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APPROXIMATION THEOREMS FOR MARKOV OPERATORS

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Let $(X, \mathscr{F}, m)$ be a totally σ-finite measure space. A Markov operator (with invariant measure m) is a positive operator T on $L_\infty(X, \mathscr{F}, m)$ such that $T1 = 1$ and $\int Tfdm = \int fdm$ for all $f \in L_1(X, \mathscr{F}, m) \cap L_\infty(X, \mathscr{F}, m)$. If φ is an invertible measure-preserving transformation of $(X, \mathscr{F}, m)$, then φ determines a Markov operator T_φ by the formula $T_\varphi f(x) = f(\varphi x)$. The set M of all Markov operators is convex and each T_φ is an extreme point.

In case $(X, \mathscr{F}, m)$ is a finite, homogeneous, nonatomic measure space, M may be identified with the set of all doubly stochastic measures on the product space $(X \times X, \mathscr{F} \times \mathscr{F}, m \times m)$. The main result of the present paper is that M is compact in the weak operator topology of operators on $L_2(X, \mathscr{F}, m)$ and that the set Φ of operators T_φ is dense in M. It follows that M is the closed convex hull of Φ in the strong operator topology. We shall further show that Φ is closed in the uniform operator topology and that the closure of Φ in the strong operator topology is the set Φ_1 of all (not necessarily invertible) measure-preserving transformations of $(X, \mathscr{F}, m)$.

We shall denote $L_p(X, \mathscr{F}, m)$, $1 \leqq p \leqq \infty$, more briefly by L_p. The operators T_φ arising from invertible measure-preserving transformations are in certain respects the most pleasant of all Markov operators (in particular, they are unitary). Therefore, it seems worthwhile to determine what role they play in the structure of M.

Interest in special cases of this problem is indicated by the attention which has been devoted to the solution of Birkhoff's Problem 111, which is concerned with the case of a denumerable space X with a measure m uniformly distributed on the points of X. Thus J. R. Isbell [4], B. A. Rattray and J. E. L. Peck [12] and D. G. Kendall [6] have given approximation theorems for doubly stochastic matrices in terms of convex combinations of permutation matrices. A second type of solution has been given by Kendall [6] and by Isbell [5]. They have shown that the set of permutation matrices coincides with the set of extreme points of M. Still a third type of solution has been offered by P. Révész [13], who has shown (essentially) that every

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doubly stochastic matrix is an integral over the set of permutation matrices.

In the case of the real line X with Lebesgue measure m, Peck [11] has given a solution of the first type above for the set M of what he calls doubly stochastic measures. He also alludes to the corresponding result for the unit interval with Lebesgue measure. In § 3 we shall show that, for finite measure spaces, the set of doubly stochastic measures can be identified with the set of Markov operators. This is *not* the case for a nonfinite measure space. However, approximation theorems for Markov operators always imply the corresponding approximation theorems for doubly stochastic measures or matrices.

In this paper we shall restrict our attention to finite measure spaces. The σ-finite case is more complicated and will be considered at a later time.

In § 4 we give a solution of the above-mentioned problem (Theorem 1) for the case of a nonatomic, finite measure space. This is an approximation theorem like that of Peck [11], but is a stronger result in that the convex closure of the set Φ of invertible measure-preserving transformations of $(X, \mathscr{F}, m)$ is replaced by the closure of Φ. The topology on M is the weak operator topology for operators on L_2. Simple examples can be constructed to show that the second and third types of solutions mentioned above for doubly stochastic matrices do not extend to doubly stochastic measures on a nonatomic measure space. The Choquet representation theorem can be invoked to give solutions of the third type as integrals over the set of extreme points of M. These extreme points have recently been characterized by J. Lindenstrauss [7], but this approach will not be pursued here.

In § 5 we consider the closure and the convex closure of Φ in the strong operator topology and the uniform operator topology. The results of that section are obtained directly from Theorem 1 and an interesting geometric characterization (Theorem 5) of the operators T_φ arising from measure-preserving transformations.

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2. Markov operators. A Markov process with discrete time parameter and state space $(X, \mathscr{F})$ is determined (cf. [1], p. 190) by a *stochastic transition function* $P(x, B)$, i.e. a nonnegative function of $x \in X$, $B \in \mathscr{F}$ such that

(i) $P(x, B)$ is a probability measure in B for each fixed $x \in X$;

(ii) $P(x, B)$ is a measurable function of x for each fixed $B \in \mathscr{F}$.
We assume, in addition, that m is an *invariant measure* for $P(x, B)$ in the sense that

$$\text{(iii)} \qquad \int_X P(x, B)m(dx) = m(B) \ , \qquad B \in \mathscr{F} \ .$$

Under these conditions $P(x, B)$ defines a bounded linear operator T on L_∞ by the formula

$$(2) \qquad Tf(x) = \int_X f(y)P(x, dy) \ .$$

The operator T clearly has the properties:

$$(3) \qquad f \geqq 0 \Rightarrow Tf \geqq 0$$

$$(4) \qquad T1 = 1$$

$$(5) \qquad \int_X Tf(x)m(dx) = \int_X f(x)m(dx) \ , \qquad f \in L_\infty \ .$$

For instance, equation (5) follows from condition (iii) when f is a characteristic function of a measurable set and more generally by an approximation argument. Likewise, (4) follows from (i). We shall refer to any linear operator T on L_∞ which satisfies (3), (4) and (5) as a *Markov operator with invariant measure m*, or simply a Markov operator.

It follows from (3) and (4) that T is a positive operator on L_∞ with $\| T \|_\infty = 1$ and from (5) that T may be uniquely extended to a positive operator on L_1 with $\| T \|_1 = 1$. This extension is again given by (2). According to the Riesz convexity theorem, T is a contraction operator on L_p for each p, $1 \leqq p \leqq \infty$. That is, T is a positive operator with $\| T \|_p \leqq 1$.

The adjoint T^* of T defined by

$$(6) \qquad (T^*f, g) = (f, Tg) = \int_X f(x)Tg(x)m(dx)$$

for $f \in L_p$, $g \in L_q$, $1 \leqq p$, $q \leqq \infty$, $1/p + 1/q = 1$, is also a Markov operator. Indeed, equation (5) is equivalent to

$$(5^*) \qquad T^*1 = 1 \ .$$

Thus equations (3), (4) and (5*) may be taken as the definition of a Markov operator.

In general, a Markov operator can not be defined in terms of a stochastic transition function. However, under suitable separability restrictions on $(X, \mathscr{F}, m)$ there will always exist a stochastic transition function $P(x, B)$ such that (2) holds almost everywhere for each $f \in L_\infty$ (cf. [1], pp. 29 ff).

A measurable transformation (function, mapping) φ from $(X, \mathscr{F})$ into itself is said to be *measure-preserving* if

$$(7) \qquad m(\varphi^{-1}B) = m(B) , \qquad B \in \mathscr{F} .$$

It follows that φ is essentially onto, i.e. the complement of its range has measure zero. As usual we shall identify functions which are equal almost everywhere. Hence we may assume that φ is onto. If it is also one-to-one and if φ^{-1} is measurable, we shall say that φ is an *invertible measure-preserving transformation*. It follows that φ^{-1} is also measure-preserving.

The correct notion for our purposes is actually that of a measure-preserving set function from $\mathscr{F}$ into $\mathscr{F}$. However, for the measure spaces that we shall be considering in this paper, namely, products of the unit interval with the product Lebesgue measure, every measure-preserving set transformation ψ is given by a measure-preserving point transformation φ, $\psi(B) = \varphi^{-1}(B)$.

Let us denote by Φ the set of all invertible measure-preserving transformations of $(X, \mathscr{F}, m)$ (or rather all equivalence classes of such transformations modulo null transformations) and by M the set of all Markov operators. Then Φ may be identified with a subset of M by the correspondence $\varphi \to T_\varphi$ where

$$(9) \qquad T_\varphi f(x) = f(\varphi x) , \qquad f \in L_p .$$

We shall show in § 4 that, subject to a homogeneity condition on $(X, \mathscr{F}, m)$, Φ is dense in M in the weak operator topology.

3. Doubly stochastic measures. It is well known that any finite measure space $(X, \mathscr{F}, m)$ with $m(X) = 1$ which is nonatomic and for which there exists a countable class $\mathscr{F}_0$ of measurable sets that generates $\mathscr{F}$ is measure-theoretically equivalent to the unit interval. That is, there exists a one-to-one mapping (modulo sets of measure zero) of $\mathscr{F}$ onto the class of Borel subsets of the unit interval which preserves set operations and the measure ([3], p. 173).

More generally, if m is any finite measure, it can be shown [8] that X is essentially a countable union of measurable sets X_n which are measure-theoretically equivalent either to a single point or to a product of intervals with the product Lebesgue measure. The measure spaces $(X_n, \mathscr{F} \cap X_n, m)$ are *homogeneous* in the sense that

(i) there exists a class $\mathscr{F}_n$ of measurable subsets of X_n which generates $\mathscr{F} \cap X_n$ and

(ii) for each subset Y of X_n of positive measure, the σ-algebra

$\mathscr{F} \cap Y$ is not generated by any class of smaller cardinality than that of $\mathscr{F}_n$. The cardinality of $\mathscr{F}_n$ is called the *character* of X_n and determines the number of copies of the unit interval which go into its representation as a product measure space. It is clear that, except for the atoms (character 1), each of the spaces X_n may be assumed to be of different character, hence not measure-theoretically equivalent to each other. In particular, any invertible measure-preserving transformation of X must leave invariant each of the nonatomic X_n as well as their union.

Now suppose that $X = X_1 \cup X_2$ where X_1 and X_2 are disjoint and X_1 is either one of the nonatomic X_n of the preceding paragraph or the union of all of the atoms. Consider the Markov operator T defined on L_p by

$$Tf(x) = \int_X f(y)m(dy) .$$

For this operator we have

$$(\chi_{X_1}, T\chi_{X_2}) = \int_X \chi_{X_1}(x)T\chi_{X_2}(x)m(dx) = m(X_1)m(X_2) .$$

(Here and in the remainder of the paper χ_A denotes the characteristic function of the set A.) On the other hand, if φ is any measure preserving transformation of X, then we know that

$$(\chi_{X_1}, T_\varphi \chi_{X_2}) = m(X_1 \cap \varphi^{-1}X_2) = 0 .$$

Thus T cannot be a limit of convex combinations of elements of Φ in any operator topology. We therefore assume that $(X, \mathscr{F}, m)$ is nonatomic and homogeneous. As noted above, this implies that $(X, \mathscr{F}, m)$ is essentially a product of unit intervals with the product Lebesgue measure. As such it has a natural topology, the product topology, in which X is compact, $\mathscr{F}$ coincides with the class of Borel sets of X and m is a regular Borel measure.

Now let T be a Markov operator on L_∞. We shall denote the product measure space $(X \times X, \mathscr{F} \times \mathscr{F})$ by $(X^2, \mathscr{F}^2)$. We shall further denote the algebra of finite unions of measurable rectangles $A \times B$, $A, B \in \mathscr{F}$, by $\mathscr{F}_0^2$. For each such rectangle we define

$$(10) \qquad \lambda(A \times B) = (\chi_A, T\chi_B) .$$

Since λ is additive in A and B individually, it follows that it can be uniquely extended, by additivity, to a finitely additive, nonnegative set function on $\mathscr{F}_0^2$. Moreover,

$$(11) \qquad \lambda(A \times X) = \lambda(X \times A) = m(A) , \qquad A \in \mathscr{F}.$$

We shall show that λ is countably additive. This is a special case of a theorem of E. Marczewski and C. Ryll-Nardzewski [9, 10] on non-direct products of compact measures. For completeness and because the special case is much simpler than the general case, we include a proof.

Any nonnegative, finitely additive set function λ on $\mathscr{F}_0^2$ which satisfies (11) will be said to be *doubly stochastic.*

LEMMA. *Let λ be a doubly stochastic set function on $\mathscr{F}_0^2$. Then λ is countably additive and regular, hence it has a unique extension to a doubly stochastic measure on $\mathscr{F}^2$.*

Proof. If A and B are Borel subsets of X, then by the regularity of m there exist compact sets A_1 and B_1 such that $A_1 \subset A$, $B_1 \subset B$, $m(A - A_1) < \varepsilon$ and $m(B - B_1) < \varepsilon$, where ε is any preassigned positive number. It follows that $A_1 \times B_1 \subset A \times B$, $A_1 \times B_1$ is compact, and

$$\begin{aligned}\lambda(A \times B - A_1 \times B_1) &\leqq \lambda(A \times (B - B_1)) + \lambda((A - A_1) \times B) \\ &\leqq m(B - B_1) + m(A - A_1) < 2\varepsilon .\end{aligned}$$

Thus any set in $\mathscr{F}_0^2$ can be approximated from the inside by compact sets, i.e. λ is regular. It follows by Alexandroff's theorem ([2], p. 138) that λ is countably additive. The existence and uniqueness of the extension then follow from the Hahn extension theorem.

COROLLARY. *The relation* (10) *determines a one-to-one correspondence between the set M of Markov operators on L_∞ and the set of all doubly stochastic measures on $\mathscr{F}^2$.*

Proof. We have shown that each Markov operator T determines a unique doubly stochastic measure λ satisfying (10). Conversely, suppose that λ is a doubly stochastic measure. Let $g \in L_\infty$ and let f be a simple function on $(X, \mathscr{F}, m)$. Set

$$G(f) = \int f(x)g(y)\lambda(dx, dy) . \tag{12}$$

Then

$$\begin{aligned}|G(f)| &\leqq \|g\|_\infty \int |f(x)| \lambda(dx, dy) \\ &= \|g\|_\infty \|f\|_1 .\end{aligned}$$

Thus (12) defines for each $g \in L_\infty$ a bounded linear functional G on a dense subset of L_1. It follows that there exists a function $Tg \in L_\infty$ such that

$$(f,\ Tg) = \int f(x)g(y)\lambda(dx,\ dy) \tag{13}$$

for all $f \in L_1$. If $g \geqq 0$, then G is a positive functional so that $Tg \geqq 0$. Thus T is a positive linear operator on L_∞. Clearly, $T1 = 1$. Moreover,

$$\int Tg(x)m(dx) = \int g(y)\lambda(dx,\ dy)\ .$$

Since λ is doubly stochastic, it follows that $\int Tg = \int g$ for all simple functions g and hence for all $g \in L_\infty$. Thus T is a Markov operator. Since (10) is clearly a special case of (13), the proof is complete.

Suppose that $T = T_\varphi$ is determined by a measure-preserving transformation $\varphi \in \Phi$. Then according to (10) and (9) φ determines the doubly stochastic measure λ_φ defined by

$$\lambda_\varphi(A \times B) = m(A \cap \varphi^{-1}B)\ . \tag{14}$$

Thus λ_φ is concentrated on the graph of φ. Such measures are sometimes referred to as *permutation measures*.

4. Weak approximation. We are now in a position to prove the main result of this paper.

THEOREM 1. *M is a compact convex set of operators and Φ is dense in M in the weak operator topology of L_p, $1 < p < \infty$. If $(X, \mathscr{F}, m)$ is a separable measure space, then M is metrizable.*

Proof. The convexity of M is clear. Let us show that M is compact. Suppose that T belongs to the closure of M in the weak operator topology of bounded linear operators on L_p. Since $L_\infty \subset L_p \cap L_q$, where $1/p + 1/q = 1$, and since $(f,\ Tg)$ is a continuous function of T for each fixed $f \in L_p$, $g \in L_q$, it follows that T has the properties (3)–(5) of § 2. For instance, (3) is equivalent to

$$f \geqq 0,\ g \geqq 0 \Rightarrow (f,\ Tg) \geqq 0\ .$$

It follows from (3) and (4) that T maps L_∞ into itself. Thus T is a Markov operator and M is closed. Since L_p is reflexive, the closed unit sphere in the space of bounded linear operators on L_p is compact ([2], p. 512). Since $\|T\|_p \leqq 1$ for each $T \in M$, it follows that M is compact.

Note that each of the weak operator topologies corresponding to different values of p is stronger than the weak topology on M determined by the functionals $(f,\ Tg)$ for $f,\ g \in L_\infty$. Since the latter is, nevertheless, a Hausdorff topology and since M is compact in each of the weak operator topologies, it follows that they all coincide and hence there is no ambiguity in referring to the weak operator topology on M.

Now suppose that $(X, \mathscr{F}, m)$ is separable. Then L_p and L_q are separable in their norm topologies. Let $\{f_n\}$ and $\{g_n\}$ be dense sequences in L_p and L_q, respectively. The series

$$\rho(S, T) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2^{n+m}} \cdot \frac{|(f_n, (T-S)g_m)|}{1 + |(f_n, (T-S)g_m)|}$$

is uniformly convergent in S and T and thus defines a continuous (weak operator topology) metric on M. It follows that the resulting metric topology is weaker than the weak operator topology and hence, by the compactness of M, that the two topologies coincide.

It only remains to show that Φ is dense in M. A basis for the weak operator topology of M is given by all sets of the form

$$\{T: |(f_k, Tg_k) - (f_k, T_0 g_k)| < \varepsilon,\ k = 1, \cdots, n\} \tag{15}$$

where f_k and g_k run through a dense subset of L_2, $T_0 \in M$ and $\varepsilon > 0$. In particular, we may take f_k and g_k to be continuous. (Recall that X has a natural topology for which X is compact and m is regular.) In this case they are bounded and, by the arbitrariness of ε in (15), we may assume that they are bounded by 1. Let $T_0 \in M$. We shall show that there exists a measure-preserving transformation φ such that T_φ belongs to the set (15).

For $A, B \in \mathscr{F}$ we introduce the notation

$$\lambda_0(A \times B) = (\chi_A, T_0 \chi_B)$$
$$\lambda_\varphi(A \times B) = (\chi_A, T_\varphi \chi_B) = m(A \cap \varphi^{-1}B)\ , \qquad \varphi \in \Phi\ .$$

According to the lemma of § 3, λ_0 and λ_φ may be extended to doubly stochastic measures on $(X^2, \mathscr{F}^2)$. Let us set $h_k(x, y) = f_k(x) g_k(y)$, $k = 1, \cdots, n$. According to (13), we have

$$(f_k, T_0 g_k) = \int_{X^2} h_k d\lambda_0$$

and

$$(f_k, T_\varphi g_k) = \int_{X^2} h_k d\lambda_\varphi\ .$$

Since each h_k is uniformly continuous on X^2, it follows that there exist disjoint sets $X_1, \cdots, X_s \in \mathscr{F}$ such that $X = \bigcup_{i=1}^{s} X_i$ and the oscillation of each h_k is less than $\varepsilon/3$ on each rectangle $X_i \times X_j$, $i, j = 1, \cdots, s$. Since λ_0 is doubly stochastic, we can choose, for each $i = 1, \cdots, s$, disjoint measurable subsets X_{ij} of X_i such that $m(X_{ij}) =$

$\lambda_0(X_i \times X_j)$, $j = 1, \cdots, s$. (Recall that X is nonatomic.) Similarly, for each $j = 1, \cdots, s$ we can choose disjoint measurable subsets Y_{ij} of X_j such that $m(Y_{ij}) = \lambda_0(X_i \times X_j)$, $i = 1, \cdots, s$. For each i and j there exists an invertible measure-preserving transformation φ_{ij} of X_{ij} onto Y_{ij} since X_{ij} and Y_{ij} are both homogeneous with the same character and the same measure. Define φ on X by

$$\varphi(x) = \varphi_{ij}(x) , \qquad x \in X_{ij} .$$

We shall show that T_φ belongs to the set (15).

Since φ maps each X_{ir} onto $Y_{ir} \subset X_r$ and since $X_i = \bigcup_{r=1}^{s} X_{ir}$, it follows that

$$\lambda_\varphi(X_i \times X_j) = m(X_i \cap \varphi^{-1}X_j) = m(X_{ij}) = \lambda_0(X_i \times X_j)$$

for each $i, j = 1, \cdots, s$. Recalling that the oscillation of each of the functions h_k is less than $\varepsilon/3$ on each of the sets $X_i \times X_j$ and that $|h_k(x, y)| \leqq 1$, we have

$$\begin{aligned}
|(f_k, T_\varphi g_k) - (f_k, T_0 g_k)| &= \left| \int_{X^2} h_k d\lambda_\varphi - \int_{X^2} h_k d\lambda_0 \right| \\
&\leqq \sum_{i,j=1}^{s} \left| \int_{X_i \times X_j} h_k d\lambda_\varphi - \int_{X_i \times X_j} h_k d\lambda_0 \right| \\
&\leqq \sum_{i,j=1}^{s} \{\varepsilon/3[\lambda_\varphi(X_i \times X_j) + \lambda_0(X_i \times X_j)] \\
&\quad + |\lambda_\varphi(X_i \times X_j) - \lambda_0(X_i \times X_j)|\} \\
&= 2\varepsilon/3 \sum_{i,j=1}^{s} \lambda_0(X_i \times X_j) < \varepsilon
\end{aligned}$$

and the proof of Theorem 1 is complete.

5. **Strong and uniform approximation.** Since convex sets have the same closure in the weak operator and the strong operator topologies ([2], p. 447), we immediately obtain the following approximation theorem.

THEOREM 2. *M is the closed convex hull of Φ in the strong operator topology.*

It is natural to ask whether Theorems 1 and 2 can be strengthened to give M as the closure of Φ in the strong operator topology. The answer, at least on L_2, is negative as we shall now show.

Henceforth, all operator topologies will refer to operators on L_2.

Let us denote by Φ_1 the set of all (not necessarily invertible) measure-preserving transformations of $(X, \mathscr{F}, m)$. Again we identify Φ_1 with a subset of M by setting $T_\varphi f(x) = f(\varphi x)$, $f \in L_2$. It follows that $\Phi \subset \Phi_1 \subset M$.

THEOREM 3. *Φ_1 is the closure of Φ in the strong operator topology.*

THEOREM 4. *Φ is closed in the uniform operator topology.*

Theorems 3 and 4 follow easily from Theorem 2 and a pair of simple algebraic propositions which we give as Theorem 5 below. Note that nothing is said about the closed convex hull of Φ in the uniform operator topology. This is apparently still an open question.

THEOREM 5. *Let T be a bounded linear operator on L_2. Then $T \in \Phi_1$ if and only if T is doubly stochastic and isometric; $T \in \Phi$ if and only if T is doubly stochastic and unitary.*

Proof. It is well known that every $T \in \Phi_1$ is isometric while every $T \in \Phi$ is unitary. Moreover, T is unitary if and only if both T and T^* are isometric. Thus the second part of the theorem follows from the first. It only remains to show that every $T \in M$ which is isometric is induced by a measure-preserving transformation φ of X.

Suppose that $T \in M$ is isometric and let A and B be measurable subsets of X. Then

$$\begin{aligned}\int_X (T\chi_A)(T\chi_B)dm &= (T\chi_A, T\chi_B) = (\chi_A, \chi_B) \\ &= (\chi_{A\cap B}, 1) = (T\chi_{A\cap B}, 1) \\ &= \int_X T\chi_{A\cap B}\, dm\ . \end{aligned} \tag{16}$$

However, since T is positive and $0 \leqq \chi_A \leqq 1$, we have that $0 \leqq T\chi_A \leqq T1 = 1$ and so

$$0 \leqq (T\chi_A)^2 \leqq T\chi_A \leqq 1\ . \tag{17}$$

It follows from (17) and (16) with $A = B$ that $(T\chi_A)^2 = T\chi_A$ and hence that $T\chi_A$ is (essentially) the characteristic function of some measurable set. Let us denote this set by $\psi(A)$.

Using the positivity of T again we have, moreover, that $T\chi_{A\cap B} \leqq \min\{T\chi_A, T\chi_B\}$ so that

$$0 \leqq T\chi_{A\cap B} = (T\chi_{A\cap B})^2 \leqq (T\chi_A)(T\chi_B)\ . \tag{18}$$

From (16) and (18) we see that

$$T\chi_{A\cap B} = (T\chi_A)(T\chi_B)$$

or

$$\psi(A \cap B) = \psi(A) \cap \psi(B)$$

for all measurable sets A and B. Thus ψ preserves finite intersections. From the relations $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ and $\chi_{A'} = 1 - \chi_A$ and the fact that $T1 = 1$, it follows that ψ preserves finite unions and complements as well. From the relation

$$m(\psi(A)) = (T\chi_A, T1) = (\chi_A, 1) = m(A)$$

it follows that ψ preserves countable unions and intersections as well as the measure.

It follows that there exists a measure-preserving transformation φ of $(X, \mathscr{F}, m)$ such that $\psi(A) = \varphi^{-1}(A)$ for all $A \in \mathscr{F}$. Thus $T\chi_A(x) = \chi_{\psi(A)}(x) = \chi_A(\varphi x)$. It follows that T coincides with T_φ on all simple functions and hence on all of L_2. This completes the proof of Theorem 5.

It follows immediately from Theorem 5 that Φ is closed in the uniform operator topology and that Φ_1 is closed in the strong operator topology. According to Theorem 2, every element $T \in \Phi_1$ is the limit in the weak operator topology of a convergent net T_α of elements of Φ. It follows that

$$\|(T - T_\alpha)f\|_2^2 = 2(f, f) - (Tf, T_\alpha f) - (T_\alpha f, Tf) \rightarrow 0$$

for each $f \in L_2$. Thus $T_\alpha \rightarrow T$ and Φ is dense in Φ_1 in the strong operator topology. This completes the proofs of Theorems 3 and 4.

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