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HARNACK'S INEQUALITIES ON THE CLASSICAL CARTAN DOMAINS

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Recently an extensive work by L. K. Hua on harmonic analysis in Cartan domains, which are called the classical domains, has been translated into English. Here we give Harnack's inequalities for the four main types of Cartan domains treated by Hua.

Harnack's inequality on a type of Cartan domain was obtained [6] for the case of square matrix spaces. Some of these inequalities are application and extension of the results of [6]. I am grateful to Professor J. Mitchell for her encouragement and comments on writing this paper.

Let z be a matrix of complex entries, $z^* = \bar{z}'$ the complex conjugate of the transposed matrix z' and I the identity matrix. Also $H > 0$ means that a hermitian matrix H is positive definite. The first three types of Cartan domains are defined by $D_k = \{z : I - zz^* > 0\}$, $k = 1, 2, 3$, where for $D_1 \equiv D_1(m, n)$, z is an (m, n) matrix (Since the conditions $I - zz^* > 0$ and $I - z^*z > 0$ are equivalent we assume for definiteness that $m \leqq n$.), for $D_2 \equiv D_2(n)$, z is a symmetric matrix of order n and for $D_3 \equiv D_3(n)$, z is a skew-symmetric matrix of order n. The fourth type, $D_4 \equiv D_4(1, n)$, is the set of all $(1, n)$ matrices, or n-dimensional vectors $(n > 2)$, of complex numbers satisfying the conditions

$$(1) \qquad 1 + |zz'|^2 - 2zz^* > 0\,, \qquad |zz'| < 1\,.$$

It is known that each of the domain D_k possesses a distinguished boundary [1] or characteristic manifold [2, p. 6] $C_k : C_1 \equiv C_1(m, n)$ consists of the (m, n) matrices u satisfying the condition $uu^* = I$. $C_2 \equiv C_2(n)$ consists of all symmetric unitary matrices of order n. $C_3 \equiv C_3(n)$ [2, p. 71] consists of all matrices u of the form $u = w's_1w$, where w is an n-rowed unitary matrix and

$$(2) \qquad s_1 = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for even } n \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus 0 & \text{for odd } n. \end{cases}$$

$C_4 \equiv C_4(1, n)$ consists of all $(1, n)$ matrices u of the form

$$(3) \qquad u = e^{i\theta}x\,, \qquad xx' = 1\,, \qquad 0 \leqq \theta \leqq \pi$$

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where x is a real vector.

We denote the Poisson kernel on D_k by $P_k(z, u)$, its explicit forms [3, 4] being given in following sections. The following Dirichlet problem is solved on D_k [2, 3]: Given a real-valued continuous function $f(u)$ on C_k, the Poisson integral

$$\phi(z) = \int_{C_k} f(u) P_K(z, u) \dot{u} \tag{4}$$

where $\dot{u}$ is Euclidean volume element on C_k, gives the unique function which is harmonic, in the sense given in [3], on the closure of D_k and takes the given boundary values $f(u)$ on C_k. We obtain Harnack's inequality on each D_k as a consequence of evaluating upper and lower bounds for $P_k(z, u)$.

2. Harnack's inequalities on D_1 and D_2. The Poisson kernel on D_1 [3, 4] is

$$P_1(z, u) = \frac{1}{V_1} \frac{[\det (I - zz^*)]^n}{|\det (I - zu^*)|^{2n}} \tag{5}$$

where $z \in D_1$, $u \in C_1$ and V_1 is the Euclidean volume of C_1. It is known [4, p. 411] that $P_1 > 0$ and $\int_{C_1} P_1 \dot{u} = 1$. In [6], we obtained bounds of the Poisson kernel (5) for the case $m = n$:

$$\frac{1}{V_1} \prod_{k=1}^{n} \left(\frac{1 - r_k}{1 + r_k}\right)^n \leqq P_1(z, u) \leqq \frac{1}{V_1} \prod_{k=1}^{n} \left(\frac{1 + r_k}{1 - r_k}\right)^n \tag{6}$$

where $z = u_0 R v_0 \in D_1(n, n)$, $u \in C_1(n, n)$, u_0 and v_0 are unitary matrices and $R = (\delta_{jk} r_k)$ is a diagonal matrix with $0 \leqq r_k < 1$ for $k = 1, 2, \cdots, n$. In order to obtain (6), we proved the inequality

$$\prod_{k=1}^{n} (1 - r_k)^2 \leqq |\det (I - zu^*)|^2 = |\det ((v - R)(v^* - R))| \leqq \prod_{k=1}^{n} (1 + r_k)^2 \tag{7}$$

where u, $\in C_1(n, n)$, is any unitary matrix of order n, $z \in D_1(n, n)$ and $v = u_0^* u v_0^*$.

For $z \in D_1(m, n)$ there are unitary matrices u_0 of order m and v_0 of order n such that $z = u_0(R, 0)v_0$, where R is a diagonal submatrix $(\delta_{jk} r_k)$ of order m and 0 is the $(m, n\text{-}m)$ zero submatrix [3, p. 1049]. Hence

$$\det (I^{(m)} - zz^*) = \det (I^{(m)} - RR^*) = \prod_{k=1}^{m} (1 - r_k^2) . \tag{8}$$

For the denominator of (5) we have

$$\det (I - zu^*) = \det \left(I - \begin{pmatrix} z \\ 0 \end{pmatrix} \begin{pmatrix} u \\ u_1 \end{pmatrix}^* \right) \tag{9}$$

where 0 is the $(n\text{-}m, n)$ zero submatrix and u_1 is chosen so that $U \equiv \binom{u}{u_1}$ becomes a unitary matrix of order n [5, p. 190]. If we also denote $Z \equiv \binom{z}{0}$, then $Z \in D_1(n, n)$ and $U \in C_1(n, n)$ for the case $m = n$. Therefore from (7)

$$\max_{U \in C_1} \{|\det(I - ZU^*)|^2\} \leqq \prod_{k=1}^{n} (1 + r_k)^2 = \prod_{k=1}^{m} (1 + r_k)^2 \tag{10}$$

and

$$\min_{U \in C_1} \{|\det(I - ZU)|^2\} \geqq \prod_{k=1}^{n} (1 - r_k)^2 = \prod_{k=1}^{m} (1 - r_k)^2 \tag{11}$$

since $r_{m+1} = \cdots = r_n = 0$ in Z. Finally from (9), (10) and (11) we obtain

$$\prod_{k=1}^{m} (1 - r_k)^2 \leqq |\det(I - zu^*)|^2 \leqq \prod_{k=1}^{m} (1 + r_k)^2 \tag{12}$$

where $z = u_0(R, 0)v_0 \in D_1(m, n)$ and $u \in C_1(m, n)$. This and (8) lead to

$$\frac{1}{V_1} \prod_{k=1}^{m} \left(\frac{1 - r_k}{1 + r_k}\right)^n \leqq P_1(z, u) \leqq \frac{1}{V_1} \prod_{k=1}^{m} \left(\frac{1 + r_k}{1 - r_k}\right)^n$$

for $z \in D_1$ and $u \in C_1$. Furthermore from (4) we obtain Harnack's inequality

$$\prod_{k=1}^{m} \left(\frac{1 - r_k}{1 + r_k}\right)^n \phi(0) \leqq \phi(z) \leqq \prod_{k=1}^{m} \left(\frac{1 + r_k}{1 - r_k}\right)^n \phi(0) .$$

For $z \in D_2(n)$ it is known that there is a unitary matrix u_0 such that $u_0 z u_0' = (\delta_{jk} r_k)$ where $r_1, \cdots, r_n$ are the positive square roots of the characteristic roots of $z\bar{z}$. Since $z \in D_2(n)$ implies $z \in D_1(n, n)$ and the characteristic manifold $C_2(n)$ is a subset of $C_1(n, n)$ we know that (7) and (8) hold for $z \in D_2(n)$ and $u \in C_1(n, n)$, and it can be seen from the Poisson kernel

$$P_2(z, u) = \frac{1}{V_2} \frac{[\det(I - z\bar{z})]^{(n+1)/2}}{|\det(I - z\bar{u})|^{n+1}}$$

that Harnack's inequality on $D_2(n)$ is

$$\prod_{k=1}^{n} \left(\frac{1 - r_k}{1 + r_k}\right)^{(n+1)/2} \phi(0) \leqq \phi(z) \leqq \prod_{k=1}^{n} \left(\frac{1 + r_k}{1 - r_k}\right)^{(n+1)/2} \phi(0) .$$

3. **Harnack's inequality on D_3.** The Poisson kernel on D_3 is

$$P_3(z, u) = \frac{1}{V_3} \frac{[\det(I - zz^*)]^a}{|\det(I - zu^*)|^{2a}}$$

where $a = (n-1)/2$ for even n and $a = n/2$ for odd n. For $z \in D_3(n)$ it is known [2, p. 67] that there is a unitary matrix u_0 such that

$$(13)\quad u_0 z u_0' = s = \begin{cases} \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} \dotplus \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & r_b \\ -r_b & 0 \end{pmatrix} & \text{for even } n \\ \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix} \dotplus \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & r_{[b]} \\ -r_{[b]} & 0 \end{pmatrix} \dotplus 0 & \text{for odd } n \end{cases}$$

where $b = n/2$ and $r_2^1, r_2^2, \cdots$ are characteristic roots of zz^*. Thus for even n

$$u_0 z z^* u_0^* = [r_1^2, r_1^2, \cdots, r_b^2, r_b^2] .$$

And for all n

$$\det(I - zz^*) = \det(I - u_0 z z^* u_0^*) = \prod_{k=1}^{[b]} (1 - r_k^2)^2 .$$

We denote the right hand side of (2) by u_1 with the change that the last term 0 be replaced by 1 in the case of odd n. Hence u_1 is a unitary matrix of order n and from (13) su_1^* is a diagonal matrix $R \equiv [r_1, r_1, \cdots, r_{[b]}, r_{[b]}]$ for even n and $R_0 \equiv [R, 0]$ for odd n. First we consider the case of even n. We notice from (2) that $u \in C_3$ is skew-symmetric unitary. Therefore for $z \in D_3$ and $u \in C_3$

$$|\det(I - zu^*)| = |\det(I - u_0^* s u_1^* u_1 \bar{u}_0 u^*)| = |\det(v - R)|$$

where $v = u_0 u u_0' u_1^*$ is a unitary matrix of order n. Hence from (7) we have

$$(14)\quad \prod_{k=1}^{b} (1 - r_k)^2 \leqq |\det(I - zu^*)| \leqq \prod_{k=1}^{b} (1 + r_k)^2 .$$

For the case of odd n, we notice from (2) that $\det(u) = 0$. It is known [3, p. 1073] that any $v \in C_3(n+1)$ can be written in the form

$$v = \begin{pmatrix} u & w'h' \\ -hw & 0 \end{pmatrix}, \qquad h = (0, \cdots, 0, e^{i\theta})$$

where $u = w' s_1 w \in C_3(n)$. Hence for odd n

$$\det(I - zu^*) = \det\left(I^{(n+1)} - \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & w'h' \\ -hw & 0 \end{pmatrix}^* \right)$$

and from (13)

$$\begin{pmatrix} u_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 & 0 \\ 0 & 1 \end{pmatrix}' = R \dotplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Since $n + 1$ is even we can apply (14) and obtain

$$\prod_{k=1}^{[b]}(1-r_k)^2 \leqq |\det(I-zu^*)| \leqq \prod_{k=1}^{[b]}(1+r_k)^2 .$$

As the result, this inequality is good for both odd and even n. Thus, as in § 2 we can obtain both bounds for the Poisson kernel $P_3(z, u)$ and finally Harnack's inequality on $D_3(n)$ as

$$\prod_{k=1}^{[b]}\left(\frac{1-r_k}{1+r_k}\right)^{2a}\phi(0) \leqq \phi(z) \leqq \prod_{k=1}^{[b]}\left(\frac{1+r_k}{1-r_k}\right)^{2a}\phi(0)$$

where $b = n/2$ and $a = (n-1)/2$ for even n and $a = n/2$ for odd n.

4. Harnack's inequality on D_4. The Poisson kernel on D_4 is

$$P_4(z, u) = \frac{1}{V_4}\frac{(1+|zz'|^2-2zz^*)^{n/2}}{|1+zz'\overline{uu'}-2zu^*|^n}$$

where $z \in D_4$ and $u \in C_4$. For every fixed $z \in D_4$ there is a real orthogonal matrix t such that [3, p. 1037]

(15) $$z = (z_1, z_2, 0, \cdots, 0)t .$$

Thus we have

$$1+|zz'|^2-2zz^* = 1+|z_1^2+z_2^2|^2-2(|z_1|^2+|z_2|^2) .$$

Here by denoting $z_1 - iz_2 \equiv w_1 = r_1e^{i\theta_1}$ and $z_1 + iz_2 \equiv w_2 = r_2e^{i\theta_2}$, from (1)

(16) $$0 < 1+|zz'|^2-2zz^* = (1-|w_1|^2)(1-|w_2|^2) = (1-r_1^2)(1-r_2^2)$$

and

$$0 < 1-|zz'| = 1-|z_1^2+z_2^2| = 1-|w_1w_2| = 1-r_1r_2 ,$$

hence

$$0 \leqq r_k < 1 , \qquad k = 1, 2.$$

Next, from $z \in D_4$ in (15) and $u \in C_4$ in (3)

$$zz'\overline{uu'} = (z_1^2+z_2^2)e^{-2i\theta} = w_1w_2e^{-2i\theta}$$

and

$$2zu^* = 2(z_1, z_2, 0, \cdots, 0)tx'e^{-i\theta} .$$

By denoting $t = (t_{jk})$ we have $tx' = (a_1, a_2, \cdots, a_n)'$ where $a_j = \sum_{k=1}^n t_{jk}x_k$ and since

$$\sum_{j=1}^n a_j^2 = (tx')'(tx') = xt'tx' = xx' = 1$$

we have $a_1^2 + a_2^2 \leqq 1$. Now with $z_1 = (w_1 + w_2)/2$ and $z_2 = i(w_1 - w_2)/2$, we have $2zu^* = (aw_1 + \bar{a}w_2)e^{-i\theta}$ where $a = a_1 + ia_2$ and $|a|^2 \leqq 1$. Thus

$$(17) \qquad 1 + zz'\overline{uu'} - 2zu^* = 1 + w_1w_2e^{-2i\theta} - (aw_1 + \bar{a}w_2)e^{-i\theta} .$$

We wish to find upper and lower bounds for the absolute values of expression (17). We consider the image of the closed unit disk $|a| \leqq 1$ under the mapping $f(a) = aw_1 + \bar{a}w_2$ for $a = re^{i\alpha}$. Here $f(a)$ can be written in the form

$$f(a) = r[r_1(e^{i\beta} + e^{-i\beta}) + (r_2 - r_1)e^{-i\beta}]e^{i(\theta_1+\theta_2)/2}$$

where $\beta = \alpha + (\theta_1 - \theta_2)/2$. For the case $r_1 = r_2$, $f(a)$ maps the closed unit disk onto a line segment of length $4r_1$. When $r_1 \neq r_2$, the image is a simple closed connected region. Furthermore the image of the unit circle is the line segment when $r_1 = r_2$ and is the boundary of the region when $r_1 \neq r_2$. Hence from the fact that $1 + zz'\overline{uu'} - 2zu^* \neq 0$ [3, p. 1079] we know that the maximum and the minimum of the absolute values of (17) can be found in the case $|a| = 1$. Thus

$$1 + w_1w_2e^{-2i\theta} - (aw_1 + \bar{a}w_2)e^{-i\theta} = (1 - aw_1e^{-i\theta})(1 - \bar{a}w_2e^{-i\theta})$$

and therefore

$$(1 - r_1)(1 - r_2) \leqq |1 + zz'\overline{uu'} - 2zu^*| \leqq (1 + r_1)(1 + r_2) .$$

This and (16) give us

$$\frac{1}{V_4}\prod_{k=1}^{2}\left(\frac{1 - r_k}{1 + r_k}\right)^{n/2} \leqq P_4(z, u) \leqq \frac{1}{V_4}\prod_{k=1}^{2}\left(\frac{1 + r_k}{1 - r_k}\right)^{n/2}$$

and the corresponding Harnack's inequality

$$\prod_{k=1}^{2}\left(\frac{1 - r_k}{1 + r_k}\right)^{n/2}\phi(0) \leqq \phi(z) \leqq \prod_{k=1}^{2}\left(\frac{1 + r_k}{1 + r_k}\right)^{n/2}\phi(0) .$$

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