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SOME IDENTITIES VALID IN SPECIAL JORDAN ALGEBRAS BUT NOT VALID IN ALL JORDAN ALGEBRAS

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A Jordan algebra is defined by the identities:

$$x \cdot y = y \cdot x, (x \cdot y) \cdot y^2 = (x \cdot y^2) \cdot y \ . \tag{1}$$

The algebra A_J obtained from an associative algebra A on replacing the product xy by $x \cdot y = 1/2(xy + yx)$ is easily seen to be a Jordan algebra. Any subalgebra of a Jordan algebra of this type is called special. It is known from work of Albert and Paige that the kernel of the natural homomorphism from the free Jordan algebra on three generators to the free special Jordan algebra on three generators is nonzero and consequently that there exist three-variable relations which hold identically in any homomorphic image of a special Jordan algebra but which are not consequences of the defining identities (1). Such a relation we shall call an S-identity. It is the purpose of this paper to establish that the minimum possible degree for an S-identity is 8 and to give an example of an S-identity of degree 8. In the final section we use an S-identity to give a short proof of the main theorem of Albert and Paige in a slightly strengthened form.

NOTATION. The product in a Jordan algebra will be denoted by a dot, thus $a \cdot b$, and $\{abc\}$ will denote the Jordan triple product

$$\{abc\} = a \cdot (b \cdot c) - b \cdot (c \cdot a) + c \cdot (a \cdot b) \ . \tag{2}$$

Unbracketed products $a_1 \cdot a_2 \cdot \cdots \cdot a_n$ will denote left-normed products i.e. $(\cdots((a_1 \cdot a_2 \cdot) \cdot a_3) \cdot \cdots \cdot a_n)$. When working in a special Jordan algebra we shall use juxtaposition, thus ab, to denote the product in the underlying associative algebra. Then $a \cdot b = 1/2(ab + ba)$ and $2\{abc\} = abc + cba$. The free (respectively free special) Jordan algebra on n generators, taken as $x_1, \cdots, x_n$ or as x, y, z if $n = 3$, will be denoted by $J^{(n)}$ (respectively $J_0^{(n)}$) and the kernel of the natural homomorphism ν_n (written as ν for $n = 3$) of $J^{(n)}$ onto $J_0^{(n)}$ by K_n. The subspace of $J^{(n)}$ spanned by the monomials of degree n linear in each of the generators will be denoted by L_n. The underlying associative algebra for $J_0^{(n)}$ is the free associative algebra on n generators: we shall denote this by $A^{(n)}$. Throughout the paper we work over some fixed, but arbitrary, field of characteristic not two.

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1. The following theorem has been proved by MacDonald [4]:

THEOREM 1. (*MacDonald*). *K_3 contains no (nonzero) element which is linear in one of the generators.*

We have at once the following corollaries:

COROLLARY 1. *K_3 contains no (nonzero) element of degree less than six.*

COROLLARY 2. *An element u in $J^{(3)}$ linear in one generator, or of degree less than six, can be unambiguously represented by the expansion of $u\nu$ in $A^{(3)}$.*

In this section we shall strengthen Corollary 1 to the following theorem, which I understand has previously been proved by J. Blattner:

THEOREM 2. *K_3 contains no (nonzero) element of degree less than eight.*

Proof. Let L be the subspace of $J^{(3)}$ spanned by the elements of degree two in x, two in y and two in z; M the subspace of $J^{(3)}$ spanned by the elements of degree two in x, two in y and three in z. It is sufficient to show that (i) the restriction of ν to L is one-to-one and (ii) the restriction of ν to M is one-to-one. For (i) we display a set of elements which span L but whose images are linearly independent in $L\nu$. For (ii) we prove a Lemma which implies that if (ii) does not hold, then (i) does not hold.

Let R_b denote the mapping $a \rightarrow a \cdot b$ in a Jordan algebra. Then it is well-known that:

$$R_{a\cdot b\cdot c} = R_{a\cdot b}R_c + R_{b\cdot c}R_a + R_{c\cdot a}R_b - R_aR_cR_b - R_bR_cR_a \tag{3}$$

$$R_{a\cdot b\cdot c} = R_aR_{b\cdot c} + R_bR_{c\cdot a} + R_cR_{a\cdot b} - R_aR_cR_b - R_bR_cR_a \tag{4}$$

$$R_{a\cdot b}R_c + R_{b\cdot c}R_a + R_{c\cdot a}R_b = R_aR_{b\cdot c} + R_bR_{c\cdot a} + R_cR_{a\cdot b} \tag{5}$$

So L is spanned by elements of the forms

(i)	$aR_bR_cR_dR_eR_b$	(v)	$aR_bR_cR_dR_{e\cdot f}$
(ii)	$aR_{b\cdot c}R_dR_eR_f$	(vi)	$aR_{b\cdot c}R_{d\cdot e}R_f$
(iii)	$aR_bR_{c\cdot d}R_eR_f$	(vii)	$aR_{b\cdot c}R_dR_{e\cdot f}$
(iv)	$aR_bR_cR_{d\cdot e}R_f$	(viii)	$aR_bR_{c\cdot d}R_{e\cdot f}$

where two of a, b, c, d, e, f represent x, two represent y and two represent z. Consider those of type (i). We have

$$\begin{aligned}(16\, aR_bR_cR_dR_e)\nu &= (abcde + edcba) + (bacde + edcab) \\ &+ (cabde + edbac) + (cbade + edabc) \\ &+ (dabce + ecbad) + (dbace + ecabd) \\ &+ (dcabe + ebacd) + (dcbae + eabcd) \\ &= U_1 + U_2 + \cdots + U_8 \text{ (say)} \\ &\text{(where } U_1 = abcde + edcba, \text{ etc)} .\end{aligned}$$

Cohn has shown [2] that reversible elements in $A^{(3)}$ are in $J_0^{(3)}$ so that each U_i is in $J_0^{(3)}$. Since ν is an epimorphism there exist $u_i \in J^{(3)}$ ($i = 1, \cdots, 8$) for which $u_i\nu = U_i$. Then

$$(16\, aR_bR_cR_dR_e)\nu = \Sigma U_i = \Sigma(u_i\nu) = (\Sigma u_i)\nu$$

Thus

$$16\, aR_bR_cR_dR_e = \Sigma u_i \qquad \text{(Theorem 1, Corollary 1)}$$

and

$$16\, aR_bR_cR_dR_eR_f = (\Sigma u_i)\cdot f = \Sigma(u_i \cdot f)$$

By Theorem 1, Corollary 2, we can use U_i to represent u_i without introducing ambiguity. Thus instead of $u_1 \cdot f$ we can write $U_1 \cdot f$ i.e. *$(abcde + edcba)\cdot f$, an element in $J^{(3)}$ but with notation for the part* in brackets borrowed from $J_0^{(3)}$. Treating elements of types (ii)-(viii) similarly we see that with this notational convention L is spanned by elements of the forms $(abcde + edcba)\cdot f$ and $(abcd + dcba)\cdot(e \cdot f)$. The following elements then, together with those obtained by permuting x, y and z, span L:

T-elements

1. $2x\cdot(xy^2z^2 + z^2y^2x) = 4x\cdot\{xy^2z^2\}$

2(a). $x\cdot(xyzyz + zyzyx) = 2x\cdot\{x\{yzy\}z\} = 2x\cdot\{xy\{zyz\}\}$.

(b). $x\cdot(xzyzy + yzyzx) = 2x\cdot\{x\{zyz\}y\} = 2x\cdot\{xz\{yzy\}\}$

3. $2x\cdot(xyz^2y + yz^2yx) = 4x\cdot(x\cdot\{yz^2y\})$

4. $2z\cdot(yzyx^2 + x^2yzy) = 4z\cdot(\{yzy\}\cdot x^2)$

5. $2x\cdot(yxzyz + zyzxy) = 4x\cdot\{yx\{zyz\}\}$

6. $2y\cdot(zyx^2z + zx^2yz) = 4y\cdot\{z(y\cdot x^2)z\}$

7. $2x\cdot(y^2xz^2 + z^2xy^2) = 4x\cdot\{y^2xz^2\}$

8(a). $x\cdot(yzxyz + zyxzy) = x\cdot f(x, y, z)$

(b). $y\cdot(zxyzx + xzyxz) = y\cdot f(y, z, x)$

(c). $z\cdot(xyzxy + yxzyx) = z\cdot f(z, x, y)$
where $f(x, y, z) = 4\{(y\cdot z)x(y\cdot z)\} - \{y\{zxz\}y\} - \{z\{yxy\}z\}$

9. $2y\cdot(xzyzx) = 2y\cdot\{x\{zyz\}x\}$

10(a). $x^2\cdot(y^2z^2 + z^2y^2) = 2x^2\cdot(y^2\cdot z^2)$

(b). $y^2\cdot(z^2x^2 + x^2z^2) = 2y^2\cdot(z^2\cdot x^2)$

(c). $z^2\cdot(x^2y^2 + y^2x^2) = 2z^2\cdot(x^2\cdot y^2)$

11. $x^2\cdot(yzyz + zyzy) = 2x^2\cdot(y\cdot\{zyz\}) = 2x^2\cdot(\{yzy\}\cdot z)$

12. $2x^2\cdot(yz^2y) = 2x^2\cdot\{yz^2y\}$

13. $(x\cdot y)\cdot(xyz^2 + z^2yx) = 2(x\cdot y)R_{\{xyz^2\}}$

14. $(x\cdot y)\cdot(xzyz + zyzx) = 2\{zyz\}R_xR_{x\cdot y}$

15. $(x\cdot y)\cdot(xz^2y + yz^2x) = 2(x\cdot y)R_{\{xz^2y\}}$

16. $(x\cdot y)\cdot(zxyz + zyxz) = 2(x\cdot y)\cdot\{z(x\cdot y)z\} = 2\{(x\cdot y)z(x\cdot y)\}\cdot z$

T16 is clearly redundant, while use of formulae (3), (5) and (3) respectively shows that T13, T14 and T15 are also redundant. So the set T (namely T1–T12 together with those elements obtained from T1–T12 by permuting x, y and z) spans L. We now display a set U of Jordan elements. Each U-element may be considered as an element in $J_0^{(3)}$: as such its expansion in $A^{(3)}$ appears as the corresponding V-element. Alternatively the U-element may be considered as an element in L: its expression as a linear combination of T-elements appears as the corresponding W-element. For each integer r the validity of the relation $Ur = Wr$ can be checked by appealing to MacDonald's theorem. For example, in the case of $r = 7$, $U7 = W7$ is valid in $J_0^{(3)}$ and linear

r	U-elements	V-elements	W-elements	π
1	$2x^2\cdot\{yz^2y\}$	$x^2yz^2y + yz^2yx^2$	T12	6
2	$2\{x\{yz^2y\}x\}$	$2xyz^2yx$	T3 − T12	6
3	$2\{x^2y^2z^2\}$	$x^2y^2z^2 + z^2y^2x^2$	T10a − T10b + T10c	3
4	$2\{x(y^2\cdot z^2)x\}$	$xy^2z^2x + xz^2y^2x$	T1 − W3	3
5	$2\{x(y\cdot\{zyz\})x\}$	$xyzyzx + xzyzyx$	T2a + T2b − T11	3
6	$2\{x^2\{yzy\}z\}$	$x^2yzyz + zyzyx^2$	T2a − T2b + T11	6
7	$2\{zx^2\{yzy\}\}$	$zx^2yzy + yzyx^2z$	T4 − W6	6
8	$f(x^2, y, z)$	$yzx^2yz + zyx^2zy$	T6 − W7	3
9	$2y\cdot\{x\{zyz\}x\}$	$yxzyzx + xzyzxy$	T9	6
10	$2\{xyx\}\cdot\{zyz\}$	$xyxzyz + zyzxyx$	T5 − W9	3
11	$x\cdot f(x, y, z) - y\cdot f(y, z, x) + z\cdot f(z, x, y)$	$xyzxyz + zyxzyx$	T8a − T8b + T8c	3

in $\{yzy\}$, so $U7 = W7$ is valid in $J^{(3)}$. Suppose now that the sets of U, V and W-elements have been augmented by adjoining all elements obtained from those displayed by permuting x, y and z. The column headed π shows the number of distinct elements obtained for each value of r. It is then easy to check that each T-element is a linear combination of W-elements, so the W-elements span L. But their images under ν are the V-elements which are clearly linearly independent. So $\nu \mid L$ is one-to-one. To complete the proof of the theorem we now prove the following lemma:

LEMMA 1. *Let n be an odd (positive) integer and u an element in $K_n \cap L_n$ which is expressible in the form $u = \sum_{i=1}^{n} x_i \cdot y_i$. Then $y_i \in K_n$ for each $i = 1, \cdots, n$.*

Proof. For convenience we denote ν_n by ν. For $n = 1$ there is nothing to prove. Assume $n > 1$ and let the coefficient of

$$x_{i+1}x_{i+2} \cdots x_n x_1 \cdots x_{i-1} \, (x_2 \cdots x_n \text{ if } i = 1)$$

in $y_i\nu$ be μ_i. Then the coefficient of $x_{i+1}x_{i+2} \cdots x_n x_1 \cdots x_i$ in $2u\nu$ is $\mu_i + \mu_{i+1}$. Since distinct monomials in $A^{(n)}$ are linearly independent we have $\mu_i + \mu_{i+1} = 0, i = 1, \cdots, n-1$ and $\mu_n + \mu_1 = 0$, whence (n being odd) $\mu_i = 0, i = 1, \cdots, n$. In particular $\mu_1 = 0$, i.e. the coefficient of $x_2 \cdots x_n$ in $y_1\nu$ is zero. It follows by considering suitable renumberings of $x_2, \cdots, x_n$ that $y_1\nu = 0$, i.e. $y_1 \in K_n$. Similarly $y_i \in K_n$ for $i = 2, \cdots, n$.

COROLLARY. *Let u be an element in K_3 which is homogeneous of odd degree such that $u = x \cdot a + y \cdot b + z \cdot c$. Then $a, b, c \in K_3$.*

Proof. Suppose $u = x \cdot a + y \cdot b + z \cdot c$ is of degree p in x, q in y and r in z with $p + q + r = n$ (an odd integer). Let $x_1, \cdots, x_n$ be n symbols of which p denote x, q denote y and r denote z. For convenience we denote ν_3 by ν. For $n = 1$ there is nothing to prove. We now proceed almost word for word as in the proof of the lemma. Assume $n > 1$ and let μ_i be the coefficient of $x_{i+1} \cdots x_n x_1 \cdots x_{i-1}$ ($x_2 \cdots x_n$ if $i = 1$, $x_1 \cdots x_{n-1}$ if $i = n$) and so also of $x_{i-1} \cdots x_1 x_n \cdots x_{i+1}$ ($x_n \cdots x_2$ if $i = 1$, $x_{n-1} \cdots x_1$ if $i = n$) in the expansion in $A^{(3)}$ of $a\nu$ if $x_i = x$, of $b\nu$ if $x_i = y$ and of $c\nu$ if $x_i = z$. Then the coefficient of $x_{i+1} \cdots x_n x_1 \cdots x_i$ ($x_1 \cdots x_n$ if $i = n$) in the expansion in $A^{(3)}$ of $2u\nu$ is $\mu_i + \mu_{i+1}$ ($\mu_n + \mu_1$ if $i = n$). Since distinct monomials in $A^{(3)}$ are linearly independent we have $\mu_i + \mu_{i+1} = 0, i = 1, \cdots, n-1$ and $\mu_n + \mu_1 = 0$. Whence (n being odd) $\mu_i = 0, i = 1, \cdots, n$. Since the argument goes through for any distribution of p x's, q y's and r z's amongst $x_1, \cdots, x_n$ the coefficient of each monomial in the expansion in $A^{(3)}$ of $a\nu$ is zero, i.e. $a \in K_3$. Similarly for b and c.

It is now sufficient for the proof of Theorem 2 to show that each element in M is of the form $x\cdot a + y\cdot b + z\cdot c$. Let N be the subspace of $J^{(3)}$ spanned by elements of this form. We shall write $a \equiv b$ to denote $a - b \in N$: thus we wish to show that $m \equiv 0$ for each $m \in M$. Now M is spanned by elements of the forms $a\cdot(bcdefg + gfedcb)$ and $(a\cdot b)\cdot(cdefg + gfedc)$ where two of a, b, c, d, e, f, g represent x, two represent y and three represent z. It is sufficient to show that each element of the form $(a\cdot b)\cdot(cdefg + gfedc)$ is in N, or by formulae (3) and (4) that each of the following is in N:

$$(1)\quad aR_bR_cR_dR_eR_{f\cdot g} = cR_{a\cdot b}R_dR_eR_{f\cdot g}$$

$$(2)\quad aR_bR_cR_{d\cdot e}R_{f\cdot g} = cR_{a\cdot b}R_{d\cdot e}R_{f\cdot g}$$

$$(3)\quad aR_bR_{c\cdot d}R_eR_{f\cdot g}$$

For types (1) and (2) let $t = a\cdot b\cdot c$. Then we have for (1): $(f\cdot g)R_{t\cdot d\cdot e}$, and for (2): $(f\cdot g)R_{d\cdot e\cdot t}$. Since $R_t = R_{a\cdot b\cdot c}$ it follows by two applications of formula (3) in each case that elements of types (1) and (2) are in N. Since any element in M can be written as zP where P is an operator generated by the right multiplications R_u, $u \in J^{(3)}$, it will be sufficient in the case of elements of type (3) to consider $a = z$. The possibilities, modulo interchange of x and y, are:

(i)	$zR_zR_{x\cdot x}R_zR_{y\cdot y} = xR_xR_{z\cdot z}R_zR_{y\cdot y} = xR_xR_zR_{z\cdot z}R_{y\cdot y} \equiv 0$	(type 2)
(ii)	$zR_zR_{x\cdot x}R_yR_{y\cdot z} \equiv -\frac{1}{2}zR_zR_{x\cdot x}R_zR_{y\cdot y} \equiv 0$	by (i)
(iii)	$zR_xR_{z\cdot x}R_zR_{y\cdot y} \equiv -\frac{1}{2}zR_zR_{x\cdot x}R_zR_{y\cdot y} \equiv 0$	by (i)
(iv)	$zR_xR_{z\cdot x}R_yR_{z\cdot y} \equiv -\frac{1}{2}zR_xR_{z\cdot x}R_zR_{y\cdot y} \equiv 0$	by (iii)
(v)	$zR_zR_{x\cdot y}R_zR_{x\cdot y} = xR_yR_{z\cdot z}R_zR_{x\cdot y} = xR_yR_zR_{z\cdot z}R_{x\cdot y} \equiv 0$	(type 2)
(vi)	$zR_zR_{x\cdot y}R_xR_{z\cdot y} \equiv -\frac{1}{2}zR_zR_{x\cdot x}R_yR_{z\cdot y} \equiv 0$	by (ii)
(vii)	$zR_xR_{z\cdot y}R_zR_{x\cdot y} \equiv -\frac{1}{2}zR_zR_{x\cdot y}R_zR_{x\cdot y} \equiv 0$	by (v)
(viii)	$zR_xR_{z\cdot y}R_xR_{y\cdot z} \equiv -\frac{1}{2}zR_zR_{x\cdot y}R_xR_{y\cdot z} \equiv 0$	by (vi)
(ix)	$zR_yR_{x\cdot x}R_zR_{z\cdot y} = xR_xR_{z\cdot y}R_zR_{z\cdot y} \equiv -\frac{1}{2}xR_xR_{z\cdot z}R_yR_{z\cdot y} \equiv 0$	by (ii)
(x)	$zR_yR_{x\cdot x}R_yR_{z\cdot z} \equiv -2zR_yR_{x\cdot x}R_zR_{y\cdot z} \equiv 0$	by (ix)
(xi)	$zR_xR_{x\cdot y}R_zR_{z\cdot y} \equiv -\frac{1}{2}zR_yR_{x\cdot x}R_zR_{z\cdot y} \equiv 0$	by (ix)
(xii)	$zR_xR_{x\cdot y}R_yR_{z\cdot z} \equiv -\frac{1}{2}zR_yR_{x\cdot x}R_yR_{z\cdot z} \equiv 0$	by (x)
(xiii)	$zR_xR_{z\cdot z}R_xR_{y\cdot y} = xR_zR_{z\cdot z}R_xR_{y\cdot y} = xR_{z\cdot z}R_zR_xR_{y\cdot y} \equiv 0$	(type 1)
(xiv)	$zR_xR_{z\cdot z}R_yR_{x\cdot y} = xR_zR_{z\cdot z}R_yR_{x\cdot y} = xR_{z\cdot z}R_zR_yR_{x\cdot y} \equiv 0$	(type 1)

This completes the proof of Theorem 2.

It is possible to avoid the use of MacDonald's theorem in the proof of Theorem 2 by using the following result, tabulating bases for each subspace spanned by homogeneous elements of degree six and applying the Corollary to Lemma 1 to each subspace spanned by homogeneous elements of degree seven. This process is straightforward, if somewhat tedious, and is in any case largely a special case of MacDonald's theorem. We include Theorem 3, however, as it would appear to be of independent interest, in providing easy verification of proposed five-variable identities linear in each variable.

THEOREM 3. *$K_n \cap L_n = \{0\}$ for $n \leqq 5$.*

Proof. The cases $n = 1, 2, 3$ follow at once from the case $n = 4$ with which we begin, taking the generators of $J^{(4)}$ as x, y, z, t. Let R_b, S_{bc}, U_{bc} denote the mappings $a \to a \cdot b$, $a \to \{abc\}$, $a \to \{bac\}$ respectively. Then $S_{bc} = R_{b \cdot c} + R_b R_c - R_c R_b$, and $U_{bc} = R_b R_c + R_c R_b - R_{b \cdot c}$. Since L_4 is spanned by the elements $tR_x R_y R_z$, $tR_x R_{y \cdot z}$, $tR_{x \cdot y} R_z$ and all others obtained from these by permuting x, y and z and $R_{b \cdot c} = R_b R_c + R_c R_b - U_{bc}$, $2R_b R_c = S_{bc} + U_{bc}$, we have that (again to within permutations of x, y and z) L_4 is spanned by $tR_x S_{yz}$, $tR_x U_{yz}$, $tU_{xy} R_z$. Now let $u \in K_4 \cap L_4$ and suppose that

$$u = \Sigma t(\alpha_{xyz} R_x S_{yz} + \beta_x R_x U_{yz} + \gamma_z U_{xy} R_z)$$

where the summation is over permutations of x, y and z. Since $u \in K_4$ and distinct monomials in $A^{(4)}$ are linearly independent we have

(1) $\alpha_{xyz} = 0$ (coefficient of $txyz$ in $A^{(4)}$) and similarly each α-coefficient is zero, and

(2) $\beta_y + \gamma_z = 0$ (coefficient of $xtyz$ in $A^{(4)}$) and similarly for each pair of distinct subscripts. So $\beta_x = \beta_y = \beta_z = -\gamma_x = -\gamma_y = -\gamma_z$ and u is a scalar multiple of

$$t(R_x U_{yz} + R_y U_{zx} + R_z U_{xy} - U_{xy} R_z - U_{yz} R_x - U_{zx} R_y)$$

which is zero by (5). So $K_4 \cap L_4 = \{0\}$.

The result for $n = 5$ now follows by Lemma 1 and the fact, already noted in the proof of Theorem 2, that L_5 is spanned by the elements $a \cdot b$ where a is a generator and b is linear in each of the other generators.

2. In order to establish the existence of an S-identity of degree 8 we now examine the situation discussed by Albert and Paige in the paper [1] mentioned in the introduction.

Let D be an algebra with an identity element 1 and an involution $d \to \bar{d}$. In the algebra D_n of $n \times n$ matrices with entries in D we can define an involution $M \to \bar{M}'$ by taking $(\bar{M}')_{ij} = (\overline{M_{ji}})$, i.e. $\bar{M}'$ is the conjugate transpose of M. Further, we can define an involution $M \to M^*$ in D_n by choosing a diagonal matrix $\Gamma = \mathrm{diag}\,\{\gamma_1, \cdots, \gamma_n\}$

where the γ_i are self-adjoint ($\bar{\gamma}_i = \gamma_i$), in the nucleus of D and have inverses, and defining $M^* = \Gamma^{-1}\bar{M}'\Gamma$. Such an involution is called a canonical involution in D_n. The particular case in which Γ is the identity matrix reduces to the first involution defined and this is called a standard involution. It is clear that the subset of D_n of matrices self-adjoint under a canonical involution (i.e. $M^* = M$) is closed under the product $A \cdot B = 1/2(AB + BA)$ where AB is the usual matrix product and forms an algebra relative to this product and the usual addition and scalar multiplication. We denote this algebra by $H(D_n, \Gamma)$ or simply $H(D_n)$ if Γ is the identity matrix. With this notation the main theorem proved by Albert and Paige can be stated as:

THEOREM 4. (*Albert and Paige*). *If $H(D_3)$ is the homomorphic image of a special Jordan algebra then D is associative.*

Our first step will be to obtain a three-variable relation, $S(x, y, z) = 0$, which will be easily seen to hold in $J_0^{(3)}$ and so in any homomorphic image of a special Jordan algebra. Substitution of suitable elements x, y, z from $H(D_3)$ will immediately show that D is associative, giving an independent proof of the Albert-Paige result and simultaneously showing that $S(x, y, z) = 0$ is not valid in every Jordan algebra, since an example is known (with D as the eight-dimensional Cayley algebra) of a Jordan algebra $H(D_3)$ in which D is not associative. The homogeneous part of $S(x, y, z) = 0$ of degree 3 in x, 2 in y and 3 in z then gives the required S-identity of degree 8. Lemmas 2 and 3 are essentially due to Albert and Paige.

LEMMA 2. *Let θ be a homomorphism from a special Jordan algebra H, embedded in an associative algebra U, onto a Jordan algebra J such that*

(1) *H is generated by elements X, Y, Z and I (I an identity in U) and*

(2) *H contains elements $E_1, \cdots, E_k$ ($k \geqq 3$) such that $E_iE_j = E_jE_i$ in U and such that $e_1, \cdots, e_k$ ($e_i = E_i\theta$) form a set of orthogonal idempotents in J whose sum is the identity $f = I\theta$ of J. Then, for α, β in the set $1, \cdots, k$ and A a monomial in U generated by X, Y, Z and I we have $(F_\alpha AF_\beta + F_\beta A^* F_\alpha)\theta \in J_{\alpha\beta}$ where $F_\alpha = E_\alpha^2$, $F_\beta = E_\beta^2$, A^* is the reverse of A and $J_{\alpha\beta}$ is the α, β component of J in the Pierce decomposition determined by the e_i's.*

Proof. Let $B = E_\alpha AE_\beta + E_\beta A^* E_\alpha$, $C = A + A^*$. Then

$$\begin{aligned} F_\alpha AF_\beta + F_\beta A^* F_\alpha &= E_\alpha BE_\beta + E_\beta BE_\alpha - (E_\alpha E_\beta)C(E_\alpha E_\beta) \\ &= 2\{E_\alpha BE_\beta\} - \{(E_\alpha \cdot E_\beta)C(E_\alpha \cdot E_\beta)\} \end{aligned}$$

So

$$(F_\alpha A F_\beta + F_\beta A^* F_\alpha)\theta = 2\{e_\alpha(B\theta)e_\beta\} - \{(e_\alpha \cdot e_\beta)(C\theta)(e_\alpha \cdot e_\beta)\} \in J_{\alpha\beta}$$

LEMMA 2′. *With H, J, θ and condition* (1) (*but not condition* (2)) *as in Lemma* 2 *suppose that* $E_1 = 1/2(X^2 + X)$, $E_2 = I - X^2$, $E_3 = 1/2(X^2 - X)$ *and* $X\theta = x$, $I\theta = f$, $E_1\theta = e_1$, $E_2\theta = e_2$, $E_3\theta = e_3$. *Then if* (2)′ $x^3 = x$, *we have that* (a) e_1, e_2, e_3 *are orthogonal idempotents with sum f and* (b) $(E_\alpha A E_\beta + E_\beta A^* E_\alpha)\theta \in J_{\alpha\alpha} + J_{\alpha\beta} + J_{\beta\beta}$.

Proof. (a) This follows immediately from the definitions of e_1, e_2, e_3 and condition (2)′.

(b) Let $B = XA(I - X) + (I - X)A^*X$. Then

$$\begin{aligned} 2(E_1 A E_2 + E_2 A^* E_1)\theta &= \{(I + X)B(I + X)\}\theta \\ &= \{(2E_1 + E_2)B(2E_1 + E_2)\}\theta \\ &= \{(2e_1 + e_2)(B\theta)(2e_1 + e_2)\} \in J_{11} + J_{12} + J_{22} \end{aligned}$$

Similarly for other choices of α and β.

LEMMA 3. *With notation as in Lemma* 2′:

$$\begin{aligned} &2[(E_\alpha A E_\beta + E_\beta A^* E_\alpha)\cdot(E_\beta D E_\gamma + E_\gamma D^* E_\beta)]\theta \\ &= [E_\alpha A E_\beta E_\beta D E_\gamma + E_\gamma D^* E_\beta E_\beta A^* E_\alpha]\theta \end{aligned}$$

where D is a monomial in U generated by X, Y, Z and I, and α, β, γ are distinct integers chosen from 1, 2, 3.

Proof.

$$\begin{aligned} &2[(E_\alpha A E_\beta + E_\beta A^* E_\alpha)\cdot(E_\beta D E_\gamma + E_\gamma D^* E_\beta)]\theta \\ &= (E_\alpha A E_\beta E_\beta D E_\gamma + E_\gamma D^* E_\beta E_\beta A^* E_\alpha)\theta \\ &\quad + (E_\alpha A E_\beta E_\gamma D^* E_\beta + E_\beta D E_\gamma E_\beta A^* E_\alpha)\theta \\ &\quad + (E_\beta A^* E_\alpha E_\beta D E_\gamma + E_\gamma D^* E_\beta E_\alpha A E_\beta)\theta \\ &\quad + (E_\beta A^* E_\alpha E_\gamma D^* E_\beta + E_\beta D E_\gamma E_\alpha A E_\beta)\theta \,. \end{aligned}$$

Now, since α, β and γ are distinct, $J_{\alpha\beta}J_{\beta\gamma} \subseteqq J_{\alpha\gamma}$. So, by Lemma 2′, the left-hand-side is in $J_{\alpha\gamma}$. The result now follows from Lemma 2′ and the disjointness of the Peirce decomposition.

COROLLARY.

$$\begin{aligned} &4[\{E_1 Z E_2\}\cdot(E_2 A E_2 E_2 Z E_3 + E_3 Z E_2 E_2 A^* E_2)]\theta \\ &= 4[(E_1 Z E_2 E_2 A E_2 + E_2 A^* E_2 E_2 Z E_1)\cdot\{E_2 Z E_3\}]\theta \end{aligned} \tag{6}$$

Equation (6) suggests the following relation in U:

$$
(7)\quad
\begin{aligned}
&4[\{E_1ZE_2\}\cdot(E_2CE_2E_2ZE_3 + E_3ZE_2E_2C^*E_2)]\\
&\quad - (E_2ZE_1E_2CE_2E_2ZE_3 + E_3ZE_2E_2C^*E_2E_1ZE_2)\\
&\quad - (E_1ZE_2E_3ZE_2E_2C^*E_2 + E_2CE_2E_2ZE_3E_2ZE_1)\\
&\quad - (E_2ZE_1E_3ZE_2E_2C^*E_2 + E_2CE_2E_2ZE_3E_1ZE_2)\\
&= 4[(E_1ZE_2E_2CE_2 + E_2C^*E_2E_2ZE_1)\cdot\{E_2ZE_3\}]\\
&\quad - (E_1ZE_2E_2CE_2E_3ZE_2 + E_2ZE_3E_2C^*E_2E_2ZE_1)\\
&\quad - (E_2C^*E_2E_2ZE_1E_2ZE_3 + E_3ZE_2E_1ZE_2E_2CE_2)\\
&\quad - (E_2C^*E_2E_2ZE_1E_3ZE_2 + E_2ZE_3E_1ZE_2E_2CE_2)
\end{aligned}
$$

where, for reasons which will appear later, we take $C = YXZY$ and $C^* = YZXY$. In turn, (7) suggests the following relation in $J_0^{(3)}$, (this is the relation referred to previously as $S(x, y, z) = 0$)

$$
(8)\quad
\begin{aligned}
&4\{e_1ze_2\}\cdot p_1 + \{(e_2 + 2e_3)q_1(e_2 + 2e_3)\}\\
&\quad - \{(2e_1 + e_2)r_1(2e_1 + e_2)\} - \{e_2s_1e_2\}\\
&= 4\{e_2ze_3\}\cdot p_2 - \{(2e_1 + e_2)q_2(2e_1 + e_2)\}\\
&\quad + \{(e_2 + 2e_3)r_2(e_2 + 2e_3)\} - \{e_2s_2e_2\}
\end{aligned}
$$

where $e_1 = 1/2(x^2 + x)$, $e_2 = 1 - x^2$, $e_3 = 1/2(x^2 - x)$ and $p_1, 2q_1, 2r_1, s_1, p_2, 2q_2, 2r_2, s_2$ are Jordan elements in $J_0^{(3)}$ equal respectively in $A^{(3)}$ to

$$
\begin{aligned}
&e_2yxzye_2e_2ze_3 + e_3ze_2e_2yzxye_2\ ,\\
&(1 + x)ze_1e_2yxzye_2e_2zx + xze_2e_2yzxye_2e_1z(1 + x)\ ,\\
&xze_2e_3ze_2e_2yzxy(1 - x) + (1 - x)yxzye_2e_2ze_3e_2zx\ ,\\
&ze_1e_3ze_2e_2yzxy + yxzye_2e_2ze_3e_1z\ ,\\
&e_1ze_2e_2yxzye_2 + e_2yzxye_2e_2ze_1\ ,\\
&xze_2e_2yxzye_2e_3z(1 - x) + (1 - x)ze_3e_2yzxye_2e_2zx\ ,\\
&(1 + x)yzxye_2e_2ze_1e_2zx + xze_2e_1ze_2e_2yxzy(1 + x)\ ,\\
&yzxye_2e_2ze_1e_3z + ze_3e_1ze_2e_2yxzy\ .
\end{aligned}
$$

Now, (8) is an S-identity. By construction it holds in $J_0^{(3)}$ and we may see that it does not hold in $H(C_3)$, where C is the eight-dimensional Cayley algebra, by substituting

$$
x = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 \end{pmatrix},\quad
y = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix},\quad
z = \begin{pmatrix} \cdot & u & v \\ \bar{u} & \cdot & w \\ \bar{v} & \bar{w} & \cdot \end{pmatrix}
$$

where u, v and w are arbitrary elements in C, and examining the 1, 3 element on each side of (8). The calculation is quite simple: by choice of x, the only nonzero contribution on each side arises from the first term. Further, p_1 and p_2 are of degree two in z and so may be evaluated as though C were associative, that is by substituting directly into their equivalent associative forms displayed above. The result is $u[(v-\bar{v})w]$ on the left and $[u(v-\bar{v})]w$ on the right. Since self-adjoint elements in C are in any case in the nucleus we have $u[(v+\bar{v})w] = [u(v+\bar{v})]w$ Whence $u(vw) = (uv)w$. But C is not associative. So (8) does not hold in the Jordan algebra $H(C_3)$ and is thus an S-identity.

The relation (8) can be written as $\sum_{i=3}^{15} f_i(x, y, z) = 0$, where $f_i(x, y, z)$ is a Jordan polynomial of degree i in x. Now $f_i(x, y, z)$ can be expanded in $A^{(3)}$ as a linear combination of monomials in x, y, z of degree i in x. Since $A^{(3)}$ is free, $f_i(x, y, z) = 0$ for each i. We consider the case $i = 3$.

The parts of the terms of (8) which are of degree 3 in x are equal respectively in $A^{(3)}$ to:

(a) $\quad -4(x\cdot z)\cdot(yxzyzx + xzyzxy)$

(b) $\quad zxyxzyzx + xzyzxyxz$

(c) $\quad xzxzyzxy + yxzyzxzx$

(d) $\quad zxxzyzxy + yxzyzxxz$

(e) $\quad -4(z\cdot x)\cdot(xzyxzy + yzxyzx)$

(f) $\quad xzyxzyxz + zxyzxyzx$

(g) $\quad yzxyzxzx + xzxzyxzy$

(h) $\quad yzxyzxxz + zxxzyxzy$

We now make the following choices for Jordan expressions of the above:

$$\text{(a)} + \text{(c)} + \text{(d)}:\ -4\{(x\cdot z)y\{x\{zyz\}x\}\}$$
$$\text{(b)}:\ 4\{\{z\{xyx\}z\}y(z\cdot x)\} - 2\{z\{x\{y(x\cdot z)y\}x\}z\}$$
$$\text{(e)} + \text{(f)} + \text{(g)} + \text{(h)}:\ -2\{x\{z\{y(x\cdot z)y\}z\}x\}$$

and obtain the following relation which clearly holds identically in $J_0^{(3)}$:

$$\begin{aligned} &4\{\{z\{xyx\}z\}y(z\cdot x)\} - 2\{z\{x\{y(x\cdot z)y\}x\}z\} \\ &\quad = 4\{(x\cdot z)y\{x\{zyz\}x\}\} - 2\{x\{z\{y(x\cdot z)y\}z\}x\} \end{aligned} \tag{9}$$

Substitution in (9) of the same elements as were substituted in (8) shows that (9) is an S-identity.

3. In $H(D_3)$ let

$$g = \begin{pmatrix} \alpha & p & q \\ \bar{p} & \beta & r \\ \bar{q} & \bar{r} & \gamma \end{pmatrix}, \quad x = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad y = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}.$$

Then we have

$$\{xgx\} = \begin{pmatrix} \beta & \bar{p} & \cdot \\ p & \alpha & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad \text{and} \quad \{ygy\} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \gamma & \bar{r} \\ \cdot & r & \beta \end{pmatrix},$$

while xgy (ordinary matrix multiplication) is equal to

$$\begin{pmatrix} \cdot & r & \beta \\ \cdot & q & p \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

With these results in mind, (8) suggests the following candidate for an S-identity:

$$\begin{aligned} 2\{xzx\}\cdot\{x\{zy^2z\}y\} &- \{x\{z\{x\{yzy\}y\}z\}x\} \\ = 2\{x\{zx^2z\}y\}\cdot\{yzy\} &- \{y\{z\{y\{xzx\}x\}z\}y\} \end{aligned} \tag{10}$$

We verify that (10) is an S-identity by using it to prove the Albert-Paige Theorem in a slightly strengthened form. (Albert and Paige mention that their method will give the stronger result but do not give the details.)

THEOREM 4(a). *If $H(D_n, \Gamma)$, $n \geqq 3$, is the homomorphic image of a special Jordan algebra then D is associative.*

[Theorem 4(a) is also a stronger form of a theorem due to Jacobson [3] viz: If $H(D_n, \Gamma)$, $n \geqq 3$, is a special Jordan algebra then D is associative.]

Proof of Theorem 4(a). It is sufficient to prove the result for $n = 3$. Since $H(D_3, \Gamma)$ is the homomorphic image of a special Jordan algebra the relation (10), which clearly holds in $J_0^{(3)}$, holds in $H(D_3, \Gamma)$. Now suppose that

$$\Gamma = \begin{pmatrix} \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot \\ \cdot & \cdot & \gamma \end{pmatrix}$$

and let

$$x = \begin{pmatrix} \cdot & \beta & \cdot \\ \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad y = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \gamma \\ \cdot & \beta & \cdot \end{pmatrix}, \quad z = \begin{pmatrix} \cdot & \bar{u}\beta & v\gamma \\ u\alpha & \cdot & \bar{w}\gamma \\ \bar{v}\alpha & w\alpha & \cdot \end{pmatrix},$$

where u, v and w are arbitrary elements in D. Substitution in (10) gives, in the first row, third column:
left hand side: $\beta u\alpha\beta(\alpha v\gamma\beta\gamma w\beta\gamma)$
right hand side: $(\beta u\alpha\beta\alpha v\gamma\beta)\gamma w\beta\gamma$

Since u, v and w are arbitrary and α, β and γ are in the nucleus of D with inverses the result follows at once.

REMARK. It can be shown by using the corollary to Lemma 1 that the S-identity (10) is generated by S-identities of degree 8. We do not give the details here as we hope to embody them in a later paper.

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