{2.6}$$

Now, the matrix $\Psi_1(t)\Psi^{-1}(\tau)$ has blocks along the main diagonal which are zero in those positions for which $j \in I_2$ and of the form (2.4) in those positions for which $j \in I_1$. Hence, using the hypothesis of the theorem of § 1, for $t_0 \leqq \tau \leqq t$ we have

(2.7)
$$|\Psi_1(t)\Psi^{-1}(\tau)R(\tau)| \leqq C[|t-\tau|^q + 1] \exp\left(-\int_{\tau}^{t} d_{kj}\right) \exp\left(\int_{\tau}^{t} Re\lambda_k\right) |R(\tau)$$
$$\leqq CM|R(\tau)| \exp \int_{\tau}^{t} Re\lambda_k ,$$

where $C$ is a suitable constant dependent only of $q$. In the same way, for $t \leqq \tau < \infty$,

$$(2.8) \qquad |\Psi_2(t)\Psi^{-1}(\tau)R(\tau)| \leqq CM[|t-\tau|^q + 1]|R(\tau)| \exp -\int_t^\tau Re\lambda_k .$$

Using the estimates (2.6), (2.7) and (2.8) we arrive at the estimate

$$(2.9) \qquad \begin{aligned} &|\phi^2 - \phi^1| \exp -\int_{t_0}^t Re\lambda_k \\ &\leqq CM\left\{\int_{t_0}^t |R(\tau)| \sum_{j=0}^{l-1} \tau^j/j!d\tau + \int_t^\infty |R(\tau)|[|t-\tau|^q + 1] \sum_{j=0}^{l-1} \tau^j/j!d\tau\right\} . \end{aligned}$$

Now using the fact that $\tau^{2q}|R(\tau)| \in L^1$ we can choose $t_0$ so large so that

$$(2.10) \qquad |\phi^2 - \phi^1| \exp -\int_{t_0}^t Re\lambda_k \leqq 1/2 \quad \text{for} \quad t \geqq t_0 .$$

Using (2.7), (2.8) and (2.10) and proceeding by induction we find that for $j \geqq 1$,

$$(2.11) \qquad \begin{aligned} &|\phi^{j+1} - \phi^j| \exp - \int_{t_0}^t Re\lambda_k \\ &\leqq (1/2)^{j-1}CM\left\{\int_{t_0}^t |R(\tau)|d\tau + \int_t^\infty [|t-\tau|^q + 1]|R(\tau)|d\tau\right\} \\ &\leqq (1/2)^j . \end{aligned}$$

This means that there exists a function $\phi$ so that on every compact subinterval of $[t_0, \infty)$, $\phi^j$ goes uniformly to $\phi$, and indeed, using (2.6),

$$(2.12) \qquad |\phi - \phi^j| \leqq (1/2)^{j-1} \exp \int_{t_0}^t Re\lambda_k, \; |\phi| \leqq C[t^q + 1] \exp \int_{t_0}^t Re\lambda_k .$$

The estimates (2.12) taken together with the estimate (2.8) shows that the integrands in (2.5) are in $L^1$ and that indeed $\phi$ is a solution of that equation.

We claim that

$$(2.13) \qquad [\phi(t) - \Psi(t)e_i] \exp - \int_{t_0}^t \lambda_k \to 0 \quad \text{as} \quad t \to \infty .$$

To show this, it is enough to show that

$$(2.14) \qquad \exp\left(-\int_{t_0}^t Re\lambda_k\right)\int_{t_0}^t \Psi_1(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau \to 0 \quad \text{as } t \to \infty, \text{ and}$$

$$(2.15) \qquad \exp\left(-\int_{t_0}^t Re\lambda_k\right)\int_t^\infty \Psi_2(t)\Psi^{-1}(\tau)R(\tau)\phi(\tau)d\tau \to 0 \quad \text{as } t \to \infty .$$

Using (2.12) and (2.8) we see that the norm of (2.15) is less than or equal to

$$C^2M\int_t^\infty [|t-\tau|^q+1][\tau^q+1]\,|R(\tau)|\,d\tau,$$

which goes to zero as $t\to\infty$. To prove (2.14) we use the fact that $t^{-q}\exp\int_{t_0}^t d_{kj}\to\infty$. Choose $t_1$ so that $CM\int_{t_1}^\infty |R(\tau)|\,|\phi(\tau)|\,d\tau<\varepsilon$. Then the norm of (2.14) is less than or equal to

$$\varepsilon+\exp\left(-\int_{t_0}^t Re\lambda_k\right)|\Psi_1(t)|\int_{t_0}^{t_1}|\Psi^{-1}(\tau)|\,|R(\tau)|\,|\phi(\tau)|\,d\tau\,.$$

Now,

$$\exp\left(-\int_{t_0}^t Re\lambda_k\right)|\Psi_1(t)|\leqq Ct^q\sum_{j\in I_1}\exp-\int_{t_0}^t d_{kj}\to 0\quad\text{as } t\to\infty\,.$$

Hence we see that (2.14) is valid.

The vector $\left[\exp-\int_{t_0}^t \lambda_k\right]\Psi(t)e_i$ has the entry $t^{l-j-1}/(l-j-1)!$ in the $i+j$ position, $0\leqq j\leqq l-1$, and zero elsewhere. Hence

$$\left\{\frac{t^{l-1}}{(l-1)!}\exp\int_{t_0}^t\lambda_k\right\}^{-1}\phi(t)-e_i\to 0\quad\text{as } t\to\infty\,. \tag{2.16}$$

Let us designate the solution we have obtained in the previous considerations by $\phi_i$. Then the set of solutions $\{\phi_i\}_1^n$ is a fundamental system for (2.1). Indeed, it is clear that the determinant of the matrix $\Phi$ with the vectors $\phi_i$ as columns is nonzero for $t$ sufficiently large.

3. In order to use the results of § 2 to prove the theorem of § 1 it will be necessary to establish the following.

LEMMA. *Suppose the matrix $A+V(t)$ satisfies the conditions of the theorem of* § 1. *Then for all sufficiently large $t$ there exists a differentiable and invertible matrix $P(t)$ such that $t^{2q}|P^{-1}(t)P'(t)|\in L^1$, $P(t)[A+V(t)]P^{-1}(t)$ is a Jordan canonical form, $P(t)\to P$ and $P^{-1}(t)\to P^{-1}$ as $t\to\infty$, where $PAP^{-1}$ is a corresponding Jordan canonical form for $A$, and the columns of $P^{-1}$ are a given set of principal vectors for $A$.*

*Proof.* Let $\lambda_1,\lambda_2,\cdots,\lambda_m$ be the distinct eigenvalues of $A$. Since the coefficients of the characteristic polynomial of $A+V(t)$ are continuous functions of $t$ in a neighborhood of $\infty$, using the hypothesis of the theorem, there exists a neighborhood of $\infty$ so that $A+V(t)$ has eigenvalues $\lambda_1(t),\cdots,\lambda_m(t)$ which are continuous for all $t$ in that neighborhood. In particular, this means that $\lambda_k(t)\to\lambda_k$ as $t\to\infty$.

In fact, for $t$ sufficiently large, each $\lambda_k(t)$ is $(q+1)$-times continuously differentiable. To see this, we consider the characteristic polynomial

$$F(\lambda, t) = \sum_{j=0}^{n} f_j(t)\lambda^{n-j} = (-1)^n \prod_{j=1}^{m} (\lambda - \lambda_j(t))^{n_j} , \tag{3.1}$$

where $f_j(t)$ is $(q + 1)$-times continuously differentiable. If we set $G_k(\lambda, t) = \partial^{n_k-1}F(\lambda, t)/\partial\lambda^{n_k-1}$, then $G_k(\lambda_k(\tau), \tau) = 0$, but $\partial G_k(\lambda_k(\tau), \tau)/\partial\lambda \neq 0$. Hence, the implicit function theorem tells us that there exists a neighborhood about $\tau$ and a $(q + 1)$-times continuously differentiable function $\mu_k$, defined in this neighborhood, so that $\mu_k(\tau) = \lambda_k(\tau)$ and $G_k(\mu_k(t), t) = 0$. Moreover, if any other continuous function satisfies the last two conditions, then this other function coincides with $\mu_k$ in some neighborhood of $\tau$. Hence $\lambda_k(t) = \mu_k(t)$ in some neighborhood of $\tau$, which proves our assertion.

Let $\{q_{kj}; 1 \leqq j \leqq n_k\}$ be a given set of principal vectors for $\lambda_k$ and let $Q$ be the matrix whose columns are $\{q_{11}, \cdots, q_{1n_1}, q_{21}, \cdots, q_{2n_2}, \cdots, q_{m1}, \cdots, q_{mn_m}\}$, in the given order. Then, since the minimal and characteristic polynomials of $A$ are of the same degree, $Q^{-1}AQ$ is in the Jordan canonical form (see e.g. [1], Ch. XVII). If $V_k$ is the subspace generated by $\{g_{kj}; 1 \leqq j \leqq n_k\}$, then $A$ is reduced by $V_k$. Hence, if we set

$$\pi_k(A) = \prod_{j \neq k} (A - \lambda_j)^{n_j} ,$$

then this matrix is reduced by $V_k$ and the restriction of $\pi_k(A)$ to $V_k$ has an inverse. Let us set $h_k = \pi_k^{-1}(A)q_{kn_k}$, where by $\pi_k^{-1}(A)$ we mean the inverse of the restriction of $\pi_k(A)$ to $V_k$.

Let us write the minimal polynomial, $\chi(\lambda, t)$, of $A + V(t)$ as

$$\chi(\lambda, t) = (\lambda - \lambda_k(t))^{n_k}\pi_k(\lambda, t) ,$$

where

$$\pi_k(\lambda, t) = \prod_{j \neq k} (\lambda - \lambda_j(t))^{n_j} .$$

Set $q_{kn_k}(t) = \pi_k(A + V(t), t)h_k$; then since $\pi_k(A + V(t), t) \to \pi_k(A)$ as $t \to \infty$, it follows that if we set

$$q_{kj}(t) = (A + V(t) - \lambda_k(t))^{n_k-j}q_{kn_k}(t)$$

the set $\{q_{kj}(t)\}_1^{n_k}$ forms a set of principal vectors for the eigenvalue $\lambda_k(t)$, provided $t$ is sufficiently large. Indeed for $t$ sufficiently large,

$$(A + V(t) - \lambda_k(t))^{n_k-1}q_{kn_k}(t) \neq 0 ,$$

but

$$(A + V(t) - \lambda_k(t))^{n_k}q_{kn_k}(t) = \chi(A + V(t), t)h_k = 0 .$$

If $Q(t)$ is the matrix whose columns are the vectors

$$\{q_{11}(t), \cdots, q_{1n_1}(t), q_{21}(t), \cdots, q_{2n_2}(t), \cdots, q_{m1}(t), \cdots, q_{mn_m}(t)\} ,$$

in the order given, then $Q^{-1}(t)[A + V(t)]Q(t)$ is in the Jordan canonical form ([1]).

Notice that the elements of $Q(t)$ are polynomial functions in $\{\lambda_k(t)\}_1^m$ and the elements of $A + V(t)$, and hence the elements of $Q^{-1}(t)$ are rational functions in these variables, where the denominator of each rational function is $\det Q(t)$. Hence, if we set $P(t) = [\det Q(t)]Q^{-1}(t)$, then the elements of $P(t)$ are polynomials in the previously mentioned variables and $P(t)[A + V(t)]P^{-1}(t)$ is in the Jordan canonical form. Further, from the assumptions of the lemma, and the manner of construction of $Q(t)$, it is clear that $Q(t) \to Q$, where $Q^{-1}AQ$ is in the Jordan canonical form. Hence $P(t) \to P$, where $PAP^{-1}$ is in the Jordan canonical form.

Since $P^{-1}(t) \to P^{-1}$, it is clear that $P^{-1}(t)$ is bounded in a neighborhood of infinity. Hence, if we can show that $t^{2q} | P'(t) | \in L^1$ we will have proved the lemma. The elements of $P'(t)$ are linear functions of $\{\lambda_k'(t)\}_1^m$ and $\{v_{ij}'(t)\}$ (the entries of $V'(t)$) with coefficients which are bounded in a neighborhood of infinity. Since, by hypothesis $t^{2q} | v_{ij}'(t) | \in L^1$, if we can show that $t^{2q} | \lambda_k'(t) | \in L^1$ we will be done.

Use (3.1) to obtain

$$\begin{aligned}\frac{\partial^{n_k} F(\lambda_k(t), t)}{\partial t^{n_k}} &= \sum_{j=1}^{n} f_j^{(n_k)}(t)\lambda_k^{n-j}(t) \\ &= (-1)^{n+n_k}\Big[n_k! \prod_{j\neq k} (\lambda_k(t) - \lambda_j(t))^{n_j}\Big][\lambda_k'(t)]^{n_k} .\end{aligned}$$

Since $\prod_{j\neq k} (\lambda_k(t) - \lambda_j(t))^{n_j}$ is uniformly bounded away from zero and $\lambda_k(t)$ is bounded, in a neighborhood of $\infty$, it follows that there exists a constant $N$ such that

$$(3.2) \qquad | \lambda_k'(t) | \leqq N\Big[\sum_1^n | f_j^{(n_k)}(t) |\Big]^{1/n_k} \leqq N \sum_1^n | f_j^{(n_k)}(t) |^{1/n_k} .$$

Each function $f_j$ is the sum of suitably signed products of elements of $A + V(t)$. A typical term in the sum representing $f_j$ is say $a_1(t) \cdots a_j(t)$, where $a_i(t)$ is an entry of $A + V(t)$. The $n_k$ derivative of this product is given by

$$\sum C_{i_1,\cdots,i_j} a_1^{(i_1)}(t) \cdots a_j^{(i_j)}(t) ,$$

where $C_{i_1,\cdots,i_j}$ are the constants which appear in the multinomial expansion of $(x_1 + \cdots + x_j)^{n_k}$ and the sum is taken over all $j$-tuples of nonnegative integers, $(i_1, \cdots, i_j)$, whose sum is $n_k$. Hence if

$$t^{2q} \mid a_1^{(i_1)} \cdots a_j^{(i_j)} \mid^{1/n_k} \in L^1 \tag{3.3}$$

it will follow that $t^{2q} \mid \lambda_k' \mid \in L^1$ and hence $t^{2q} \mid P'(t) \mid \in L$.

If $\sum_{r=1}^{j} i_r = n_k$, we may apply Holder's inequality to get,

$$\int_{t_0}^{\infty} t^{2q} \left| \prod_{r=1}^{j} a_r^{(i_r)} \right|^{1/n_k} \leqq \prod_{r=1}^{j} \left[ \int_{t_0}^{\infty} t^{2q} \mid a_r^{(i_r)} \mid^{1/i_r} \right]^{i_r/n_k} , \tag{3.4}$$

where we make the convention that if $i_r = 0$, then

$$\| a_r \|_\infty = \sup_{t \geqq t_0} \mid a_r(t) \mid = \left[ \int_{t_0}^{\infty} t^{2q} \mid a_r^{(i_r)} \mid^{1/i_r} \right]^{i_r/n_k} .$$

From the hypothesis of the lemma it follows from (3.4) that (3.3) is satisfied and hence lemma is proved.

4. Using the results of § 2 and § 3 it is now an easy matter to finish the proof of the theorem stated in § 1. Make the transformation $x(t) = P(t)y(t)$ in (1.1) and we get the equation

$$x' = [P(A + V)P^{-1} - P^{-1}P' + PRP^{-1}]x . \tag{4.1}$$

The matrix $P(A + V)P^{-1}$ is in the Jordan form of the matrix $A(t)$ of (2.1) and $t^{2q} \mid PRP^{-1} - P^{-1}P' \mid \in L^1$. Hence, we may apply the results of § 2 and for $i = l + \sum_{j=1}^{k-1} n_j$, $1 \leqq l \leqq n_k$, we find a solution $x_i$ such that

$$\left[ \frac{t^{l-1}}{(l-1)!} \exp \int_{t_0}^{t} \lambda_k \right]^{-1} x_i(t) - e_i \longrightarrow 0 \quad \text{as } t \longrightarrow \infty .$$

Hence, if $y_i(t) = P^{-1}(t)x_i$, we get

$$\left[ \frac{t^{l-1}}{(l-1)!} \exp \int_{t_0}^{t} \lambda_k \right]^{-1} y_i(t) - P^{-1}e_i \longrightarrow 0 \quad \text{as } t \longrightarrow \infty ,$$

where $P^{-1} = \lim_{t \to \infty} P^{-1}(t)$.

The vector $P^{-1}e_i$ is the $i$th column of $P^{-1}$ which by Lemma 3 can be taken to be the given principal vector $q_{kl}$. Since the vectors $\{q_{kl}; 1 \leqq l \leqq n_k, 1 \leqq k \leqq m\}$ are linearly independent, the vectors $\{y_i(t)\}_1^m$ form a fundamental set of solutions of (1.1). This completes the proof of the theorem.

*Note added in proof.* The theorem of this paper can be generalized in the following way. Using the same notation as in the theorem let $p$ be a real number satisfying the inequality $0 \leqq p \leqq q$. Suppose further that for each given $k$ all integers $j$, $1 \leqq j \leqq n$, fall into two classes $I_1$ and $I_2$ where $I_1$ is the same as in the hypothesis of the theorem but now $I_2$ is the collection of $j$ so that

$$(|t - \tau|^p + 1) \exp \int_\tau^t d_{kj} < M < \infty \quad \text{for} \quad t \geqq \tau \geqq 0\,.$$

Then under the hypothesis that $t^{2q-p}|v_{ij}^{(r)}(t)|^{1/r}$, $1 \leqq r \leqq q+1$, and $t^{2q-p}|R(t)|$ are summable, the conclusion of the theorem holds. The proof of the generalized theorem follows the proof given in the text *mutatis mutandis*.

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