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THE UNIFORMIZING FUNCTION FOR CERTAIN SIMPLY CONNECTED RIEMANN SURFACES

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This paper contains a definition of a class of simply connected Riemann surfaces, the determination of the type of a surface from this class, and a representation of the uniformizing function and its derivative as infinite products of quotients as well as quotients of infinite products.

Definition of the class of surfaces. Let $\{a_{2n-1}\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that for $n \geqq 1$,

$$0 < a_{2n-1} < b_{2n-1} < b_{2n}$$

and $b_{2n+1} < b_{2n}$. A surface F of the class to be discussed consists of sheets S_n, $n = 1, 2, 3, \cdots$, over the w-sphere, where for S_n a copy of the w-sphere,

(a) S_1 is slit along the real axis from a_1 to b_1.

(b) For $n \geqq 1$, S_{2n} is slit along the real axis from a_{2n-1} to b_{2n-1} and from b_{2n} to $+\infty$.

(c) For $n \geqq 1$, S_{2n+1} is slit along the real axis from a_{2n+1} to b_{2n+1} and from b_{2n} to $+\infty$.

(d) For $n \geqq 1$, S_n is joined to S_{n+1} along the slits to make the b_n coincide and to form first order branch points at the endpoints of the slits.

The uniformizing function. Because F is simply connected and noncompact, there exists a unique function g which maps F schlichtly and conformally onto $\{|z| < R \leqq \infty\}$, where for $f(z) = g^{-1}(z)$, $f(0) = 0 \in S_1$ and $f'(0) = 1$. Two surfaces of hyperbolic type are obtained by slitting each sheet of F along the uncut parts of the real axis, and an application of the reflection principle to the uniformizing function of one of these surfaces shows that $f(z)$ is real for real z. Let $f(\alpha_{2k-1}) = a_{2k-1}$, $f(-\beta_k) = b_k$, $f(\gamma_{2k}) = \infty \in S_{2k}$ and S_{2k+1}, $f(-\gamma_1) = \infty \in S_1$, and $f(\delta_k) = 0 \in S_k$. The image of F in the z-plane satisfies the following properties. The image of S_n is a region which is symmetric about the real axis. S_1 is mapped onto a domain containing the origin and bounded by a simple closed curve C_1 which intersects the real axis at $-\beta_1$ and α_1. For $n \geqq 2$, S_n is mapped onto an annular region about the origin and bounded by two simple closed curves C_{n-1} and C_n, which

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are images of cuts. For n odd, C_n intersects the real axis at $-\beta_n$ and α_n, while for n even, C_n intersects the real axis at $-\beta_n$ and γ_n. Furthermore, for $k \geqq 1$,

$$-\beta_{k+1} < -\beta_k < -\gamma_1 < 0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \delta_{2k+1} < \alpha_{2k+1} .$$

The approximating closed surfaces. Let F_n be the surface formed from the first $2n+2$ sheets of F with the slit in S_{2n+2} from b_{2n+2} to ∞ deleted, so that F_n is a compact, simply connected surface.

NOTATION. $\alpha_\varphi^* = 1 - z/\alpha_\varphi$, $\beta_\varphi^* = 1 + z/\beta_\varphi$,
$\gamma_\varphi^* = 1 - z/\gamma_\varphi$, $\delta_\varphi^* = 1 - z/\delta_\varphi$.

LEMMA 1. *Let R_n be the unique rational function which maps the z-sphere one-to-one onto the simply connected compact surface F_n with $R_n(0) = 0 \in S_1$, $R_n'(0) = 1$, and $R_n(\infty) = \infty \in S_{2n+2}$. Then*

$$R_n(z) = [z/(1 + z/\gamma_{1,n})]\left[\prod_{k=2}^{2n+2} \delta_{k,n}^*\right]\Big/\left[\prod_{k=1}^{n} (\gamma_{2k,n}^*)^2\right]$$

and

$$R_n'(z) = [1/(1 + z/\gamma_{1,n})^2]\left[\prod_{k=0}^{n} \alpha_{2k+1,n}^*\right]\left[\prod_{k=1}^{2n+1} \beta_{k,n}^*\right]\Big/\left[\prod_{k=1}^{n} (\gamma_{2k,n}^*)^3\right] .$$

Proof. The representations of R_n and R_n' must contain factors shown and can contain no more. The $\alpha_{2k+1,n}$, $-\beta_{k,n}$, $\gamma_{2k,n}$, and $\delta_{k,n}$, which are ordered in the same manner as the α_{2k+1}, $-\beta_k$, γ_{2k}, and δ_k, are images of a_{2k+1}, b_k, ∞, and 0, respectively, under R_n^{-1}.

LEMMA 2. *F is parabolic.*

Proof. Suppose that F is hyperbolic, and thus g maps F onto $\{|z| < R < \infty\}$. If D_n is the z-plane slit along the real axis from $-\beta_{2n+1,n}$ to $-\infty$, then $\zeta = \psi_n(z) = g[R_n(z)]$ defines a Schlicht mapping of D_n onto a simply connected region Δ_n of the ζ-plane bounded by C_{2n+2} and the segment $(-\beta_{2n+2}, -\beta_{2n+1})$. If $T_n(z) = z(1 - z/4\beta_{2n+1,n})^{-2}$, then $\zeta = \psi_n[T_n(z)]$ defines a properly normalized, Schlicht mapping of $\{|z| < 4\beta_{2n+1,n}\}$ onto Δ_n such that if the Koebe Distortion Theorem is applied to this map, then $\beta_{2n+1,n} \leqq d(0, C_{2n+2}) \leqq R < \infty$, where $d(0, C_{2n+2})$ is the distance from $\zeta = 0$ to the curve C_{2n+2}. Thus there exists a subsequence $\{\beta_{2n_j+1,n_j}\}$ such that $\beta_{2n_j+1,n_j} \rightarrow A \leqq R$ as $j \rightarrow \infty$, and ψ_{n_j} is a Schlicht mapping of D_{n_j} onto Δ_{n_j}. If D is the z-plane slit along the negative real axis from $-A$ to $-\infty$, then $\{\psi_{n_j}\}$ forms a family of functions which is normal in D, and hence there exists a subsequence $\{\psi_i\}$ such that as $i \rightarrow \infty$, $\psi_i(z) \rightarrow \psi(z)$ uniformly on any compact subset of D. Because $D_i \to D$ and $\psi_i(z) \to \psi(z)$ as $i \to \infty$, then $\Delta_i \to \{|z| < R\}$ and ψ maps D onto $\{|\zeta| < R\}$ in a one-to-one manner. ([1], p. 18). Then $R_i(z) = f[\psi_i(z)] \to f[\psi(z)] = H(z)$ uniformly on any compact subset of D as $i \to \infty$, where H is meromorphic in D, while $H(z) \not\equiv \infty$ because $R_i(0) = 0$. H maps D onto F.

Now let D^* be the z-plane slit along the real axis from $-A$ to $+\infty$. For i sufficiently large, $R_i(z)$ assumes no negative real values in any compact subset of D^*, and thus $\{R_i\}$ is a family of functions which is normal in D^*. Therefore, there exists a subsequence $\{R_m\}$ of $\{R_i\}$ such that as $m \to \infty$, $R_m(z) \to G(z)$ uniformly on any compact subset of D^*. H and G have a common domain of convergence, so that G is the analytic continuation of H. Then $w = G(z)$ defines a mapping of the z-plane punched at $z = A$ and ∞ one-to-one and conformally onto an open doubly connected Riemann surface F^* of which F is a subsurface obtained by inserting some slits in F^* over the real axis. This is impossible, as is clear from the definition of F. Hence $R = \infty$.

LEMMA 3. *$R_n(z) \to f(z)$ uniformly on any compact subset of the z-plane as $n \to \infty$.*

Proof. Because $\Delta_n \to \{|\zeta| < \infty\}$ as $n \to \infty$, it follows ([1], p. 18) that $z = R_n^{-1}[f(\zeta)] \to \zeta = g[R_n(z)]$ uniformly on any compact subset of the ζ-plane as $n \to \infty$. Also, $D_n \to \{|z| < \infty\}$ and $R_n(z) \to f(z)$ uniformly on any compact subset of the z-plane as $n \to \infty$.

LEMMA 4. *$\alpha_{2k-1,n} \to \alpha_{2k-1}$, $\beta_{k,n} \to \beta_k$, $\gamma_{2k,n} \to \gamma_{2k}$, and $\delta_{k,n} \to \delta_k$ as $n \to \infty$.*

Proof. This is a consequence of Hurwitz's Theorem.

LEMMA 5. *The infinite product*

$$\pi(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^* \delta_{2k+1}^*/(\gamma_{2k}^*)^2]$$

converges uniformly on any compact subset of the z-plane.

Proof. Since $\gamma_{2k} \to \infty$ and $\delta_k \to \infty$ as $k \to \infty$, then for any $R > 0$, there exists $n_0 = n_0(R)$ such that for $k \geqq n_0$, $\delta_k > R$ and $\gamma_{2k} > R$. Then consider

$$M_p(z) = \prod_{k=n_0}^{n_0+p} [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] .$$

M_p is holomorphic for $|z| \leqq R$ and $M_p(z) \neq 0$ for $|z| \leqq R$. A sufficient condition for the uniform convergence of $M_p(z)$ in $E = \{|z| \leqq R\}$ as $p \to \infty$ is the uniform convergence in E of

$$\sum_{k=n_0}^{n_0+p} \log [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \text{ as } p \to \infty ,$$

where each logarithm is the principal value. By the Cauchy criterion, this last sequence converges uniformly in E provided for $z \in E$ and for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that for $n > N(\varepsilon)$ and $p > 0$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \right| < \varepsilon .$$

Now since $\delta_{2k} < \delta_{2k+1}$ and since $\gamma_{2k} < \delta_{2k+2} < \delta_{2k+3}$, then for $m \geqq 1$ and $p > 0$,

$$0 < \sum_{k=n_0+n}^{n_0+n+p} [1/(\delta_{2k})^m + 1/(\delta_{2k+1})^m - 2/(\gamma_{2k})^m] < 2/(\delta_{2n_0+2n})^m.$$

Then for all $p > 0$ and $z \in E$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log [\delta_{2k}^* \ \delta_{2k+1}^*/(\gamma_{2k}^*)^2] \right| = \left| -\sum_{m=1}^{\infty} [z^m/m] \sum_{k=n_0+n}^{n_0+n+p} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right|$$
$$\leqq \sum_{m=1}^{\infty} [R^m/m][2/(\delta_{2n_0+2n})^m] \leqq 2 \sum_{m=1}^{\infty} [R/(\delta_{2n_0+2n})]^m = 2R/(\delta_{2n_0+2n} - R) .$$

Since $\delta_{2n_0+2n} \to \infty$ as $n \to \infty$, the Cauchy criterion is satisfied and M_p converges uniformly in E. Thus $\Pi(z)$ converges uniformly in any compact subset of the z-plane.

LEMMA 6. $\pi(z) = f(z)$.

Proof. As a consequence of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$, while each of these quotients defines a function which is holomorphic for $|z| < r$ and takes the value 1 at $z = 0$. Thus using the principal value of the logarithm, for $|z| < r$,

$$\log [R_n(z)/z] - \log [\pi(z)/z] = \log [R_n(z)/\pi(z)] = \log [(1 + z/\gamma_1)/(1 + z/\gamma_{1,n})]$$
$$- \sum_{m=1}^{\infty} \{z^m/m\} \left\{ \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^{n} (1/\delta_{2k+1,n}^m) - \sum_{k=1}^{n} (2/\gamma_{2k,n}^m) \right.$$
$$\left. - \sum_{k=1}^{\infty} [(1/\delta_{2k}^m) + (1/\delta_{2k+1}^m) - 2/\gamma_{2k}^m] \right\} .$$

Therefore, for $n_0 > 2$, as $n \to \infty$,

$$0 \leqq \lim \sup \left| \sum_{k=1}^{n+1} (1/\delta_{2k,n}^m) + \sum_{k=1}^{n} (1/\delta_{2k+1,n}^m) \right.$$

$$- \sum_{k=1}^{n}(2/\gamma_{2k,n}^{m}) - \sum_{k=1}^{\infty}[(1/\delta_{2k}^{m}) + (1/\delta_{2k+1}^{m}) - 2/\gamma_{2k}^{m}]\Big|$$
$$\leqq \limsup \Big|\sum_{k=n_0}^{n+1}(1/\delta_{2k,n}^{m}) + \sum_{k=n_0}^{n}(1/\delta_{2k+1,n}^{m}) - \sum_{k=n_0}^{n}(2/\gamma_{2k,n}^{m})$$
$$- \sum_{k=n_0}^{\infty}[(1/\delta_{2k}^{m}) + (1/\delta_{2k+1}^{m}) - 2/\gamma_{2k}^{m}]\Big|$$
$$\leqq \limsup |(1/\delta_{2n_0,n}^{m}) + (1/\delta_{2n_0+1,n}^{m}) + (1/\delta_{2n_0}^{m}) + (1/\delta_{2n_0+1}^{m})|$$
$$= (2/\delta_{2n_0}^{m}) + (2/\delta_{2n_0+1}^{m}) .$$

Since $\delta_{2n_0} \to \infty$ and $\delta_{2n_0+1} \to \infty$ as $n_0 \to \infty$, it follows that the limit as $n \to \infty$ of each coefficient of the preceding expansion of $\log[R_n(z)/\pi(z)]$ is zero. Furthermore, because as $n \to \infty$, $\{\log[R_n(z)/\pi(z)]\}_{n=1}^{\infty}$ converges uniformly on $\{|z| < r\}$, then $\log[R_n(z)/\pi(z)] \to 0$ as $n \to \infty$. Thus $\pi(z) = \lim_{n\to\infty} R_n(z) = f(z)$.

LEMMA 7. *$\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$, $\sum_{k=1}^{\infty} 1/\beta_k < \infty$, $\sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty$, and*

$$\sum_{k=2}^{\infty} 1/\delta_k < \infty .$$

Proof. Again by Lemma 4, there exists $r > 0$ such that $f'(z) \neq 0$ and $R_n'(z) \neq 0$ for $|z| < r$. Since $R_n(z) \to f(z)$, it follows that $R_n'(z) \to f'(z)$ and thus $\log R_n'(z) \to \log f'(z)$ uniformly in $\{|z| < r\}$ as $n \to \infty$. Thus for $|z| < r$, $\log R_n'(z)$

$$= \sum_{m=1}^{\infty}[z^m/m]\Big[- \sum_{k=0}^{n} 1/\alpha_{2k+1,n}^{m}$$
$$+ \sum_{k=1}^{2n+1}(-1)^{m+1}/\beta_{k,n}^{m} + 2(-1)^m/\gamma_{1,n}^{m} + \sum_{k=1}^{n} 3/\gamma_{2k,n}^{m}\Big] .$$

Hence, for $m = 1$,

$$\lim_{n\to\infty}\Big| - \sum_{k=0}^{n} 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n} + \sum_{k=1}^{n} 3/\gamma_{2k,n}\Big| < \infty .$$

Because $0 < \gamma_{1,n} < \beta_{1,n}$ and $0 < \gamma_{2k,n} < \alpha_{2k+1,n}$, then

$$0 < \sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^{n} 2/\gamma_{2k,n}$$
$$< - \sum_{k=0}^{n} 1/\alpha_{2k+1,n} + \sum_{k=1}^{2n+1} 1/\beta_{k,n} - 2/\gamma_{1,n}$$
$$+ \sum_{k=1}^{n} 3/\gamma_{2k,n} + 1/\alpha_{1,n} + 2/\gamma_{1,n} .$$

Therefore, as $n \to \infty$,

$$0 \leqq \limsup \Big[\sum_{k=1}^{2n+1} 1/\beta_{k,n} + \sum_{k=1}^{n} 2/\gamma_{2k,n}\Big] < \infty ,$$

$$\lim\sup \sum_{k=1}^{2n+1} 1/\beta_{k,n} < \infty, \text{ and } \lim\sup \sum_{k=1}^{n} 1/\gamma_{2k,n} < \infty .$$

Furthermore, because for

$$k \geqq 1,\ \gamma_{2k,n} < \delta_{2k+1,n} < \alpha_{2k+1,n} < \delta_{2k+2,n} ,$$
$$\lim_{n\to\infty}\sup \sum_{k=3}^{2n+2} 1/\delta_{k,n} \leqq \lim_{n\to\infty}\sup \sum_{k=1}^{n} 2/\gamma_{2k,n} < \infty$$

and

$$\lim_{n\to\infty}\sup \sum_{k=1}^{n} 1/\alpha_{2k+1,n} \leqq \lim_{n\to\infty}\sup \sum_{k=1}^{n} 1/\gamma_{2k,n} < \infty .$$

Hence

$$\lim_{n\to\infty}\sup \sum_{k=0}^{n} 1/\alpha_{2k+1,n} < \infty \quad\text{and}\quad \lim_{n\to\infty}\sup \sum_{k=2}^{2n+2} 1/\delta_{k,n} < \infty .$$

For all $N > 0$, as $n \to \infty$,

$$\sum_{k=0}^{N} 1/\alpha_{2k+1} = \sum_{k=0}^{N} \lim 1/\alpha_{2k+1,n} \leqq \lim\sup \sum_{k=0}^{n} 1/\alpha_{2k+1,n} < \infty ,$$

and thus $\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty$. The convergence of the other series is established in a similar manner.

LEMMA 8. *Each of the three infinite products in*

$$P(z) = [1/(1 + z/\gamma_1)^2]\left[\prod_{k=0}^{\infty} \alpha^*_{2k+1} \prod_{k=1}^{\infty} \beta^*_k / \prod_{k=1}^{\infty} (\gamma^*_{2k})^3\right]$$

converges uniformly on any compact subset of the z-plane.

Proof. This is a consequence of Lemma 7.

LEMMA 9. $f'(z) = [\exp(\delta z)][P(z)]$ *where* δ *is real.*

Proof. By Lemma 4, there exists $r > 0$ such that for $|z| < r$, $R'_n(z) \neq 0$ and $f'(z) \neq 0$. For $m \geqq 1$, consider the coefficient of z^m/m in the Taylor expansion of $\log [R'_n(z)/P(z)]$ about $z = 0$ for $|z| < r$. Because of Lemma 7, there exists $M > 0$ such that for all $n \geqq 1$,

$$\sum_{k=1}^{n} 1/\gamma_{2k,n} < M \quad\text{and}\quad \sum_{k=1}^{\infty} 1/\gamma_{2k} < M .$$

Then because of the ordering of the $\gamma_{k,n}$ and γ_k, for each $k < n$, $k/\gamma_{2k,n} < M$ and $k/\gamma_{2k} < M$. Thus for each $N > 1$, as $n \to \infty$,

$$\limsup \left| \sum_{k=1}^{n} 1/\gamma_{2k,n}^{m} - \sum_{k=1}^{\infty} 1/\gamma_{2k}^{m} \right|$$
$$\leqq \limsup \left| \sum_{k=N}^{n} 1/\gamma_{2k,n}^{m} - \sum_{k=N}^{\infty} 1/\gamma_{2k}^{m} \right| \leqq 2M^{m} \sum_{k=N}^{\infty} 1/k^{m},$$

which implies for $m \geqq 2$, as $n \to \infty$

$$\lim \left[\sum_{k=1}^{n} 1/\gamma_{2k,n}^{m} - \sum_{k=1}^{\infty} 1/\gamma_{2k}^{m} \right] = 0 .$$

Similarly, the other terms in the coefficient of z^m/m have a limit of zero for $m \geqq 2$, and the coefficient of z is real. Then as $n \to \infty$, $\log [R_n'(z)/P(z)] \to \log [f'(z)/P(z)] = \delta z$, and thus $f'(z) = [\exp(\delta z)][P(z)]$.

LEMMA 10. $\delta = 0$.

Proof. Because the factors of $P(z)$ are canonical products of genus zero with real zeros, for $\varepsilon > 0$ and $0 < \rho \leqq |\arg z| \leqq \pi - \rho$, $P(z) = 0[\exp(\varepsilon |z|)]$ and $1/P(z) = 0[\exp(\varepsilon|z|)]$. Then if $\arg z$ satisfies the preceding conditions and $|z|$ is sufficiently large, then

$$\exp[\delta \mathscr{R}(z) - \varepsilon |z|] \leqq |f'(z)| \leqq \exp[\delta \mathscr{R}(z) + \varepsilon |z|] .$$

Let $A_1 = \{z \mid \pi/4 \leqq \arg z \leqq \pi/3\}$ and $A_2 = \{z \mid 2\pi/3 \leqq \arg z \leqq 3\pi/4\}$. If $\delta > 0$, then there exists $\varphi_1 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geqq \exp(\varphi_1 |z|)$ when $z \in A_1$ and $|f'(z)| \leqq \exp(-\varphi_1 |z|)$ when $z \in A_2$. Thus as $z \to \infty$ in A_2, $f'(z) \to 0$, and because $f(z) \geqq b_{2n} > 0$ for z on the curve C_{2n}, $f(z) \to k \geqq 0$ as $z \to \infty$ in A_2. Thus for n sufficiently large, $b_{2n} < k + 1$. Since $f'(z)dz > 0$ in the positive sense on the part of the curve C_{2n+1} in A_1, $b_{2n+1} - a_{2n+1} \to \infty$ as $n \to \infty$, where $a_{2n+1} > 0$ and thus $b_{2n+1} \to \infty$ as $n \to \infty$. Because $b_{2n+1} < b_{2n}$, a contradiction has been reached and $\delta \not> 0$. If $\delta < 0$, then there exists $\varphi_2 > 0$ such that for $|z|$ sufficiently large $|f'(z)| \geqq \exp(\varphi_2 |z|)$ when $z \in A_2$ and $|f'(z)| \leqq \exp(-\varphi_2 |z|)$ when $z \in A_1$. Similarly, $\delta \not< 0$.

THEOREM. *A Riemann surface of the class defined is parabolic and its mapping function f is given by*

$$f(z) = [z/(1 + z/\gamma_1)] \prod_{k=1}^{\infty} [\delta_{2k}^{*} \ \delta_{2k+1}^{*}/(\gamma_{2k}^{*})^2]$$

where

$$f'(z) = [1/(1 + z/\gamma_1)^2] \left[\prod_{k=0}^{\infty} \alpha_{2k+1}^{*} \prod_{k=1}^{\infty} \beta_k^{*} / \prod_{k=1}^{\infty} (\gamma_{2k}^{*})^3 \right] .$$

Furthermore,

$$\sum_{k=0}^{\infty} 1/\alpha_{2k+1} < \infty, \ \sum_{k=1}^{\infty} 1/\beta_k < \infty, \ \sum_{k=1}^{\infty} 1/\gamma_{2k} < \infty, \ \textit{and} \ \sum_{k=2}^{\infty} 1/\delta_k < \infty .$$

REMARKS. Lemmas 5 and 6 establish the representation of $f(z)$ as the product of quotients, while Lemmas 8 and 9 show a representation of $f'(z)$ as a quotient of products. However, Lemma 7 can be used to show that the representation of $f(z)$ can also be considered as the quotient of products.

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