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THE DEFICIENCY INDEX OF ORDINARY SELF-ADJOINT DIFFERENTIAL OPERATORS

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This paper is concerned with the computation of the deficiency index of an ordinary self-adjoint differential operator with real coefficients. The operator, L, is supposed defined on $[0, \infty)$ and is regular at the origin. The deficiency index counts the number of L^2 solutions to the equation $Ly = zy$, where z is any nonreal complex number.

The results obtained include as rather special cases almost all of the results known to the author when the order of L is larger than two.

The principal tool used is an asymptotic theorem of N. Levinson.

We are interested in computing the deficiency index of an ordinary self-adjoint differential operator,

$$Ly = (-1)^n \frac{d^n}{dt^n}\left(q_0 \frac{d^n y}{dt^n}\right) + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}}\left(q_1 \frac{d^{n-1} y}{dt^{n-1}}\right) + \cdots + q_n y \;, \tag{1.1}$$

defined on the interval $[0, \infty)$, with the coefficients q_k real and measurable. We shall suppose that L is regular at the origin which means that $1/q_0, q_1, \cdots, q_n$ belong to L^1 on every finite interval $[0, T]$.

The number m in the deficiency index (m, m) of the minimal operator L_0 associated with the formal operator (1.1) is the dimension of the linear space of L^2 solutions to any equation

$$Ly = zy \;, \qquad Im z \neq 0 \;. \tag{1.2}$$

As is well known, and easy to show, it is always true that $n \leqq m \leqq 2n$.

In the case where the order of the operator in (1.1) is two, fairly sophisticated tests are now available, [1], [5], which tell when the deficiency index in (1, 1). For an order larger than two very little seems to be known. Some results are due to M.A. Neumark [8] who obtains conditions that the deficiency index shall be either (n, n) or $(n+1, n+1)$. Other results are due to S. A. Orlov [9] and F.A. Neimark [7] who obtain the deficiency index of L_0 when the coefficients q_k in (1.1) are essentially of polynomial growth as $t \to \infty$. These results will appear as rather special cases of the considerations which we shall present in this paper. As a by product we can obtain the result, originally proved by Glasmann [4], that the number m can

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take *any* value between n and $2n$, and indeed our considerations will show how to explicitly and easily construct any number of differential operators with a given deficiency index (m, m).

2. The equation (1.2) is equivalent to the vector equation

$$\frac{du(t)}{dt} = B(t)u(t) , \tag{2.1}$$

where

$$B = \begin{bmatrix} 0 & 1 & & & & & & & & 0 \\ & 0 & 1 & & & & & & & \cdot \\ & & \cdot & \cdot & & & & & \cdot & \\ & & & \cdot & \cdot & & & \cdot & & \\ & & & & \cdot & 1 & 0 & & & \\ & & & & & 0 & 1/q_0 & & & \\ & & & & & q_1 & 0 & -1 & & \\ & & & & \cdot & & & \cdot & \cdot & \cdot \\ & & & \cdot & & & & & \cdot & \cdot & -1 \\ & & \cdot & & & & & & & \\ & q_{n-1} & & & & & & & & \\ q_n - z & & & & & & & & & 0 \end{bmatrix} ,$$

all functions being evaluated at t. The equivalence is in the sense that if y is a solution to $Ly = zy$, then $u = (y, y^{[1]}, \cdots, y^{[2n-1]})$ is a solution to (2.1), where $y^{[k]} = d^k y/dt^k$ for $0 \leqq k \leqq n-1$, $y^{[n]} = q_0 d^n y/dt^n$, and $y^{[n+k]} = q_k d^{n-k} y/dt^{n-k} - dy^{[n+k-1]}/dt$ for $1 \leqq k \leqq n$. These are the so called quasi-derivatives of y. Conversely, if $u = (u_1, \cdots, u_{2n})$ is a solution of (2.1), then u_1 is a solution to $Ly = zy$ and $u_2, \cdots, u_{2n}$ are the corresponding quasi-derivatives of u_1.

Let Q_0 be a real nonnegative measurable function so that $Q_0^{-1} \in L^1$ on every finite interval $[t_0, t]$, t_0 fixed, and

$$\int_{t_0}^{\infty} Q_0^{-1} = \infty .$$

Set $s = s(t) = \int_{t_0}^{t} Q_0^{-1}$ and let $t(s)$ be the inverse of $s(t)$; i.e. $t(s(t)) = t$. If u is a solution of (2.1), set $v(s) = u(t(s))$ and we get a vector equation

$$\frac{dv}{ds} = Q_0 B v . \tag{2.2}$$

Next, suppose that $Q_1, \cdots, Q_n$ are positive differentiable functions of t. Set

$$C(s)=\begin{bmatrix} Q_n & & & & & \\ & \ddots & & & & \\ & & Q_1 & & & \\ & & & Q_1^{-1} & & \\ & & & & \ddots & \\ & & & & & Q_n^{-1} \end{bmatrix},$$

where each Q_k is evaluated at $t(s)$ and the entries of C which are not on the main diagonal are zero. Make the transformation

$$v = Cw$$

and from (2.2) we get the differential equation

$$\frac{dw}{ds} = \left[Q_0C^{-1}BC - C^{-1}\frac{dC}{ds}\right]w = Dw\ , \tag{2.3}$$

where

$$D=\begin{bmatrix} -b_n & d_{n-1} & & & & & & 0 \\ & \ddots & \ddots & & & & \cdot & \\ & & \ddots & \ddots & & \cdot & & \\ & & & \cdot\ d_1 & 0 & & & \\ & & & -b_1 & a_0 & & & \\ & & & a_1 & b_1 & -d_1 & & \\ & & \cdot & & & \ddots & \ddots & \\ & \cdot & & & & & \cdot & -d_{n-1} \\ a_{n-1} & & & & & & & \cdot \\ a_n - zQ_0Q_n^2 & & & & & & & b_n \end{bmatrix}, \tag{2.4}$$

$$\begin{aligned} & a_0 = Q_0(Q_1^2q_0)^{-1}\ , \qquad a_k = Q_0Q_k^2q_k\ , \qquad 1 \leqq k \leqq n\ , \\ & b_k = Q_k^{-1}dQ_k/ds = Q_0Q_k^{-1}dQ_k/dt\ , \\ & d_k = Q_0Q_k/Q_{k+1}\ , \end{aligned} \tag{2.5}$$

and the other entries of D are zero.

We can now state and prove several theorems in terms of the functions a_k, b_k and d_k which will give the deficiency index of the differential operator (1.1). Before we do this let us record several different instances of the matrix D of (2.4):

(i) Take $Q_0 = Q$ and $Q_{n-k} = Q^{\rho-k}$, ρ real, $0 \leqq k \leqq n-1$. Then $a_0 = 1/Q^{2(\rho-n)+1}q_0$, $a_{n-k} = Q^{2(\rho-k)+1}q_{n-k}$, $0 \leqq k \leqq n-1$, $b_k = (\rho - n + k)dQ/dt$ and $d_k = 1$.

(ii) Replace Q and ρ by Q^{-1} and $-\rho$ respectively in (i) and we get $a_0 = 1/Q^{2(\rho+n)-1}q_0$, $a_{n-k} = Q^{2(\rho+k)-1}q_{n-k}$, $0 \leqq k \leqq n-1$,

$$b_k = -(\rho + n - k)Q^{-2}dQ/dt$$

and $d_k = 1$.

(iii) Take $Q_0 = 1$, $Q_k = Q$. Then $a_0 = 1/Q^2 q_0$, $a_k = Q^2 q_k$, $1 \leqq k \leqq n$, $b_k = Q^{-1} dQ/dt$ and $d_k = 1$.

We have singled out these special cases since, as we shall see later, the theorems obtained in [7], [8] and [9] fall under these cases.

3. In this section we shall prove the following:

THEOREM 1. *Let* $\{Q_k\}_0^n$, $\{a_k\}_0^n$, $\{b_k\}_1^n$ *and* $\{d_k\}_1^{n-1}$ *be functions as described in* §2. *Suppose we can write the matrix of* (2.4) *as*

$$D(t, z) = A + V(t) + R(t) + S(t, z) \ ,$$

where A *is a real constant matrix given by*

$$(3.1) \qquad A = \begin{bmatrix} \beta_n & 1 & & & & & & 0 \\ \cdot & & \beta_{n-1} & & & & & \cdot \\ & \cdot & \cdot & & & & \cdot & \\ & & \cdot & \cdot & & \cdot & & \\ & & & \cdot & 1 & \cdot & & \\ & & & & \beta_1 & 1/\alpha_0 & & \\ & & & & \alpha_1 & -\beta_1 & -1 & \\ & & & \cdot & & \cdot & \cdot & \\ & & \cdot & & & & \cdot & \cdot \\ & \cdot & & & & & & \cdot \\ & \alpha_{n-1} & & & & & & -1 \\ \alpha_n & & & & & & & -\beta_n \end{bmatrix} ,$$

$V(t)$ *and* $R(t)$ *are real so that* $V(t) \to 0$ *as* $t \to \infty$, $| V'(t) |$ *and* $| R(t) | Q_0^{-1}(t)$ *are summable*,[1] *and* $S(t, z)$ *has everywhere zero entries except in the position* $(1, 2n)$ *where it has the entry* $-zQ_0Q_n^2$. *Suppose further that if* $V(t) = 0$[2] *the* n *roots of the polynomial equation*

$$(3.2) \qquad P(\mu) = \sum_{k-0}^{n-1} (-1)^k \alpha_k \prod_{j=k+1}^{n} (\mu - \beta_j^2) + (-1)^n \alpha_n = 0 \ ,$$

are all nonzero and simple; and if $V(t) \neq 0$[2] *we have the added condition that those roots which lie in the half-plane* $\{z: Im z > 0\} \cup \{x : x \geqq 0\}$ *have square roots all of whose real parts are different. Then we have the following*:

(a) *If* Q_n^2 *is summable let* m *be the number of roots of* $P(\lambda^2)$ *which lie in the half-plane* $\mathscr{P} = \cup \{z: Re z \leqq \delta$ *and* $Q_n^2 \exp 2\delta \int_{t_0}^t Q_0^{-1} \in L^1\}$. *The deficiency index of the minimal operator* L_0 *associated with* (1.1) *is either* $(m-2, m-2)$, (m, m) *or* $(m+2, m+2)$; *if* $V(t) = 0$ *the deficiency index is* (m, m). *If* $\mathscr{P} = \{z: Re z \leqq 0\}$, *then the deficiency*

[1] If A is a matrix with entries a_{ij} we write $| A | = \sum_{i,j} | a_{ij} |$. If a function is defined on $[o, \infty)$ we shall say it is summable if it is summable in the usual sense on some interval $[t_0, \infty)$ for t_0 sufficiently large.

[2] By this we mean in some interval $[t_0, \infty)$ for t_0 sufficiently large.

index of L_0 is $(n+k, n+k)$ if and only if $P(\mu)$ has k negative real roots.

(b) *If Q_n^2 is not summable, but $Q_0Q_n^2 = \alpha + v(t) + r(t)$, where α is constant, $v(t) \to 0$ as $t \to \infty$, and dv/dt and $Q_0^{-1}r$ are summable, then the deficiency index of L_0 is (n, n) provided either $\alpha \neq 0$, or if λ_1 and λ_2 are roots of $G(\lambda) = P(\lambda^2) = 0$ and $Re(\lambda_1 - \lambda_2) = 0$, then $G'(\lambda_1) \neq G'(\lambda_2)$ and sgn $v(t)$ remains ultimately nonnegative or nonpositive. In case $\alpha \neq 0$, the result is valid even if the roots of $P(\lambda^2)$ are not assumed simple.*

Proof. After a certain amount of preparation it will become apparent that the theorem is a consequence of an asymptotic theorem of Levinson [6], [2, p. 92]. We shall first prove part (a). An easy computation shows that the characteristic polynomial of A is $(1/\alpha_0)P(\lambda^2)$, where P is the polynomial of (3.2). Because of the conditions put on $V(t)$, the characteristic polynomial of $A + V(t)$ can be written as

$$(1/\alpha_0)P(\lambda^2) + o(\lambda^2, t) ,$$

where $o(\lambda^2, t)$ is a polynomial in λ^2 of degree at most $n-1$ with coefficients which are functions of t which go to zero as $t \to \infty$. We shall write $P(\lambda^2, s) = P(\lambda^2) + o(\lambda^2, t(s))]$, where, recall from § 2, $s = \int_{t_0}^{t} Q_0^{-1}$.

By assumption, all of the roots of $P(\mu)$ are simple and hence for all s sufficiently large the roots of $P(\mu, s)$ are simple. This means that the number of real roots of $P(\mu, s)$ is the same as the number of real roots of $P(\mu)$. To see this, let $\mu_1, \cdots, \mu_n$ be the roots of $P(\mu)$ and $\mu_1(s), \cdots, \mu_n(s)$ the roots of $P(\mu, s)$. The latter roots can be chosen to be continuous functions of s in a neighborhood of infinity, including the point at infinity. In particular this means that $\mu_j(s) \to \mu_j$ as $s \to \infty$. Now, if $\mu_1, \cdots, \mu_l$ are the real roots of $P(\mu)$, then for all s sufficiently large, $\mu_1(s), \cdots, \mu_l(s)$ must be real. Otherwise, since the coefficients of $P(\mu, s)$ are real, its nonreal roots come in conjugate pairs and since $\mu_{l+1}(s), \cdots, \mu_n(s)$ are close to $\mu_{l+1}, \cdots, \mu_n$, they are not real and we would get too many roots for the nth order polynomial $P(\mu, s)$. Indeed, this argument shows that $\mu_1(s), \cdots, \mu_l(s)$ are exactly the real roots of $P(\mu, s)$.

The $2n$ roots of $P(\lambda^2, s)$ are $\{\pm\sqrt{\mu_j}\}_0^n$ where some fixed branch of the square root has been chosen. We shall designate these $2n$ roots by $\{\lambda_{\pm j}; j = 1, \cdots, n\}$, where $\lambda_{-j} = -\lambda_j$. We label the roots of $P(\lambda^2, s)$ in a corresponding manner; the $2n$ roots of this polynomial are $\{\pm\sqrt{\mu_j(s)}\}_1^n$, where the same branch of the square root has been chosen as before, and we correspondingly designate these roots by $\{\lambda_{\pm j}(s)\}_1^n$. We can suppose we have chosen the branch of the square root in such a way that these latter roots are continuous in a neighborhood of infinity, including the point at infinity. We also suppose we have labeled the roots in such a way so that $Re\lambda_j(s) \geqq 0, 1 \leqq j \leqq n$.

We would now like to apply Levinson's asymptotic theorem [6], [2, p. 92] to our situation. Before we can do this we must make sure that all of the hypotheses of his theorem are fulfilled. In addition to the hypotheses on the matrices $V(t)$ and $R(t)$ the following two conditions must be fulfilled for the roots of $P(\lambda^2)$ and $P(\lambda^2, s)$:

(i) The $2n$ roots of $P(\lambda^2)$ are all simple;

(ii) For a given j let $d_{jm}(s) = Re(\lambda_j(s) - \lambda_m(s))$ and suppose that all m, $1 \leqq |m| \leqq n$, fall into one of two classes I_1 and I_2, where

$$m \in I_1 \text{ if } \int_0^s d_{jm}(\sigma)d\sigma \longrightarrow \infty \quad \text{as} \quad s \longrightarrow \infty \text{ and}$$

$$\int_{s_1}^{s_2} d_{jm}(\sigma)d\sigma \geqq -K \quad (s_2 \geqq s_1 \geqq 0) ,$$

$$m \in I_2 \text{ if } \int_{s_1}^{s_2} d_{jm}(\sigma)d\sigma \leqq K , \quad (s_2 \geqq s_1 \geqq 0) ,$$

where K is a constant.

Condition (i) is fulfilled by virtue of the hypothesis of Theorem 1(a). We must therefore examine condition (ii). If $Re(\lambda_j - \lambda_m) \geqq 2\delta > 0$, then for s sufficiently large $Re(\lambda_j(s) - \lambda_m(s)) \geqq \delta$ and hence $m \in I_1$. If $Re(\lambda_j - \lambda_m) \leqq 2\delta < 0$, then for s sufficiently large,

$$Re(\lambda_j(s) - \lambda_m(s)) \leqq \delta < 0 ,$$

and $m \in I_2$. If $Re(\lambda_j - \lambda_m) = 0$, then from the hypothesis concerning the simplicity of the roots and the hypothesis about the square roots of the roots of $P(\mu)$ we must have $\lambda_j = \bar{\lambda}_m$ or λ_j and λ_m are purely imaginary. For s sufficiently large we have in the first case $\lambda_j(s) = \bar{\lambda}_m(s)$, and in the second case $\lambda_j(s)$ and $\lambda_m(s)$ purely imaginary. (Recall the discussion about the real roots given several paragraphs back!) In either case, for s sufficiently large, $Re(\lambda_j(s) - \lambda_m(s)) = 0$, and $m \in I_2$ Therefore, in all cases the condition (ii) is fulfilled.

Finally, we note that $V(t(s)) \longrightarrow 0$ as $s \longrightarrow \infty$,

$$\int_0^\infty \left| \frac{dV}{ds} \right| ds = \int_{t_0}^\infty \left| \frac{dV}{dt} \right| dt < \infty ,$$

and

$$\int_0^\infty |R + S| \, ds \leqq \int_0^\infty \{|R| + |S|\} ds$$

$$= \int_{t_0}^\infty \{|R| Q_0^{-1} + |z| Q_n^2\} dt < \infty .$$

Hence all of the conditions of Levinson's theorem are satisfied.

Let $\{p_k; k = \pm 1, \cdots, \pm n\}$ be a set of linearly independent eigenvectors for A corresponding to the eigenvalues $\{\lambda_k; k = \pm 1, \cdots, \pm n\}$ respectively. Apply Levinson's theorem and we find $2n$ linearly independent solutions $\{w_j; j = \pm 1, \cdots, \pm n\}$ to (2.3) and an $0 \leqq s_0 < \infty$ such that

$$w_j(s) \sim p_j \exp \int_{s_0}^{s} \lambda_j(\sigma) d\sigma . \tag{3.7}$$

This means, in particular, that if $w_{jm}(s)$ and p_{jm} are the mth components of $w_j(s)$ and p_j, respectively, then

$$w_{jm}(s) \sim p_{jm} \exp \int_{s_0}^{s} \lambda_j(\sigma) d\sigma . \tag{3.7$'$}$$

Before we proceed further let us note that for any j we can always choose p_j so that $p_{j1} = 1$. Indeed if we write out the set of linear equations corresponding to $Ap_j = \lambda_j p_j$ we see immediately that if $p_{j1} = 0$, then all of the other components p_j must be zero. Hence, we shall suppose from now on that p_j is that eigenvector with $p_{j1} = 1$.

Returning to the considerations of § 2, we recall that

$$v_j(s) = C(t(s)) w_j(s) ,$$

and these functions are linearly independent for $j = \pm 1, \cdots, \pm n$, since C is nonsingular for each t. Further, $u_j(t) = v_j(s(t))$ and the set $\{u_{j1}; j = \pm 1, \cdots, \pm n\}$ is a set of linearly independent solutions to the equation (2.1).

Now,

$$v_{j1}(s(t)) = Q_n(t) w_{j1}(s(t)) ,$$

and hence from (3.7$'$),

$$| v_{j1}(s(t)) | = \{| Q_n(t) | \exp \int_{s_0}^{s} Re\lambda_j(\sigma) d\sigma\}\{1 + o(t)\} . \tag{3.8}$$

Suppose the half-plane (open or closed) of part (a) has as boundary the line $Rez = \delta > 0$. If $Re\lambda_j < \delta$ or $Re\lambda_j > \delta$, then for σ sufficiently large, $Re\lambda_j(\sigma) < \delta$ or $Re\lambda_j(\sigma) > \delta$ respectively. If no roots of $P(\lambda^2)$ lie on $Rez = \delta$, then it is an immediate consequence of (3.8) that the deficiency index is (m, m). By the hypotheses on the real parts of the roots of $P(\lambda^2)$, there can be at most two roots of this polynomial which lie on $Rez = \delta$, and these roots are complex conjugate. Since it is not known how the corresponding roots of $P(\lambda^2, s)$ approach these roots, the only thing we can say is that the solutions corresponding to these roots are both in L^2 or neither is in L^2. Hence, any of the three cases mentioned is possible. If $V(t) = 0$, the roots $\lambda_j(s) \equiv \lambda_j$ and the result is an immediate consequence of (3.8).

To prove the second part of statement (a) we first notice that if $Re\lambda_j \leqq 0$, then $Re\lambda_j(\sigma) \leqq 0$ for σ sufficiently large. If there are k negative real roots of (3.2), then $P(\lambda^2)$ has $2k$ purely imaginary roots, $n-k$ roots with real part less than zero and $n-k$ roots with real part greater than zero. Hence, (3.8) shows that the deficiency index of the minimal operator associated with (1.1) is at least $(n+k, n+k)$. On other hand it is not difficult to show that any linear combination, not identically zero, of the solutions corresponding to roots with positive real part does not lie in L^2 (see e.g. [8, p. 300)]. Hence, the deficiency index is precisely $(n+k, n+k)$. Conversely, since the number of roots of $P(\lambda^2)$ with negative real part is the same as the number of roots with positive real part, if the deficiency index of L_0 is $(n+k, n+k)$, $P(\mu)$ must have k negative real roots. This completes the proof of part (a).

We shall now prove part (b) which will complete the proof of Theorem 1. Because of the way we have assumed we can decompose $Q_0Q_n^2$, we are interested in solutions to the equation

$$P(\lambda^2, s) = z[\alpha + v(t(s))] = \zeta(z, s) \;. \tag{3.9}$$

If $\alpha \neq 0$, then we may choose a nonreal z so that the roots of $P(\lambda^2) - z\alpha = 0$ all are simple and have different real parts, *regardless of the multiplicity of the roots of P.* Moreover, since P has real coefficients, $P(\lambda^2) - z\alpha$ can have no purely imaginary roots and n roots must have positive real part and n roots must have negative real part. Hence, it follows from (3.8) that there are n solutions which do not belong to L^2. But we know by general considerations that n solutions always belong to L^2 Hence the deficiency index is (n, n).

In the other case we may as well suppose $\alpha = 0$. From (3.9) and the fact that the roots of $P(\lambda^2) = 0$ are simple it follows from the implicit function theorem that we can choose $2n$ distinct solutions, $\{\lambda_j(s, z)\}$, to (3.9) which are continuous in a neighborhood of the point $(\infty, 0)$ in the (s, z) space, and for fixed s is analytic in z. If we differentiate $G(\lambda_j(s, z)) = P(\lambda_j^2(s, z), s) = \zeta(z, s)$ with respect to z we get

$$\frac{d\lambda_j(s, z)}{dz} = v(t(s))/G'(\lambda_j(s, z)) \;. \tag{3.10}$$

We can use (3.10) to find all the derivatives of $\lambda_j(s, z)$, and each derivative is $v(t(s))$ times a function consisting of derivatives $G^{(k)}(\lambda_j(s, z))$. Using this last fact and recalling that $\lambda_j(s, o) = \lambda_j(s)$, the Taylor expansion of $\lambda_j(s, z)$ about $z = 0$ becomes

$$\lambda_j(s, z) = \lambda_j(0) + zv(t(s))\{G'(\lambda_j(s))^{-1} + o(s, z)\} \;, \tag{3.11}$$

where $o(s,z) \to 0$, as $z \to 0$, uniformly in some neighborhood of $s = \infty$. From (3.11) we get

$$(3.12) \qquad Re[\lambda_j(s, z) - \lambda_k(s, z)] = Re[\lambda_j(0) - \lambda_k(0)] + v(t(s)) Re z[\{G'(s))^{-1} - G'(\lambda_k(s))^{-1}\} + o(s, z)] .$$

From (3.12) we see that if $Re[\lambda_j(0) - \lambda_k(0)] \neq 0$, then for fixed z, $Re[\lambda_j(s, z) - \lambda_k(s, z)]$ always has the same sgn, provided s is sufficiently large. If $Re[\lambda_j(0) - \lambda_k(0)] = 0$, then we claim that for all sufficiently small nonreal z except on a finite number of straight lines through the origin, $Re[\lambda_j(s, z) - \lambda_k(s, z)]$ remains nonnegative or nonpositive, irrespective of j and k, provided s is sufficiently large. Indeed, since $G'(\lambda_j) - G'(\lambda_k) \neq 0$ $Re z\{G'(\lambda_j)^{-1} - G'(\lambda_k)^{-1}\} = 0$ only for those z which lie on the straight line

$$x Re\{G'(\lambda_j)^{-1} - G'(\lambda_k)^{-1}\} - y Im\{G'(\lambda_j)^{-1} - G'(\lambda_k)^{-1}\} = 0 .$$

Take z, sufficiently small and not on these lines, so that

$$Re z[G'(\lambda_j(s))^{-1} - G'(\lambda_k(s)^{-1} + o(s, z)]$$

maintains its sgn for s sufficiently large. From the hypothesis on $v(t(s))$, and (3.12), our statement about $Re[\lambda_j(s, z) - \lambda_k(s, z)]$ follows.

We may consequently apply Levinson's asymptotic theorem. Also, we see from (3.10) that for z sufficiently small $Re\lambda_j(s, z)$ remains nonpositive or nonnegative in a neighborhood of $s = \infty$. Since (3.9) has at least n solutions with nonnegative real part, it follows from (3.8) that the deficiency index of L_0 is (m, m) where $m \leqq n$. On the other hand, we know already from general considerations that $m \geqq n$. Hence, we have completed the proof of Theorem 1.

4. Since Theorem 1 has a relatively complicated statement it would be well to pause here and examine in more detail the examples given at the end § 2. Specifically we want to show how Theorem 1 contains the known results which we have previously mentioned. We start with a theorem of M.A. Neumark [8, p. 293].

THEOREM. *Let the following conditions be fulfilled:*
1. $|q_n(t)| \to \infty$ *for* $t \to \infty$;
2. q'_n, q''_n *maintain their sgn in a neighborhood of infinity;*
3. *for* $t \to \infty$,

$$q'_n = 0(|q_n|^\gamma), \quad 0 < \gamma < 1 + \frac{1}{2n} ;$$

4. q'_0/q_0, $q_1|q_n|^{-1/2n}$, $q_2|q_n|^{-3/2n}$, $\cdots$, $q_{n-1}|q_n|^{-(2n-3)/2n}$ *are summable;*
5. $\lim_{t\to\infty} q_0(t) > 0$.

If $q_n(t) \to \infty$ for $t \to \infty$, then L_0 has the deficiency index (n, n).

If $q_n(t) \to -\infty$ for $t \to \infty$, then L_0 has the deficiency index (n, n) or $(n + 1, n + 1)$ if the integral

$$\int^\infty | q_n |^{-1+1/2n} dt$$

diverges or converges, respectively.

We claim that this theorem falls under our Theorem 1. To see this we take $Q = | q_n |^{-1/2n}$, $Q_0 = Q$ and $Q_{n-k} = Q^{\rho-k}$, where $\rho = (2n - 1)/2$. Since $| q_n | \to \infty$, it is clear that $\int_{t_0}^t Q_0^{-1} \to \infty$ as $t \to \infty$. This then falls under case (i) at the end of § 2. Recalling the terminology used in § 2 and § 3, we find that $a_k = q_k | q_n |^{-k/n}$, $1 \leqq k \leqq n$, $a_0 = 1/q_0$, and $b_k = \{(2k - 1)/2\} d | q_n |^{-1/2n}/dt$.

Now, by hypothesis 4,

$$Q_0^{-1} a_k = q_k | q_n |^{-(2k-1)/2n}$$

is summable for $1 \leqq k \leqq n - 1$ and hence we may take $\alpha_k = 0$ for these values of k. Further $a_n = 1$ or $a_n = -1$ for t sufficiently large and hence we may take $\alpha_n = 1$ or $\alpha_n = -1$ respectively. From the fact that q_0'/q_0 is summable it follows that $\lim_{t\to\infty} q_0(t)$ exists as a finite number. We take this limit as α_0^{-1} and hypothesis 5 tells us that $\alpha_0 > 0$. Let us write

$$\frac{1}{q_0} = \alpha_0 + v_0(t) .$$

Then $v_0(t) \to 0$ as $t \to \infty$ and $v_0'(t) = q_0'/q_0^2$ is summable by hypotheses 4 and 5.

It remains to examine the functions $b_k(t)$. The facts that q_n is twice differentiable and q_n ultimately maintains its sgn means that b_k exists and

$$b_k = \pm C_k | q_n |^{-(1+1/2n)} q_n', \quad C_k = (2k - 1)/4n .$$

From hypothesis 3 it follows that $b_k(t) \to 0$ as $t \to \infty$. Further

$$b_k' = C_k\{(1 + 1/2n) | q_n |^{-(2+1/2n)} (q_n')^2 \pm | q_n |^{-(1+1/2n)} q_n''\} .$$

Now use the hypotheses 1, 2, and 3; if $\gamma = 1 + 1/2n - \varepsilon$, $\varepsilon > 0$, then

$$\int_{t_0}^\infty | q_n |^{-(2+1/2n)} | q_n' |^2 \leqq C \int_{t_0}^\infty | q_n |^{-1-\varepsilon} | q_n' |$$

$$= \pm C \int_{t_0}^\infty | q_n |^{-1-\varepsilon} | q_n |' < \infty ,$$

where C is some positive constant and t_0 is sufficiently large so that $q_n(t_0) \neq 0$. Further

$$\pm \int_{t_0}^{\infty} | q_n |^{-(1+1/2n)} q_n'' = \int_{t_0}^{\infty} | q_n |^{-(1+1/2n)} | q_n |''$$

$$= - | q_n(t_0) |^{-(1+1/2n)} | q_n(t_0) |'$$

$$+ \left(1 + \frac{1}{2n}\right) \int_{t_0}^{\infty} | q_n |^{-(2+1/2n)} \{| q_n |'\}^2 < \infty .$$

Hence $b_k' \in L^1$ and we may take $\beta_k = 0$.

In this case the polynomial (3.2) becomes

$$P(\mu) = \alpha_0 \mu^n + (-1)^n \alpha_n, \alpha_0 > 0 .$$

If $q_n(t) \to \infty$ as $t \to \infty$, then $\alpha_n = 1$ and we are considering the roots to the equation $\alpha_0 \mu^n + (-1)^n = 0$. If n is odd or even, we can have no negative real roots and the conditions on the real parts of the roots needed for the application of Theorem 1 are clearly satisfied. If $Q_n^2 = | q_n |^{-1+1/2n}$ is not summable, it is a simple matter to check that

$$Q_n^2(t) \exp \varepsilon \int_{t_0}^{t} Q^{-1} = Q^{\rho}(t) \exp \varepsilon \int_{t_0}^{t} Q^{-1}$$

is not summable for any ρ and any $\varepsilon > 0$. Hence part (a) of Theorem 1 applies. In either case the deficiency index is (n, n).

If $q_n(t) \to -\infty$ as $t \to \infty$, then $\alpha_n = -1$ and we are interested in the roots of the equation $\alpha_0 \mu^n + (-1)^{n+1} = 0$. If n is odd or even, we have exactly one negative real root. Hence if Q_n^2 is summable, the deficiency is $(n + 1, n + 1)$ and otherwise it is (n, n). The completes the proof of Neumark's theorem.

We now want to give a theorem of S.A. Orlov, [9], or more precisely, an improved version due to F.A. Neimark [7].

THEOREM. *Suppose that* $q_k(t) = t^{2(n-k)+\nu}[\alpha_k + r_k(t)]$, $1 \leqq k \leqq n$ *and* $q_0(t) = t^{2n+\nu}[1/\alpha_0 + r_0(t)]^{-1}$, *where* $t^{-1} r_k(t)$ *is summable,* $0 \leqq k \leqq n$, *and* $\nu \geqq 0$. *Set*

$$F(\lambda, \nu) = \sum_{k=0}^{n} (-1)^k \alpha_{n-k} \prod_{j=0}^{k-1} \left[\left(\lambda + \frac{\nu}{2}\right)^2 - \left(\frac{\nu + 1}{2} + j\right)^2 \right] + \alpha_n ,$$

and suppose all the roots of this polynomial are simple for $\nu > 0$ *and all roots of* $F(\lambda, 0) - z$ *are simple for* z *some complex number with nonzero imaginary part. Then the number of linearly independent solutions of* (1.2) *which belong to* L^2 *are for* $\nu > 0$ *the number of roots* $F(\lambda, \nu)$ *with* $Re\lambda < 0$, *and for* $\nu = 0$ *are the number of roots of* $F(\lambda, 0) - z = 0$ *with* $Re\lambda < 0$.

Let us show that this theorem also is a special case of our Theorem 1. Take $Q = 1/t$, $Q_0 = t$ and $Q_0Q_k^2 = t^{-2(n-k)-\nu}$. Clearly, $Q_0Q_k^2 = Q^{2(\rho+n-k)-1}$, where $\rho = (\nu + 1)/2$ and hence this situation falls under case (ii) given at the end of § 2.

We have $a_0(t) = 1/\alpha_0 + r_0(t)$, $a_k = \alpha_k + r_k(t)$, and

$$b_k = -[(\nu + 1)/2 + n - k)] = \beta_k .$$

Further, $Q_0^{-1}r_k = r_k(t)/t \in L^1$ by hypothesis. Hence $V(t) = 0$ in the decomposition of the matrix $D(t, z)$ of Theorem 1. The polynomial (3.2) becomes

$$\begin{aligned} P(\lambda^2) &= \sum_{k=1}^{n} (-1)^k \alpha_{n-k} \prod_{j=0}^{k-1} (\lambda^2 - [(\nu + 1)/2 + j]^2 + \alpha_n \\ &= F(\lambda - \nu/2, \nu) . \end{aligned}$$

Now, $Q_n^2 \exp 2\delta \int_1^t Q_0^{-1} = t^{2\delta-1-\nu}$, and this belongs to L^1 for $\delta < \nu/2$. If $\nu > 0$, then $Q_n^2 \in L^1$, and since $V(t) = 0$ part (a) implies one part of the present theorem. If $\nu = 0, Q_n^2 \notin L^1$, but $Q_0Q_n^2(t) = 1$. An easy analysis of this situation gives the remaining part of the present theorem. However, part (b) of our Theorem 1 applies here and shows that *if* $\nu = 0$, *the deficiency index is always* (n, n), which is of course a somewhat sharper result that given by Neimark's theorem.

Let us now list two results given by Neumark [8] which fall under case (iii) of § 2. The statements are immediate consequences of Theorem 1 (b) by taking $Q = 1$ in (iii).

I. *If* $(1/q_0)'$, $q_1, \cdots, q_n$ *are summable and* $\lim_{t\to\infty} q_0(t) > 0$, *then* L_0 *has the deficiency index* (n, n).

II. *If there are constants,* $\alpha_0 \neq 0$, $\alpha_1, \cdots, \alpha_n$ *such that*

$$\frac{1}{q_0} - \frac{1}{\alpha_0}, \quad q_1 - \alpha_1, \cdots, q_n - \alpha_n$$

are summable, then L_0 *has deficiency index* (n, n).

5. In this section we shall obtain a theorem in which the hypothesis that the characteristic polynomial has only simple roots can be relaxed. This relaxation is obtained at the expense of having some of the other hypotheses more stringent. The basic tool in the proof of the theorem of this section, as in the previous section, is an asymptotic theorem obtained in [3], which is an extension of Levinson's asymptotic theorem.

THEOREM 2. *Let* $\{Q_k\}_0$, $\{a_k\}_0^n$, $\{b_k\}_1^n$, *and* $\{d_k\}_1^{n-1}$ *be the functions described in* § 2. *Suppose we can write the matrix of* (2.4) *as*

$$D(t, z) = A + R(t) + S(t, z) , \tag{5.1}$$

where $S(t, z)$ *is as in Theorem* 1, $R(t)$ *has real entries and* A *is a real constant matrix given by* (3.1). *Let* $\{\lambda_{\pm k}; 1 \leqq k \leqq r\}$, $\lambda_{-k} = -\lambda_k$, *be the distinct characteristic roots of* A *with the multiplicity of* λ_j *being* n_j *and let* $q + 1 = \max n_j$. *We make the further assumption that* $\left[\int_{t_0}^t Q_0^{-1}\right]^q [Q_0^{-1} | R | + Q_n^2]$ *is summable. For* $j \geqq 1$ *let* P_j *be the half-plane*

$$\cup \left\{z: Re\, z \leqq \delta \text{ and } Q_n^2 \left[\int_{t_0}^t Q_0^{-1}\right]^{(j-1)} \exp 2\delta \int_{t_0}^t Q_0^{-1} \in L^1\right\} ,$$

$m_{kj} = \min (n_k, j)$ *if* $\lambda_k \in P_j$ *and* $m_{kj} = 0$ *if* $\lambda_k \notin P_j$, $m_k = \max_j m_{kj}$, *and* $m = \sum m_k$. *Then the deficiency index of* L_0 *is* (m, m).

Proof. By hypothesis $s^q Q_0 Q_n^2$ is summable with respect to the measure ds and hence $s^q | R + S |$ is summmable. Further, it is a simple matter to check that the minimal and characteristic polynomials of A have the same degree. Hence we may apply the asymptotic theorem proved in [3] which says the following: Let $\{q_{kj}; 1 \leqq j \leqq n_k\}$ be a set of (linearly independent) "principal vectors" for the eigenvalue λ_k; i.e., $q_{kj} = (A - \lambda_k I)^{n_k - j} q_{kn_k}$, $(A - \lambda_k)^{n_k - 1} q_{kn_k} \neq 0$, and $(A - \lambda_k I)^{n_k} q_{kn_k} = 0$. Then there exists an s_0 and a fundamental set $\{w_{kj}; 1 \leqq j \leqq n_k, 1 \leqq |k| \leqq r\}$ of solutions to (2.4) so that

$$w_{kj}(s) \sim \left[\frac{s^{j-1}}{(j-1)!} \exp \lambda_k (s - s_0)\right] q_{kj} . \tag{5.2}$$

In particular this means that (5.2) remains true when w_{kj} and q_{kj} are replaced by their first components.

Before we proceed further, let us show that we can always choose q_{kn_k} so that the first component of q_{kj}, $1 \leqq j \leqq n_k$, is different from zero. To simplify the notation, set $\lambda = \lambda_k$, $q_m = q_{kn_k}$, $q_j = (A - \lambda I)^{m-j} q_m$ and denote the components of q_j by q_{jl}, $0 \leqq l \leqq 2n$. The set of equations

$$\begin{aligned} &(A - \lambda I)^{m-j} q_m - q_j = 0 , \qquad j = 1, \cdots, m - 1 , \\ &(A - \lambda I)^m q_m = 0 \end{aligned} \tag{5.3}$$

is a set of $2nm$ linear equations in the $2nm$ variables $\{q_{jl}; 1 \leqq l \leqq 2n, 1 \leqq j \leqq m\}$. This set of equations can also be written as

$$\begin{aligned} &(A - \lambda I) q_{j+1} - q_j = 0 , \qquad j = m - 1, \cdots, 1 , \\ &(A - \lambda I) q_1 = 0 . \end{aligned} \tag{5.3'}$$

The matrix, $\mathscr{A}$, of the set (5.3′) of linear equations has m blocks down the main diagonal, each block being the matrix $A - \lambda I$, $m - 1$ blocks down the superdiagonal, each block being the $2n \times 2n$ matrix $-I$, and zeros elsewhere. Since $\det(A - \lambda I) = 0$, the set of equations (5.3′) has nontrivial solutions. Further, since the rank of $(A - \lambda I)$ is $(2n - 1)$ the rank of $\mathscr{A}$ must be $(2n - 1)m$. If we remove from $\mathscr{A}$ the rows and columns of index $2kn + 1$, $0 \leqq k \leqq m - 1$, the resulting matrix has nonzero determinant. This means we can choose the numbers $\{q_{j1}\}_{j=1}^{n}$ as we will and solve for the other variables by Gramer's rule so as to obtain a solution to (5.3′). We choose the first component of q_{kj} as $(j - 1)!$

Now, from § 2 we have

$$v_{kj}(s) = C(t(s))w_{kj}(s)$$

and looking at the first components of the vectors on both sides we get

$$v_{kj1}(s(t)) = Q_n(t)w_{kj1}(s)t)) .$$

Hence, from (5.2) we get

$$(5.4) \qquad |\, v_{kj1}(s)\,| = \{|\, Q_n(t(s))\,|\, s^{j-1} \exp Re\lambda_k(s - s_0)\}\{1 + o(s)\} ,$$

for $1 \leqq j \leqq n_k$, $1 \leqq |\,k\,| \leqq r$. The equation (5.4) is of course the analogue of (3.8). If $\lambda_k \in P_j$, then $u_{kj1}(t) = v_{kj1}(s(t))$ is, by (5.4), in L^2, and indeed all functions u_{kl1}, $1 \leqq l \leqq j$, are in L^2. Therefore, there will be m_k solutions corresponding to λ_k which are in L^2. This proves Theorem 2.

REMARK. The asymptotic theorem in [3] would allow us to include a term $V(t)$ in (5.1) which has suitable differentiability properties. However, a theorem of a general nature on the deficiency index becomes very complicated to state and for the sake of simplicity we have preferred to suppose that $V(t) = 0$.

Let us conclude with an illustration of the application of Theorem 2, namely a generalization of the Neimark-Orlov theorem given in § 4. Let $q + 1$ be the maximum of the multiplicities of the roots of the polynomial $F(\lambda, \nu)$ of § 4 and assume that the functions $r_k(t)$ of that theorem satisfy the condition $[\log t]^q r_k(t)/t \in L^1$. Recall we had taken $Q = 1/t$, $Q_0 = t$ and $Q_n^2 = 1/t^{1+\nu}$. If $\nu > 0$, it is immediate that

$$\left[\int_1^t Q_0^{-1}\right]^q Q_n^2(t) = [\log t]^q/t^{1+\nu} \in L^1 .$$

Hence we may apply Theorem 2 and we no longer need the hypothesis that the roots of $F(\lambda, \nu)$ are simple. If $\nu = 0$ the situation is covered by Theorem 1(b) and deficiency index is (n, n).

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