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COMMUTATIVE F-ALGEBRAS

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We extend several theorems for commutative Banach algebras to topological algebras with a sequence of semi-norms (F-algebras). The question of what functions "operate" on an F-algebra is considered. It is proven that analytic functions in several complex variables operate by applying a theorem due to Waelbroeck. If all continuous functions operate on an F-algebra, then it is an algebra of continuous functions. However, unlike the situation for Banach algebras [6], it is not true that if $\sqrt{\ }$ operates the algebra is $C(\Delta)$. This will be shown by an example. A theorem due to Curtis [4], concerning continuity of derivations when the algebra is regular is extended to F-algebras. The result is applied to an algebra of Lipschitz functions to show that it has only a trivial derivation.

Preliminaries. Throughout this paper the letter A will stand for a commutative F-algebra. An F-algebra is a topological algebra with topology determined by a sequence of algebraic semi-norms. The nth semi-norm of an element x in A will be written $\| x \|_n$. We may and shall always assume that for all x in A, $\| x \|_n \leqq \| x \|_{n+1}$. Δ^+ will denote the topological space of all continuous multiplicative linear functionals on A with the weak* topology. Δ will denote Δ^+ minus the zero functional with the relativized topology. For x in A, $\hat{x}$ will be the function in $C(\Delta^+)$ (the continuous functions on Δ^+ with the compact-open topology) defined by $\hat{x}(\varphi) = \varphi(x)$. A will be called regular if given φ_0 in Δ and V a neighborhood of φ_0, there is an element x in A such that $\varphi_0(x) = 1$ and $\varphi(x) = 0$ for $\varphi \notin V$. A will be called semi-simple if $\hat{x} = 0$ implies $x = 0$.

A basic device in the study of F-algebras is to represent A as the inverse limit of a sequence of Banach algebras $\{A_n\}$ where A_n is the completion of A/I_n with norm $\| x + I_n \| = \| x \|_n$ and I_n is the ideal of all x in A such that $\| x \|_n = 0$. The homomorphism $\pi_{m,n}$: $A_n \rightarrow A_m$ for $m \leqq n$ is defined as the completion of the mapping $x + I_n \rightarrow x + I_m$. This representation enables one to construct an element in A by constructing a sequence $\{x_n\}$ such that for each n,

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$x_n \in A_n$ and $\pi_{m,n}x_n = x_m$. The homomorphism $\pi_n: A \to A_n$ is defined as $x \to x + I_n$. Then π_n^*: (multiplicative linear functionals in $A_n) \to \Delta^+$ is continuous and one-to-one and so its range, which we shall denote by Δ_n^+ is a compact subset of Δ^+. If K is an arbitrary compact subset of Δ^+, there is an integer n such that $K \subseteqq \Delta_n^+$ [9].

The following theorem, due to Silov, is also valid for F-algebras. *If C is a closed and open subset of Δ^+ and the zero homomorphism is not in C, then there is an idempotent e in A such that $C = \{\varphi \in \Delta^+: \varphi(e) = 1\}$.* The extension to F-algebras is proven via the device of the previous paragraph. With the aid of Silov's theorem the proof that if A is regular, then A is normal is essentially the same as for Banach algebras.

Since so many of the theorems true for Banach algebras are also true for F-algebras with almost the same proofs, it is perhaps appropriate to remark that the difficulties introduced by the sequence of semi-norms are sometimes quite subtle. For example such a seemingly innocuous question as whether a multiplicative linear functional is necessarily continuous is still unanswered.

Functions that operate on a commutative semi-simple F-algebra. A function $f: D \subseteqq \mathbf{C} \to \mathbf{C}$ is said to "operate" on an F-algebra A if $f \circ \hat{x} \in \hat{A}$ whenever $x \in A$ and the range $\hat{x} \subseteqq D$. It is not difficult to adapt Katznelson's proof in [5] to show that if every continuous function operates on A, then $A = C(\Delta)$. However another theorem due to Katznelson which states: *If A is a self-adjoint Banach algebra and $\sqrt{\ }$ operates on the positive functions in $\hat{A}$, then $A = C(\Delta)$* is no longer true for F-algebras; as the following example shows.

Let H be the subalgebra of l^∞ consisting of those sequences $\{a_n\}$ for which there is a number, a such that $|a_n - a|^{1/n} \to 0$. Let H' be the subalgebra of H consisting of those sequences for which $a = 0$. Let τ be the linear transformation from H' to the entire functions defined by $\tau(\{a_n\})(\lambda) = \sum_{n=0}^{\infty} a_n\lambda^n$. For each integer N and for $\{a_n\} \in H'$ defined $\|\{a_n\}\|_N = \sup[|\tau(\{a_n\})(\lambda)| : |\lambda| \leqq N]$. $\|-\|_N$ is evidently a vector space norm. It is also algebraic; for suppose $\{a_n\}$ and $\{b_n\} \in H'$, $f = \tau(\{a_n\})$, $g = \tau(\{b_n\})$ and $F = \tau(\{a_nb_n\})$. Then

$$F(\lambda) = (1/2\pi i)\int_{|w|=M} f(w)g(\lambda/w)dw/w \, .$$

H' is a complete F-algebra under the sequence of norms defined above and H is the F-algebra obtained by adjoining a unit to H'.

For $n = 0, 1, 2, \cdots$, define z_n as the sequence which is 1 in the nth coordinate and 0 in all the other coordinates. These elements generate H' (since the polynomials are dense in the entire functions) and together with the unit of H generate H. $\Delta(H)$ is homeomorphic to the one-point compactification of the integers, the point corresponding to the integer n being the functional sending z_n into 1.

It is evident that $\hat{H}$ is a self-adjoint subalgebra of $C(\Delta(H))$, and that H is semi-simple and regular. Yet, although $\sqrt{}$ operates on the nonnegative elements of $\hat{H}$, $H \neq C(\Delta(H))$.

For U an open subset of $\mathbf{C}^n$ let $H(U)$ be the F-algebra of all holomorphic functions on U with the compact-open topology. For σ an arbitrary subset of $\mathbf{C}^n$, let $H(\sigma)$ be the direct limit of the F-algebras $H(U)$ for U ranging over open sets containing σ directed as follows: $H(U) \geqq H(V)$ if $U \subseteqq V$.

Let $a_1, \cdots, a_n$ be elements of a commutative F-algebra, say A, with unit. For $\varphi \in \Delta = \Delta(A)$, let $\sigma(\varphi)$ be the point in $\mathbf{C}^n$ $(\varphi(a_1), \cdots, \varphi(a_n))$ and let $\sigma = \{\sigma(\varphi) : \varphi \in \Delta\}$.

THEOREM. *There is a continuous homomorphism τ from $H(\sigma)$ to A such that $\varphi(\tau f) = f(\sigma(\varphi))$ for every φ in Δ and every f in $H(\sigma)$ and $\tau(z_i) = a_i$, $i = 1, \cdots, n$. (Evidently $f \in H(\sigma)$ defines a function on σ.)*

Proof. Waelbroeck, in [11], proved that such a continuous homomorphism exists for even more general topological algebras providing the elements, $a_1, \cdots, a_n$ are regular, i.e. have compact spectrum. An element of an F-algebra needn't be regular, but an element of a Banach algebra is of course regular. We will apply Waelbroeck's theorem to each of the Banach algebras A_s where A is the inverse limit of $\{A_s\}$.

For every integer k let σ_k be defined as above for $\pi_k a_1, \cdots, \pi_k a_n$, let τ_k be the continuous homomorphism from $H(\sigma_k)$ to A_k. $\forall k\colon \sigma_k \subseteqq \sigma$ and there is a continuous homomorphism $\nu_k\colon H(\sigma) \rightarrow H(\sigma_k)$. The essence of the proof is that the sequence $\{f_k\}$ where $f_k \in A_k$ is defined as $\tau_k \circ \nu_k(f\}$ satisfies $\pi_{s,t} f_t = f_s$ for $s \leqq t$. For then the sequence $\{f_k\}$ defines an element τf in A.

If each A_k were semi-simple, then it would follow that $\pi_{s,t} f_t = f_s$ for $s \leqq t$. For Waelbroeck's theorem implies that $(\pi_{s,t} f_t)^\wedge = \hat{f}_s$. However, even if A is semi-simple, it does not follow that each A_k is semi-simple.

Let s and t be two fixed integers with $s \leqq t$. We shall examine the construction of f_s. Let $b_i = \pi_s a_i$ for $i = 1, \cdots, n$. $f \in H(\sigma)$ may be considered as a function holomorphic in a neighborhood, say W, of σ and, therefore, of σ_s. The following assertions are proven in [11].

(1) σ_s is convex in the following sense. There is a finite set of polynomials in n variables, say $p_1, \cdots, p_r$ and neighborhoods $D_1, \cdots, D_n$ of the spectrum of $b_1, \cdots, b_n$ respectively and neighborhoods $D_{n+1}, \cdots, D_{n+r}$

of the spectrum of $b_{n+1} = p_1(b_1, \cdots, b_n)$, $\cdots$, $b_{n+r} = p_r(b_1, \cdots, b_n)$ respectively such that the following two facts are true:

(a). $\sigma_s \subseteqq D \subseteqq W$ where $D = \{\lambda \in D_1 \times \cdots \times D_n : p_i(\lambda) \in D_{n+i}$ for $i = 1, \cdots r\}$.

(b). If $E = D_1 \times \cdots \times D_n \times \cdots \times D_{n+r}$ and $X = \{(\lambda, p_1(\lambda), \cdots, p_r(\lambda)) : \lambda \in D\}$, then the restriction mapping, ρ, from E to X is a continuous open homomorphism of $H(E)$ *onto* $H(X)$ with kernel the ideal generated by $\{z_{n+k} - p_k(z_1, \cdots, z_n) : k = 1, \cdots, r\}$. By (a), f is a holomorphic function on D and determines a function $F \in H(X)$ where $F(\lambda, p(\lambda))$ is defined to be $f(\lambda)$ (i.e. F depends only on the first n coordinates). By (b), $F = \rho(G)$ where $G \in H(E)$.

(2) Define $\alpha: H(E) \to A_s$ by

$$\alpha(H) = (1/2\pi i)^{n+r} \int_{\Gamma_1} \cdots \int_{\Gamma_{n+r}} H(\lambda_1, \cdots, \lambda_{n+r})(\lambda^1 - b_1)^{-1} \cdots (\lambda_{n+r} - b_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r}$$

where Γ_i is a rectifiable curve in D_i including in its interior the spectrum of b_i for $i = 1, \cdots, n + r$. α is a continuous homomorphism and $\alpha(z_i) = b_i$ for $i = 1, \cdots, n + r$. Thus, by (b), if $\rho(G_1) = \rho(G) = F$, then $\alpha(G_1) = \alpha(G)$. f_s is defined as $\alpha(G)$.

(3) If the system of polynomials $p_1, \cdots, p_r$ and the neighborhoods $D_1, \cdots, D_{n+r}$ are replaced by another system which meets the condition $\sigma_s \subseteqq D \subseteqq W$, then the same element $f_s \in A_s$ arises.

Let $\{p_1, \cdots, p_r, D_1, \cdots, D_{n+r}\}$ be a system used to to define f_t. Suppose $c_i = \pi_t a_i$ for $i = 1, \cdots, n$ and $c_{n+k} = p_k(c_1, \cdots, c_n)$ for $k = 1, \cdots, r$. Then

$$\pi_{s,t} f_t = \pi_{s,t}(1/2\pi i)^{n+r} \int \cdots \int G(\lambda)(\lambda_1 - c_1)^{-1} \cdots (\lambda_{n+r} - c_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r} = (1/2\pi i)^{n+r} \int \cdots \int G(\lambda)(\lambda_1 - b_1)^{-1} \cdots (\lambda_{n+r} - b_{n+r})^{-1} d\lambda_1 \cdots d\lambda_{n+r} = f_s .$$

For the system $\{p_1, \cdots, p_n, D_1, \cdots, D_{n+r}\}$ may be used to define f_s: $sp(b_i) \subseteqq sp(c_i) \subseteqq D_i$ for $i = 1, \cdots, n + r$ and $\sigma_s \subseteqq \sigma_t \subseteqq D \subseteqq W$. Thus τf is well defined.

If $\varphi \in \Delta$, then $\varphi \in \Delta_k$ for some integer k, say $\varphi = \pi_k^* \psi$ for $\psi \in \Delta(A_k)$, then $f(\sigma(\varphi)) = f(\sigma_k(\psi)) = \psi(f_k) = \varphi(\tau f)$. $\tau z_i = a_i$, since $(z_i)_s = \pi_s a_i$ for every integer s, for $i = 1, \cdots, n$. τ is continuous, since $f_\alpha \to f_0 \Rightarrow$ for all k $\nu_k f_\alpha \to \nu_k f_0 \Rightarrow$ for all k $\tau_k \circ \nu_k f_\alpha \to \tau_k \circ \nu_k f_0$ (i.e. for all k $(f_\alpha)_k \to (f_0)_k) \Rightarrow \tau f_\alpha \to \tau f_0$.

This theorem, except for continuity of the operational calculus, is also proven in [1] via the Arens-Calderon theorem [2].

Continuity of derivations. A derivation on an algebra A is a linear operator D satisfying $D(xy) = xDy + (Dx)y$ for every x and y in A. If A is a commutative F-algebra, a linear transformation $D: A \to C(\Delta)$ satisfying $D(xy) = \hat{x}Dy + (Dx)\hat{y}$ will be called a derivation into $C(\Delta)$. It is conjectured that a derivation on a Banach algebra must be continuous. Curtis [4] proved that if a Banach algebra is regular, then any derivation is continuous, in fact any derivation from the algebra to $C(\Delta)$ is continuous. This theorem will be extended to allow the algebra to be an F-algebra. It will then be applied to some F-algebras to determine all derivations in these algebras.

The following lemma is a modification of one in [3] and its proof is essentially the same.

LEMMA. *Let t be an algebraic homomorphism from a commutative F-algebra A to a semi-normed algebra B. Let $\{g_k\}$ and $\{h_k\}$ be two sequences of elements in A such that for all n: $g_nh_n = g_n$ and if $m \neq n$, then $h_nh_m = 0$. Then it is not possible that for all n $\| tg_n \| > n \| g_n \|_n \| h_n \|_n$.*

COROLLARY. *If D is a derivation from a regular commutative semi-simple F-algebra A to $C(\Delta)$, then D is continuous.*

Proof. Let $\{A_k\}$ and $\{\Delta_k\}$ be defined as in the preliminaries. Since every compact subset of Δ is contained in some Δ_N, it suffices to prove that if $x_n \to 0$, then $Dx_n \to 0$ uniformly on each Δ_N. The procedure will be to show:

(1) for all N there is an at most finite set $F_N \subseteqq \Delta_N$ such that $Dx_n \to 0$ uniformly on the closure of $[\Delta_N \backslash F_N]$;

(2) if φ is isolated in Δ, then $Dx(\varphi) = 0$ for every x in A; and

(3) if $\varphi \in \Delta_N$ is isolated in Δ_m for every $m \geqq N$, then φ is isolated in Δ. (1), (2), and (3) imply that $Dx_n(\varphi) \to 0$ for every φ and this together with (1) implies that $Dx_n \to 0$ uniformly on Δ_N. This is basically the some proof as in [4]. The third step is the only novel point in the proof. It does not follow from the fact that every compact set is contained in some Δ_N. The example of Arens' ([7] problem 2E) shows this. (3) may be proven as follows: Suppose $\varphi \in \Delta_N$ is isolated in Δ_m for all $m \geqq N$. By Silov's theorem, for each $m \geqq N$, there is an idempotent $e_m \in A_m$ such that $\varphi(e_m) = 1$ and $\varphi'(e_m) = 0$ if $\varphi' \in \Delta_m$ and $\varphi' \neq \varphi$ (identifying Δ_m with $\Delta(A_m)$). Then, because each e_m is an idempotent and $(\pi_{r,s}e_s)^\wedge = \hat{e}_r$ for $N \leqq r \leqq s$, $\pi_{r,s}e_s = e_r$ for $N \leqq r \leqq s$ (two idempotents in A_r equal modulo the radical are iden-

tical). Thus $\{e_m\}$ defines an idempotent e in A such that $\varphi(e) = 1$ and $\varphi'(e) = 0$ for $\varphi' \neq \varphi$ and $\varphi' \in \Delta$

Steps (1) and (2) will be sketched. Proof of (1): Let B be the semi-normed algebra which as an algebra is A, but with semi-norm $\| x \| = \| x \|_N + \| Dx \|_N$. Let $F = \{\varphi \in \Delta_N : x \to Dx(\varphi)$ is not a continuous linear functional$\}$. Since A is an F-space, the principle of uniform boundedness applies. Since for each x in A $\{Dx(\varphi) : \varphi \in \Delta_N \backslash F\}$ is bounded (by $\| Dx \|_N$), $Dx_n \to 0$ uniformly on $\Delta_N \backslash F$. F is a finite set. If not, then there is an infinite sequence $\{\varphi_n\} \subseteqq F$ with mutually disjoint neighborhoods. Since the algebra is by hypothesis regular, there are sequences $\{y_n\}, \{z_z\}$ such that $\hat{y}_n(\varphi_n) = 1$, $y_n z_n = y_n$ and $z_n z_m = 0$ if $m \neq n$. Then since $\varphi_n \in F$, there is an x_n in A such that $| Dx_n(\varphi_n) | > n \| x_n \|_n \cdot \| y_n \|_n \cdot \| z_n \|_n$. Thus letting $g_n = x_n y_n$ and $h_n = z_n$, we have $\| g_n \| \geqq \| Dg_n \|_N > n \| g_n \|_n \cdot \| h_n \|_n$ and this contradicts the previous lemma. Thus we may let F be F_N. Proof of (2): Let $\varphi \in \Delta$ be isolated. Choose, by Silov's theorem an idempotent e such that $\varphi(e) = 1$ and $\varphi'(e) = 0$ for $\varphi' = \varphi$. Then $De(\varphi) = 0$ and, by semi-simplicity, $ex = \varphi(x)e$ for any x in A. Hence

$$0 = D(ex)(\varphi) = x(\varphi)De(\varphi) + Dx(\varphi) = Dx(\varphi)$$

for any x in A.

By the closed graph theorem and the previous corollary, if D is a derivation on a regular commutative semi-simple F-algebra, then D is continuous.

Let $C^\infty(R)$ be the algebra of infinitely differentiable functions on the real line. For f in $C^\infty(R)$, let

$$\| f \|_n = \textstyle\sum_{k=0}^{n} \sup [| f^{(k)}(t) | : -n \leqq t \leqq n]/k! .$$

$C^\infty(R)$ is a regular semi-simple F-algebra. If D is a derivation on $C^\infty(R)$ and x is the function mapping t into t, then for any polynomial p in x, $Dp(x) = p'(x)Dx$. Since the polynomials in x are dense in $C^\infty(R)$ and since D is continuous, $Df = f'Dx$ for any f in $C^\infty(R)$.

As a second application of the previous corollary, we show that the following algebra of Lipschitz functions has no nontrivial derivations.

Let $\alpha \leqq 1$. Let L_α be the subalgebra of $C(R)$ consisting of functions of period 1 with finite norm $\| - \|_\alpha$ where $\| f \|_\alpha$ is defined to be

$$\sup [| f(t) | : t \in R] + \sup [| f(s + h) - f(s)|/| h |^\alpha : s \in R, h \neq 0] .$$

Let $1_\alpha = \{ f \in L_\alpha : \overline{\lim} [| f(s + h) - f(s) |/| h |^\alpha \to 0 : h \to 0]$ for $s \in R\}$. For $\alpha < 1$, L_α is a Banach space, 1_α a closed subspace, and L_α is isomorphic to 1_α^{**} [8]. Let $\alpha_n = 1 - 1/n$ and L be $\cap L_{\alpha_n}$ with the sequence of algebraic norms $\{\| - \|_{\alpha_n}\}$. L may also be defined as the inverse limit of $\{L_{\alpha_n}\}$. $L_{\alpha_{n+1}} \subseteqq 1_{\alpha_n} \subseteqq L_{\alpha_n}$ and so L is also the inverse limit of $\{1_{\alpha_n}\}$. This implies that $L = L^{**}$, however even more is true: A bounded subset of L must have compact closure, i.e., L is a Montel space. For let S be a bounded set in $L \subseteqq 1_{\alpha_n}$. 1_{α_n} is isometrically isomorphic as a Banach space with a subspace of $C(W^*)$ where W^* is a compact set obtained as follows: Let $U = \{t \in R: 0 \leqq t \leqq 1\}$, $V = \{(r, s): 0 \leqq r \leqq 1, 0 < r - s \leqq 1/2\}$ and $W = U \cup V$, then W is a locally compact space and W^* is its one-point compactification. The isomorphism $f \to \tilde{f}$ is defined by $\tilde{f}(\infty) = 0$, $\tilde{f}(t) = f(t)$, and

$$\tilde{f}(r, s) = [f(r) - f(s)]/(r - s)^{\alpha_n} .$$

To see that S is precompact in L it suffices to show that S is precompact in each 1_{α_n} or, equivalently, that $\tilde{S}$ is equicontinuous. This follows from the fact that there is a number K such that

$$f \in S \Rightarrow \|f\|_{\alpha_{n+1}} \leqq K .$$

The representation of 1_{α_n} as $C(W^*)$ is due to DeLeeuw [8].

A derivation D on L must map every element into 0. For L is a regular, commutative, semi-simple F-algebra and so it suffices to show that if $f \in L$, then $\varphi(Df) = 0$ for any $\varphi \in \Delta(L)$. $D(f - \varphi(f)) = Df$ and $f - \varphi(f)$ is in the kernel, M, of φ. So it suffices to show that $D[M] \subseteqq M$. Since M is an ideal, $D[M^2] \subseteqq M$. $M^2 \neq M$, but M^2 is dense in M and so, since D must be continuous, $D[M] \subseteqq M$. (Any maximal ideal M must be the set of all functions in L vanishing at some t_0 where $0 \leqq t_0 < 1$. The function $\sin([t - t_0]/2\pi)$ is in M but not in M^2. Sherbert [10] proved that M^2 is dense in M for the Banach algebra 1_α, in fact for algebras of Lipschitz functions on more general spaces than the unit interval. His proof works as well for L.)

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