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ADJOINT QUASI-DIFFERENTIAL OPERATORS OF EULER TYPE

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This paper treats linear quasi-differential operators of the form

$$L[y] = \sum_{j=0}^{n} p_{0j}y^{(j)} - \left(\sum_{j=0}^{n} p_{1j}y^{(j)} - \left(\cdots - \left(\sum_{j=0}^{n} p_{mj}y^{(j)}\right)' \cdots\right)'\right)',$$

based on an integrable $(m+1)\times(n+1)$ matrix function $[p_{ij}]$, $(i=0,\cdots,m;\ j=0,\cdots,n)$, about which suitable regularity assumptions are made. Results obtained by Reid (Trans. Amer. Math. Soc. Vol. 85 (1957), pp. 446–461) are extended to operators of the type considered here.

A generalized Green's function for the system $\{L[y]=0,\ y\in\mathscr{D}\}$ is defined, where $\mathscr{D}$ is a linear subspace of the domain of L. Resolvent and deterministic properties of this function are presented, together with the relationship of such a generalized Green's function to the generalized Green's function for the associated adjoint system.

For a large class of two-point boundary problems in which the boundary conditions involve the characteristic parameter linearly it is shown that there exists a simultaneous canonical representation of the boundary conditions for a given problem and those of its adjoint; in particular, in the self-adjoint case this canonical representation has the form of boundary conditions and transversality conditions for a variational problem. Finally, these results are applied to a two-point boundary problem involving a differential operator of the type considered in the paper of Reid above.

Since an important example of an operator of the form of $L[y]$ is the Euler operator in the calculus of variations, we shall refer to such operators as *quasi-differential operators of Euler type.*

Section 2 gives a more precise description of the operator, and Section 3 is concerned with a discussion of its adjoint. In particular it is shown that if $\mathscr{D}_0$ is the class of functions y in the domain of L with the property that the functions $y, y', \cdots, y^{(n-1)}$, $\tilde{y}_m \equiv \sum_{j=0}^{n} p_{mj}y^{(j)}$, $\tilde{y}_i \equiv \sum_{j=0}^{n} p_{ij}y^{(j)} - \tilde{y}'_{i+1}$, $(i = m-1, \cdots, 1)$, vanish at a and at b, and if T_0 is the restriction of L to $\mathscr{D}_0$, then the adjoint operator T_0^* is given by

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$$T_0^*[z] = L^\star[z] \equiv \sum_{i=0}^m \bar{p}_{i0} z^{(i)} - \Big(\sum_{i=0}^m \bar{p}_{i1} z^{(i)} - \Big(\cdots - \Big(\sum_{i=0}^m \bar{p}_{in} z^{(i)}\Big)' \cdots\Big)'\Big)' .$$

Section 4 is a study of extensions of the operator T_0, and their adjoints. Section 5 is devoted to generalized Green's functions for Euler type quasi-differential systems and their adjoints, and extends the results of Elliott [3] and Reid [5] to the case where the number of linearly independent boundary conditions may differ from the order of the differential equation.

Section 6 is concerned with a certain class of two-point boundary problems in which the boundary conditions involve the characteristic parameter linearly. It is shown that there exists a simultaneous canonical representation of the boundary conditions for a given problm and those of its adjoint; in particular, in the self-adjoint case this canonical representation has the form of boundary conditions and transversality conditions for a variational problem.

Finally, § 7 is devoted to an application of the results of § 6 to a two-point boundary problem involving a differential operator of the type considered by Reid in [7].

The symbol $\mathfrak{C}_n$, $(n = 0, 1, 2, \cdots)$, will signify the class of complex-valued functions defined on the compact interval $[a, b]$ which have n continuous derivatives. The set of functions y in $\mathfrak{C}_{n-1}$ for which $y^{(n-1)}$ is a.c. (absolutely continuous) is denoted by $\mathfrak{A}_n$, $(n = 0, 1, 2, \cdots)$. In particular, $\mathfrak{C}_0$ and $\mathfrak{A}_0$ will signify respectively the classes of continuous and Lebesgue integrable complex-valued functions defined on $[a, b]$. If f and g belong to $\mathfrak{A}_0$ and $f(x) = g(x)$ almost everywhere, we will simply write $f = g$. If f is a complex-valued function on $[a, b]$, then $\bar{f}$ denotes the function with domain $[a, b]$ whose value at x is the complex conjugate of $f(x)$. If u and v are functions on $[a, b]$ and $\bar{v}u \in \mathfrak{A}_0$, then we define (u, v) as

$$(u, v) = \int_a^b \bar{v}u .$$

Matrix notation will be used except where it is impracticable. If M is a matrix, then the conjugate transpose of M is denoted by M^*. Vectors are treated as matrices with one column. The symbols E_n and 0_{mn} are used to represent the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively; the subscripts will be omitted when there is no danger of confusion.

A matrix function is said to be continuous, integrable, etc. whenever each of its elements possesses the specified property. If A is an a.c. matrix function, then $A'(x)$ signifies the matrix of derivatives at values for which these derivatives exist and the zero matrix elsewhere.

2. **Description of the operator.** Suppose that $[p_{ij}]$, $(i = 0, \cdots,$

$m \geqq 1$; $j = 0, \cdots, n \geqq 1$), is an integrable $(m+1) \times (n+1)$ matrix function on a compact interval $[a, b]$ and that p_{on} and p_{mo} are essentially bounded. For suitable y in $\mathfrak{A}_n$ define functions $\tilde{y}_1, \cdots, \tilde{y}_m$ as follows:

$$\begin{aligned} &\tilde{y}_m(x) = \sum_{j=0}^{n} p_{mj}(x) y^{(j)}(x) ; \\ (2.1) \qquad &\textit{if } \tilde{y}_{j+1} \in \mathfrak{A}_1, \textit{ then } \tilde{y}_i(x) = \sum_{j=0}^{n} p_{ij}(x) y^{(j)}(x) - \tilde{y}'_{i+1}(x) , \\ &(i = m-1, \cdots, 1) . \end{aligned}$$

The class of functions y in $\mathfrak{A}_n$ for which $\tilde{y}_1, \cdots, \tilde{y}_m$ are a.c. will be denoted by $\tilde{\mathfrak{A}}_n$. For convenience the vector functions $(y^{(j-1)})$, $(j = 1, \cdots, n)$, and $(\tilde{y}_i)$, $(i = 1, \cdots, m)$, will be denoted by $\hat{y}$ and $\tilde{y}$, respectively; the $(n+m)$-vector function $(y, \cdots, y^{(n-1)}, \tilde{y}_1, \cdots, \tilde{y}_m)$ will be represented by $\hat{\tilde{y}}$.

Denote by L the operator with domain $\tilde{\mathfrak{A}}_n$ which is defined by

$$(2.2) \qquad L[y] = \sum_{j=0}^{n} p_{oj} y^{(j)} - \tilde{y}'_1 .$$

The operator L is a quasi-differential operator in the sense of Bôcher [1]; in particular, it is a generalization of the Euler operator in the calculus of variations and, as was stated in the introduction, it will be called a quasi-differential operator of the Euler type.

Let $\tilde{\mathfrak{A}}_n^0$ be the collection of functions y in $\tilde{\mathfrak{A}}_n$ for which $\hat{\tilde{y}}(a) = 0 = \hat{\tilde{y}}(b)$, and denote by T_0 the restriction of L to $\tilde{\mathfrak{A}}_n^0$. Suppose that $\mathscr{D}_0^*$ is the class of functions z in $\mathfrak{A}_0$ which are essentially bounded and have the property that there exists a function f_z in $\mathfrak{A}_0$ such that $(L[y], z) = (y, f_z)$ for all y in $\tilde{\mathfrak{A}}_n^0$.

A second operator $L^\star$ will now be defined. For suitable functions z in $\mathfrak{A}_m$ define functions $\tilde{z}_1, \cdots, \tilde{z}_n$ as follows:

$$\begin{aligned} &\tilde{z}_n(x) = \sum_{i=0}^{m} \bar{p}_{in}(x) z^{(i)}(x) ; \\ (2.3) \qquad &\textit{if } \tilde{z}_{j+1} \in \mathfrak{A}_1, \textit{ then } \tilde{z}_j(x) = \sum_{i=0}^{m} \bar{p}_{ij}(x) z^{(i)}(x) - \tilde{z}'_{j+1}(x) , \\ &(j = n-1, \cdots, 1) . \end{aligned}$$

The class of functions z in $\mathfrak{A}_m$ for which $\tilde{z}_1, \cdots, \tilde{z}_n$ are a.c. will be denoted by $\tilde{\mathfrak{A}}_m$. Let $L^\star$ be the operator with domain $\tilde{\mathfrak{A}}_m$ defined by

$$(2.4) \qquad L^\star[z] = \sum_{i=0}^{m} \bar{p}_{io} z^{(i)} - \tilde{z}'_1 .$$

If $z \in \tilde{\mathfrak{A}}_m$, then $\check{z}$ and $\tilde{z}$ will signify the vector functions $(z^{(i-1)})$, $(i = 1, \cdots, m)$, and $(\tilde{z}_j)$, $(j = 1, \cdots, n)$, respectively. The $(m+n)$-vector function $(z, \cdots, z^{(m-1)}, \tilde{z}_1, \cdots, \tilde{z}_n)$ will be denoted by $\check{\tilde{z}}$.

Except when a statement is made to the contrary, the following hypothesis will be assumed throughout this paper.

HYPOTHESIS (H). *The matrix* $[p_{ij}(x)]$, $(i = 0, \cdots, m;\ j = 0, \cdots, n)$, *is integrable and there exists an* $\varepsilon > 0$ *such that* $|p_{mn}(x)| \geqq \varepsilon$ *almost everywhere on* $[a, b]$. *Moreover,* p_{0n} *and* p_{m0} *are essentially bounded and* $p_{in}p_{mn}^{-1}p_{mj}$ *is integrable,* $(i = 1, \cdots, m-1;\ j = 1, \cdots, n-1)$.

It is to be noted that if $y \in \tilde{\mathfrak{A}}_n$ and $z \in \tilde{\mathfrak{A}}_m$, then $L[y]$ and $L^*[z]$ are integrable.

Let $\mathscr{A}_1(x)$, $\mathscr{A}_2(x)$, $\mathscr{A}_3(x)$, and $\mathscr{A}_4(x)$ be $m \times n$, $m \times m$, $n \times n$, and $n \times m$ matrices, respectively, defined as follows:

$$\mathscr{A}_1(x) = [p_{ij}(x) - p_{in}(x)p_{mn}^{-1}(x)p_{mj}(x)], \quad (i = 0, \cdots, m-1;\ j = 0, \cdots, n-1)\ ,$$

$$\mathscr{A}_2(x) = \begin{bmatrix} 0_{1\,m-1} & p_{0n}(x)p_{mn}^{-1}(x) \\ -E_{m-1} & p_{in}(x)p_{mn}^{-1}(x) \end{bmatrix}, \quad (i = 1, \cdots, m-1)\ ,$$

$$\mathscr{A}_3(x) = \begin{bmatrix} 0_{n-1\,1} & -E_{n-1} \\ p_{mn}^{-1}(x)p_{m0}(x) & p_{mn}^{-1}(x)p_{mj}(x) \end{bmatrix}, \quad (j = 1, \cdots, n-1)\ ,$$

$$\mathscr{A}_4(x) = \begin{bmatrix} 0_{n-1\,m-1} & 0_{n-1\,1} \\ 0_{1\,m-1} & -p_{mn}^{-1}(x) \end{bmatrix}.$$

If f and g belong to $\mathfrak{A}_0$, then the equation $L[y] = f$ is equivalent to the following system in the vector functions $\hat{y} = (\hat{y}_i)$, $(i = 1, \cdots, n)$, and $\check{y} = (\check{y}_j)$, $(j = 1, \cdots, m)$:

$$\begin{aligned} \hat{y}' + \mathscr{A}_3\hat{y} + \mathscr{A}_4\check{y} &= 0\ , \\ \check{y}' - \mathscr{A}_1\hat{y} - \mathscr{A}_2\check{y} &= -fe^{(m,1)}\ ; \end{aligned} \tag{2.5}$$

and the equation $L^*[z] = g$ is equivalent to the following system in the vector functions $\check{z} = (\check{z}_j)$, $(j = 1, \cdots, m)$, and $\tilde{z} = (\tilde{z}_j)$, $(i = 1, \cdots, n)$:

$$\begin{aligned} \check{z}' + \mathscr{A}_2{}^*\check{z} + \mathscr{A}_4{}^*\tilde{z} &= 0\ , \\ \tilde{z}' - \mathscr{A}_1{}^*\check{z} - \mathscr{A}_3{}^*\tilde{z} &= -ge^{(n,1)}\ , \end{aligned} \tag{2.6}$$

where $e^{(k,1)}$, $(k = 1, 2, 3, \cdots)$, is used to denote the k-dimensional vector whose first coordinate is one, and whose remaining coordinates are zero. If $\mathscr{J}$ is the $(m+n) \times (m+n)$ matrix

$$\mathscr{J} = \begin{bmatrix} 0_{mn} & -E_m \\ E_n & 0_{nm} \end{bmatrix}, \tag{2.7}$$

and $\mathscr{A}$ is the $(m+n) \times (m+n)$ matrix function defined by

$$\mathscr{A}(x) = \begin{bmatrix} \mathscr{A}_1(x) & \mathscr{A}_2(x) \\ \mathscr{A}_3(x) & \mathscr{A}_4(x) \end{bmatrix},$$

then (2.5) and (2.6) may be written as

$$\mathscr{L}[\hat{y}] \equiv \mathscr{J}\hat{y}' + \mathscr{A}\hat{y} = fe^{(m+n,1)}\,, \tag{2.8}$$

and

$$\mathscr{L}^{\star}[\breve{z}] \equiv -\mathscr{J}^{*}\breve{z} + \mathscr{A}^{*}\breve{z} = ge^{(m+n,1)}\,, \tag{2.9}$$

respectively.

Theorems on existence and uniqueness of solutions of $L[y] = f$ and $L^{\star}[z] = g$ follow from corresponding theorems for the respective first order systems (2.8) and (2.9). It also follows that $y \in \tilde{\mathfrak{A}}_n$ if and only if there exists an integrable function f such that y is the first coordinate of a vector function $\hat{y}$ satisfying (2.8), and $z \in \tilde{\mathfrak{A}}_m$ if and only if there is an integrable function g such that z is the first coordinate of a vector function $\breve{z}$ satisfying (2.9).

The differential system (2.5) is *identically normal* in the sense that if $\hat{y}(x)$ is a solution of $\mathscr{L}[\hat{y}] = 0$ with $\hat{y}(x) \equiv 0$ on a subinterval X of $[a, b]$, then $\hat{y}(x) \equiv 0$ on X. Indeed, if $\hat{y}$ is such a solution of (2.5), then $\tilde{y}$ is a solution of $\tilde{y}' - \mathscr{A}_2\tilde{y} = 0$ satisfying $\mathscr{A}_4\tilde{y} = 0$ on X. This latter condition implies that $\tilde{y}_m(x) \equiv 0$ on this subinterval, and the differential equation $\tilde{y}' - \mathscr{A}_2\tilde{y} = 0$ implies in turn that $\tilde{y}_j(x) \equiv 0$ on X for $j = m-1, \cdots, 1$. Similarly, system (2.6) is also identically normal. It follows from the identical normality of (2.5) that functions y_α in $\tilde{\mathfrak{A}}_n$ are linearly independent solutions of $L[y] = 0$ if and only if the corresponding vector functions $\hat{y}_\alpha$ are linearly independent solutions of $\mathscr{L}[\hat{y}] = 0$. Similarly, it follows from the identical normality of (2.6) that functions z_α in $\tilde{\mathfrak{A}}_m$ are linearly independent solutions of $L^{\star}[z] = 0$ if and only if the corresponding vector functions $\breve{z}_\alpha$ are linearly independent solutions of $\mathscr{L}^{\star}[\breve{z}] = 0$.

3. The adjoint operator. If $\mathscr{J}$ is the $(m+n) \times (m+n)$ matrix defined as in (2.7), then we may establish the following Lagrange identity by a simple inductive argument which does not use hypothesis (H).

LEMMA 3.1. *If $y \in \tilde{\mathfrak{A}}_n$ and $z \in \tilde{\mathfrak{A}}_m$, then*

$$\bar{z}L[y] - \bar{L}^{\star}[z]y = (\breve{z}^{*}\mathscr{J}\hat{y})'\,. \tag{3.1}$$

THEOREM 3.1. *If $f \in \mathfrak{A}_0$, then there exists a y in $\tilde{\mathfrak{A}}_n^0$ such that $L[y] = f$ if and only if z in $\tilde{\mathfrak{A}}_m$ and $L^{\star}[z] = 0$ implies that $(f, z) = 0$.*

Now if $y \in \tilde{\mathfrak{A}}_n^0$, $L[y] = f$, $z \in \tilde{\mathfrak{A}}_m$, and $L^\star[z] = 0$, then, in view of Lemma 3.1,

$$(f, z) = (L[y], z) - (y, L^\star[z]) = \breve{z}^* \mathscr{J} \hat{y} \,|_a^b = 0 \,.$$

On the other hand, suppose that $(f, z) = 0$ whenever $z \in \tilde{\mathfrak{A}}_m$ and $L^\star[z] = 0$, and let y be the function in $\tilde{\mathfrak{A}}_n$ such that $L[y] = f$ and $\hat{y}(a) = 0$. If z_j, $(j = 1, \cdots, m + n)$ are linearly independent solutions of $L^\star[z] = 0$, then the $(m + n) \times (m + n)$ matrix $\breve{Z}(x)$ with column vectors $\breve{z}_j(x)$, $(j = 1, \cdots, m + n)$, is nonsingular on $[a, b]$. From Lemma 3.1 we have the vector equation

$$0 = [(f, z_j) - (y, L^\star[z_j])] = \breve{Z}^* \mathscr{J} \hat{y} \,|_a^b = \breve{Z}^*(b) \mathscr{J} \hat{y}(b) \,,$$

and consequently $\hat{y}(b) = 0$ also.

THEOREM 3.2. *If hypothesis* (H) *holds, then* $\mathscr{D}_0^* = \tilde{\mathfrak{A}}_m$ *and* $f_z = L^\star[z]$ *on* $\mathscr{D}_0^*$.

That $\tilde{\mathfrak{A}}_m \subset \mathscr{D}_0^*$ follows from Lemma 3.1. Now let $z_0 \in \mathscr{D}_0^*$ and suppose f_{z_0} is a corresponding function in $\mathfrak{A}_0$ such that $(L[y], z_0) = (y, f_{z_0})$ when $y \in \tilde{\mathfrak{A}}_n^0$. Choose w_0 in $\tilde{\mathfrak{A}}_m$ such that $L^\star[w_0] = f_{z_0}$, and suppose that $z_i \in \tilde{\mathfrak{A}}_m$ are linearly independent solutions of $L^\star[z_i] = 0$, with $(z_i, z_j) = \delta_{ij}$, $(i, j = 1, \cdots, m + n)$. If $w = w_0 + \sum_{j=1}^{m+n} (z_0 - w_0, z_j) z_j$, then $L^\star[w] = f_{z_0}$ and $(z_0 - w, z) = 0$ when $z \in \tilde{\mathfrak{A}}_m$ and $L^\star[z] = 0$. It follows that if $y \in \tilde{\mathfrak{A}}_n^0$, then

$$(3.2) \qquad (L[y], z_0) = (y, f_{z_0}) = (y, L^\star[w]) = (L[y], w) \,,$$

so that $(L[y], z_0 - w) = 0$ when $y \in \tilde{\mathfrak{A}}_n^0$. But it follows from Theorem 3.1 that there is a function y in $\tilde{\mathfrak{A}}_n^0$ such that $L[y] = z_0 - w$. Consequently $(z_0 - w, z_0 - w) = 0$ and $z_0 = w \in \tilde{\mathfrak{A}}_m$, so that $\mathscr{D}_0^* = \tilde{\mathfrak{A}}_m$ and $f_{z_0} = L^\star[z_0]$. This result extends Theorem 4.1 of Reid [7].

Now the operator T_0^* adjoint to T_0 is defined to be the operator on $\mathscr{D}_0^*$ with value f_z at z. In view of Theorem 3.2 we have $\mathscr{D}_0^* = \tilde{\mathfrak{A}}_m$ and $T_0^*[z] = L^\star[z]$.

4. Extensions of the operator T_0. Let $\mathscr{D}$ be a linear subspace of $\tilde{\mathfrak{A}}_n$ containing $\tilde{\mathfrak{A}}_n^0$, and denote by T the restriction of L to $\mathscr{D}$. Denote by $\mathscr{D}^*$ the class of functions z in $\mathfrak{A}_0$ which are essentially bounded and for which there exists an f_z in $\mathfrak{A}_0$ such that $(L[y], z) = (y, f_z)$ for all y in $\mathscr{D}$. It follows from Theorem 3.2 that $\mathscr{D}^* \subset \tilde{\mathfrak{A}}_m$ and for each z in $\mathscr{D}^*$ there is at most one f_z, namely $L^\star[z]$, such that $(L[y], z) = (y, f_z)$ for all y in $\mathscr{D}$. The adjoint T^* of T is the operator on $\mathscr{D}^*$ defined by the formula $T^*[z] = f_z$. The operator T is said to be self-adjoint if and only if $\mathscr{D} = \mathscr{D}^*$ and $T = T^*$.

The following lemma will be helpful in describing $\mathscr{D}^*$. If $y_j \in \widetilde{\mathfrak{A}}_n$, $(j = 1, \cdots, m + n)$, then $\widehat{Y}$ will denote the matrix function defined by $\widehat{Y}(x) \equiv [\widehat{y}_j(x)]$, $(j = 1, \cdots, m+ n)$.

LEMMA 4.1. *If η and ζ are $(m + n)$-vectors, then there exists a function $y \in \widetilde{\mathfrak{A}}_n$, $(z \in \widetilde{\mathfrak{A}}_m)$, such that $\widehat{y}(a) = \eta$ and $\widehat{y}(b) = \zeta$, $(\check{z}(a) = \eta$ d an$\check{z}(b) = \zeta)$.*

Since $\widetilde{\mathfrak{A}}_n$ is a vector space it is enough to show that there exist $m + n$ functions y_j in $\widetilde{\mathfrak{A}}_n$ such that $\widehat{y}_j(a) = 0$, $(j = 1, \cdots, m + n)$ while $\widehat{Y}(b)$ is nonsingular, and to show a corresponding result with a and b interchanged. To establish the existence of functions y_j in $\widetilde{\mathfrak{A}}_n$ such that $\widehat{y}_j(a) = 0$, $(j = 1, \cdots, m + n)$, and $\widehat{Y}(b)$ is nonsingular, suppose to the contrary that for each collection of $m + n$ functions y_j in $\widetilde{\mathfrak{A}}_n$ satisfying $\widehat{y}_j(a) = 0$, $(j = 1, \cdots, m + n)$, we have $\widehat{Y}(b)$ singular. Let z_j be $m + n$ linearly independent solutions of $L^{\star}[z] = 0$, and for $j = 1, \cdots, m + n$ let y_j be the function in $\widetilde{\mathfrak{A}}_n$ such that $L[y_j] = z_j$ and $\widehat{y}_j(a) = 0$. Then there is a nonzero $(m + n)$-vector $\xi = (\xi_j)$ such that $\widehat{Y}(b)\xi = 0$. If $y(x) = \sum_{j=1}^{m+n} y_j(x)\xi_j$ and $z(x) = \sum_{j=1}^{m+n} z_j(x)\xi_j$, then $L[y] = z$, $L^{\star}[z] = 0$ and $z(x) \not\equiv 0$, moreover, $y \in \widetilde{\mathfrak{A}}_n^0$. Hence it follows from Lemma 3.1 that

$$0 = (L[y], z) - (y, L^{\star}[z]) = (z, z) ,$$

which is impossible since $z(x) \not\equiv 0$. The numbers a and b may be interchanged and the preceding argument remains valid. The result for $\widetilde{\mathfrak{A}}_m$ follows by interchanging the roles of $\widetilde{\mathfrak{A}}_n$ and $\widetilde{\mathfrak{A}}_m$, that is, by replacing $[p_{ij}]$ with $[p_{ij}]^*$.

Denote by $\mathscr{B}$ the subspace of $2(m + n)$-dimensional complex space consisting of the end values $(\hat{y}(a), \tilde{y}(a), \hat{y}(b), \tilde{y}(b))$ for functions y in $\mathscr{D}$. Similarly, $\mathscr{B}^*$ will denote the subspace of end values $(\check{z}(a), \tilde{z}(a), \check{z}(b), \tilde{z}(b))$ for functions z in $\mathscr{D}^*$. If $k < 2m + 2n$ and the dimension of $\mathscr{B}$ is $2m + 2n - k$, then let P and Q be $(m + n) \times (2m + 2n - k)$ matrices such that the columns of $[-P^* \, Q^*]^*$ form a basis for $\mathscr{B}$. If $k > 0$ also, then let M and N be $k \times (m + n)$ matrices such that the $k \times 2(m + n)$ matrix $[M N]$ has rank k and $MP - NQ = 0$. Then in view of Lemma 4.1 we have that $\mathscr{D}$ is characterized as the class of functions y in $\widetilde{\mathfrak{A}}_n$ with the property that

$$s(\widehat{y}) \equiv M\widehat{y}(a) + N\widehat{y}(b) = 0 . \tag{4.1}$$

If $k = 0$, then by Lemma 4.1 we have $\mathscr{D} = \widetilde{\mathfrak{A}}_n$.

THEOREM 4.1. *Dim* $\mathscr{B}$ + *dim* $\mathscr{B}^* = 2m + 2n$; *if* dim $\mathscr{B} > 0$ *and* P, Q *are* $(m+n) \times (2m+2n-k)$ *matrices such that the column vectors of* $[-P^* \, Q^*]^*$ *from a basis for* $\mathscr{B}$, *then* $\mathscr{D}^*$ *is the class of functions* z *in* $\tilde{\mathfrak{A}}_m$ *for which*

$$P^* \mathscr{J}^* \breve{z}(a) + Q^* \mathscr{J}^* \breve{z}(b) = 0 . \tag{4.2}$$

First note that if dim $\mathscr{B} = 0$, then $\mathscr{D}^* = \tilde{\mathfrak{A}}_m$ by Theorem 3.2, and thus by Lemma 4.1 we have dim $\mathscr{B}^* = 2m + 2n$. Now suppose that dim $\mathscr{B} > 0$, $z \in \tilde{\mathfrak{A}}_m$, and (4.2) holds. Then for y in $\mathscr{D}$ and ξ a $(2m + 2n - k)$-vector chosen so that $\hat{y}(a) = -P\xi$ and $\hat{y}(b) = Q\xi$ it follows from Lemma 3.1 that

$$(L[y], z) - (y, L^{\star}[z]) = \breve{z}^* \mathscr{J} \hat{y} \,|_a^b = \{P^* \mathscr{J}^* \breve{z}(a) + Q^* \mathscr{J}^* \breve{z}(b)\}^* \xi = 0$$

and hence $z \in \mathscr{D}^*$. On the other hand, if $z \in \mathscr{D}^*$ then it follows from Theorem 3.2 that $z \in \tilde{\mathfrak{A}}_m$, since $\tilde{\mathfrak{A}}_n^0 \subset \mathscr{D}$. Then (4.2) follows from Lemma 3.1, Lemma 4.1 and the choice of P and Q. Therefore, in view of Lemma 4.1, it follows that dim $\mathscr{B}$ + dim $\mathscr{B}^* = 2m + 2n$.

COROLLARY I. *If* dim $\mathscr{B} > 0$, *and* R *and* S *are* $(2m + 2n - k) \times (m + n)$ *matrices, then* $\mathscr{D}^*$ *is the collection of functions* z *in* $\tilde{\mathfrak{A}}_m$ *for which*

$$R\breve{z}(a) + S\breve{z}(b) = 0 \tag{4.3}$$

if and only if the $(2m + 2n - k) \times 2(m + n)$ *matrix* $[R \, S]$ *has rank* $2m + 2n - k$ *and* $M\mathscr{J}^* R^* - N\mathscr{J}^* S^* = 0$.

COROLLARY II. *The adjoint of* T^* *is* T.

The *index of compatibility* for a system $L[y] = 0$, $y \in \mathscr{D}$ is defined to be dim $\{y : y \in \mathscr{D}$ and $L[y] = 0\}$. The next two theorems are consequences of the equivalence of the equations $L[y] = f$ and $L^{\star}[z] = g$ to the systems (2.8) and (2.9), respectively, and corresponding theorems on first order systems. Analogous theorems for nth order linear differential equations are given in [2, Chapter 11], and those results may be extended to first order systems.

THEOREM 4.2. *If* dim $\mathscr{B}^* = k$ *and the index of compatibility of the system* $L[y] = 0$, $y \in \mathscr{D}$ *is* r, *then* $\rho = k + r - m - n$ *is the index of compatibility for the system* $L^{\star}[z] = 0$, $z \in \mathscr{D}^*$.

THEOREM 4.3. *If* $f \in \mathfrak{A}_0$, *then there exists a function* y *in* $\mathscr{D}$ *such that* $L[y] = f$ *if and only if* $(f, z) = 0$ *for all* z *in* $\mathscr{D}^*$ *satisfying* $L^{\star}[z] = 0$.

The next two theorems are analogues of Theorems 6.1 and 6.2 of Reid [7]. The second of the two gives necessary and sufficient conditions for the operator T to be self-adjoint when $[p_{ij}(x)]$ is Hermitian. If $y_j \in \tilde{\mathfrak{A}}_n$ and $\hat{Y} = [\hat{y}_j]$, $(j = 1, \cdots, m + n)$, then the symbols $s(\hat{Y})$ and $s^-(\hat{Y})$ are used for the $k \times (m + n)$ matrices $M\hat{Y}(a) + N\hat{Y}(b)$ and $M\hat{Y}(a) - N\hat{Y}(b)$, respectively. Similarly, if $z_j \in \tilde{\mathfrak{A}}_m$ and $\breve{Z} = [\breve{z}_j]$, $(j = 1, \cdots, m + n)$, then $t(\breve{Z})$ and $t^-(\breve{Z})$ denote $R\breve{Z}(a) + S\breve{Z}(b)$ and $R\breve{Z}(a) - S\breve{Z}(b)$, respectively.

THEOREM 4.4. *Suppose that* $2(m + n) > \dim \mathscr{B} > 0$, y_j *and* z_j, $(j = 1, \cdots, m + n)$, *are linearly independent solutions of* $L[y] = 0$ *and* $L^{\star}[z] = 0$, *respectively, and let* $\Delta = (\breve{Z}^* \mathscr{J} \hat{Y})^{-1}$. *Then* Δ *is constant on* $[a, b]$ *and* $\mathscr{D}^*$ *is the collection of functions* z *in* $\tilde{\mathfrak{A}}_m$ *satisfying* (4.3) *if and only if the* $(2m + 2n - k) \times 2(m + n)$ *matrix* $[R\, S]$ *has rank* $2m + 2n - k$ *and*

$$(4.4) \qquad s(\hat{Y})\Delta\{t^-(\breve{Z})\}^* + s^-(\hat{Y})\Delta\{t(\breve{Z})\}^* = 0 .$$

THEOREM 4.5. *Suppose that* $m = n$, $[p_{ij}(x)]$, $(i, j = 0, \cdots, n;$ $x \in [a, b])$, *is Hermitian and* $\dim \mathscr{B} = 2n$. *Let* y_j, $(j = 1, \cdots, 2n)$, *be linearly independent solutions of* $L[y] = 0$, *and let* $\Delta = (\hat{Y}^* \mathscr{J} \hat{Y})^{-1}$. *Then* Δ *is constant on* $[a, b]$, *and* T *is self-adjoint if and only if the* $2n \times 2n$ *matrix* $s^-(\hat{Y})\Delta\{s(\hat{Y})\}^*$ *is Hermitian.*

5. **Generalized Green's functions.** The subspaces $\mathscr{D}$, $\mathscr{D}^*$ of $\tilde{\mathfrak{A}}_n$ and $\tilde{\mathfrak{A}}_m$, respectively, and the subspaces $\mathscr{B}$, $\mathscr{B}^*$ of $2(m + n)$-dimensional complex space are as defined in § 4. If $0 < \dim \mathscr{B} < 2m + 2n$, then the matrices M, N, P, and Q are as specified in § 4.

If $f \in \mathfrak{A}_0$ then we are concerned with solutions of the quasi-differential system

$$(5.1) \qquad L[y] = f , \qquad y \in \mathscr{D} .$$

Of prime importance is the homogeneous system

$$(5.2) \qquad L[y] = 0 , \qquad y \in \mathscr{D} ,$$

and its adjoint system

$$(5.3) \qquad L^{\star}[z] = 0 , \qquad z \in \mathscr{D}^* .$$

By definition a generalized Green's function for the system (5.2) *is an essentially bounded and measurable function* g *on* $\square \equiv \{(x, t) : a \leqq x \leqq b,\ a \leqq t \leqq b\}$ *with the property that if* f *is a function in* $\mathfrak{A}_0$ *for which* (5.1) *has a solution, then a particular solution* y

of (5.1) *is given by*

$$y(x) = \int_a^b g(x, t) f(t) dt \, . \tag{5.4}$$

Reid [5] has shown the existence of a generalized Green's matrix for a compatible first order system with two-point boundary conditions, where the number of independent boundary conditions is equal to the number of differential equations. If $\dim \mathscr{B} = m + n$, then Reid's results could be used to obtain a generalized Green's function for (5.2). In this section the existence and some properties of a generalized Green's function will be shown when $\dim \mathscr{B}$ is not necessarily equal to $m + n$. The technique used here may be modified to extend Reid's results to the case where the number of independent boundary conditions is different from the number of differential equations.

For a νth order linear differential operator $\sum_{j=0}^{\nu} q_j(x) y^{(j)}$ with $q_j \in C_j$, $(j = 0, 1, \cdots, \nu)$, and $q_\nu(x) \neq 0$, the generalized Green's function has been treated by Greub and Rheinboldt [4] and Wyler [10]; a more comprehensive treatment of an algebraic theory of operator solutions of boundary problems, which includes this case as a special instance, is given in Wyler [11].

LEMMA 5.1. *If y_j, $(j = 1, \cdots, m + n)$, are linearly independent solutions of $L[y] = 0$, then there exist $m + n$ linearly independent solutions z_j of $L^\star[z] = 0$ such that*

$$\check{Z}^* \mathscr{J} \hat{Y} = E_{m+n} \, . \tag{5.5}$$

This result follows from Lemma 3.1 and the existence and uniqueness theorems for the equations $\mathscr{L}[\hat{y}] = 0$ and $\mathscr{L}^\star[\check{z}] = 0$.

If $y_j \in \mathfrak{A}_n$ and $z_j \in \mathfrak{A}_m$, $(j = 1, \cdots, m + n)$, then define matrix functions $\hat{Y}$, $\tilde{Y}$, $\check{Z}$, and $\tilde{Z}$ as follows: $\hat{Y}(x) = [\hat{y}_j(x)]$, $\tilde{Y}(x) = [\tilde{y}_j(x)]$, $\check{Z}(x) = [\check{z}_j(x)]$, and $\tilde{Z}(x) = [\tilde{z}_j(x)]$, $(j = 1, \cdots, m + n)$.

COROLLARY. *If y_j and z_j, $(j = 1, \cdots, m + n)$, are as in Lemma* 5.1, *then*

$$\begin{aligned} \hat{Y}(x)\check{Z}^*(x) &\equiv 0_{nm} \, , & \hat{Y}(x)\tilde{Z}^*(x) &\equiv E_n \, , \\ \tilde{Y}(x)\check{Z}^*(x) &\equiv -E_m \, , & \tilde{Y}(x)\tilde{Z}^*(x) &\equiv 0_{mn} \, . \end{aligned} \tag{5.6}$$

THEOREM 5.1. *If $\tau \in [a, b]$, ξ_j is a constant, y_j and z_j, $(j = 1, \cdots, m + n)$, are as in Lemma* 5.1, *then the solution y of $L[y] = f$ satisfying $\hat{y}(\tau) = \sum_{j=1}^{m+n} \hat{y}_j(\tau)\xi_j$ is given by the first component of the vector*

$$(5.7) \qquad \hat{y}(x) = \sum_{j=1}^{m+n} \hat{y}_j(x)\xi_j + \int_\tau^x \sum_{j=1}^{m+n} \hat{y}_j(x)\check{z}_j(t)f(t)dt \, .$$

Indeed, if $\xi = (\xi_j)$, $(j = 1, \cdots, m + n)$, and we set $\hat{y}(x) = \hat{Y}(x)u(x)$, for u an $(m + n)$-vector function, then $\hat{y}$ is a solution of $\mathscr{L}[\hat{y}] = fe^{(m+n,1)}$, $\hat{y}(\tau) = \hat{Y}(\tau)\xi$ if and only if

$$\mathscr{J}\hat{Y}(x)u'(x) = e^{(m+n,1)}f(x), \qquad u(\tau) = \xi \, .$$

Hence $u'(x) = \check{Z}^*(x)e^{(m+n,1)}f(x)$ and

$$u(x) = \xi + \int_\tau^x \check{Z}^*(s)e^{(m+n,1)}f(s)ds \, ,$$

from which the theorem follows.

Now suppose that y_j, $(j = 1, \cdots, m + n)$, are linearly independent solutions of $L[y] = 0$ and that z_j, $(j = 1, \cdots, m + n)$, are chosen as in Lemma 5.1. If $\dim \mathscr{B} = 2m + 2n - k$, $k > 0$, then $s(\hat{Y})$ and $s^-(\hat{Y})$ are $k \times (m + n)$ matrices defined as $s(\hat{Y}) = M\hat{Y}(a) + N\hat{Y}(b)$ and $s^-(\hat{Y}) = M\hat{Y}(a) - N\hat{Y}(b)$. If r is the index of compatibility for (5.2), then $s(\hat{Y})$ has rank $m + n - r$. If $r > 0$, then let S be an $(m + n) \times r$ matrix with the property that $S^*S = E_r$ and $s(\hat{Y})S = 0$. If $r > m+n-k$, then T will represent a $k \times (k - m - n + r)$ matrix such that $T^*T = E_{k-m-n+r}$ and $T^*s(\hat{Y}) = 0$. It follows that the $(k + r) \times (k + r)$ matrix

$$(5.8) \qquad \begin{bmatrix} s(\hat{Y}) & T \\ S^* & 0 \end{bmatrix}$$

is nonsingular, and its inverse is of the form

$$(5.9) \qquad \begin{bmatrix} D & S \\ T^* & 0 \end{bmatrix} .$$

The $(m + n) \times k$ matrix D is the generalized reciprocal of $s(\hat{Y})$ in the sense of E. H. Moore, (see [9, Section 14]). If $r = 0$, then the matrix S does not appear, if $r = m + n - k$, then T does not appear.

Now if $\dim \mathscr{B} < 2(m + n)$, let $G(x, t)$ be the $(m + n) \times (m + n)$ matrix defined by

$$G(x, t) = \frac{1}{2}\hat{Y}(x)\Big[\frac{|x - t|}{x - t}E_{m+n} + Ds^-(\hat{Y})\Big]\check{Z}^*(t) \, , \quad x \neq t \, ;$$

$$G(x, x) = \frac{1}{2}\hat{Y}(x)Ds^-(\hat{Y})\check{Z}^*(x) \, , \qquad x \in [a, b] \, .$$

If $\dim \mathscr{B} = 2(m + n)$, let $G(x, t)$ be defined by

$$G(x, t) = \frac{1}{2} \frac{|x - t|}{x - t} \hat{Y}(x)\breve{Z}^*(t) , \qquad x \neq t ;$$
$$G(x, x) = 0 , \qquad x \in [a, b].$$

Let g_0 be the function with domain $\square$ whose value at (x, t) is the element in the first row and first column of $G(x, t)$, that is

$$g_0(x, t) = g_{0,1}(x, t) + g_{0,2}(x, t) \qquad \textit{if } \dim \mathscr{B} < 2(m + n) ,$$
$$g_0(x, t) = g_{0,1}(x, t) \qquad \textit{if } \dim \mathscr{B} = 2(m + n) ,$$

where

$$g_{0,1}(x, t) = \frac{1}{2} \operatorname{sgn}(x - t) \sum_{i=1}^{m+n} y_i(x)\bar{z}_i(t),$$
$$g_{0,2}(x, t) = \frac{1}{2} \sum_{i,j=0}^{m+n} y_i(x)\mathscr{K}_{ij}\bar{z}_j(t) ,$$

provided $[\mathscr{K}_{ij}]$ is the matrix $Ds^-(\hat{Y})$ and $\operatorname{sgn} u = |u|/u$ for $u \neq 0$, $\operatorname{sgn} 0 = 0$.

THEOREM 5.2. *The function g_0 defined above is a generalized Green's function for* (5.2).

If $\dim \mathscr{B} = 2(m + n)$, then this result follows directly from Theorem 5.1. Now suppose that $\dim \mathscr{B} < 2(m + n)$, and f is an integrable function for which (5.1) has a solution. If y is a solution of $L[y] = f$, then for a suitable vector ξ one has

$$\hat{y}(x) = \frac{1}{2}\left[\hat{Y}(x)\xi + \int_a^x \hat{Y}(x)\breve{Z}^*(t)e^{(m+n,1)}f(t)dt - \int_x^b \hat{Y}(x)\breve{Z}^*(t)e^{(m+n,1)}f(t)dt \right] .$$

Thus, since (5.9) is the inverse of (5.8), it follows that y is a solution of (5.1) if and only if

$$T^*s^-(\hat{Y}) \int_a^b \breve{Z}^*(t)e^{(m+n,1)}f(t)dt = 0 ,$$

and for some r-vector η we have

$$\xi = Ds^-(\hat{Y}) \int_a^b \breve{Z}^*(t)e^{(m+n,1)}f(t)dt + S\eta .$$

Therefore,

$$\hat{y}(x) = \frac{1}{2}\left[\hat{Y}(x)S\eta + \hat{Y}(x)Ds^-(\hat{Y}) \int_a^b \breve{Z}^*(t)e^{(m+n,1)}f(t)dt \right.$$
$$\left. + \int_a^b \hat{Y}(x)\frac{|x - t|}{x - t} \breve{Z}^*(t)e^{(m+n,1)}f(t)dt \right] ,$$

from which the theorem follows since η may be chosen to be zero.

The symbol $g_0^{(i,j)}$ will be used to signify the partial derivative $\partial^{i+j}g_0/\partial t^j\partial x^i$. Generalized partial derivatives $g_0^{\langle\alpha,\beta\rangle}$ will now be defined for g_0. If $\alpha < n$ and $\beta < m$, then $g_0^{\langle\alpha,\beta\rangle}(x, t) = g_0^{(\alpha,\beta)}(x, t)$. If $\alpha < n$, then $g_0^{\langle\alpha,m+j\rangle}$, $(j = 0, \cdots, n-1)$, is defined as follows:

$$g_0^{\langle\alpha,m\rangle}(x, t) = \sum_{i=0}^{m} \bar{p}_{in}(t)g_0^{\langle\alpha,i\rangle}(x, t) \ ;$$

if $g^{\langle\alpha,m-1+j\rangle}$ is a.c. in its second argument, then

$$g_0^{\langle\alpha,m+j\rangle}(x, t) = \sum_{i=0}^{m} \bar{p}_{i\,n-j}(t)g_0^{\langle\alpha,i\rangle}(x, t) - \partial/\partial t\, g_0^{\langle\alpha,m-1+j\rangle}(x, t) \ , \quad (j = 1, \cdots, n-1) \ .$$

If $\beta < m$, then $g_0^{\langle n+i,\beta\rangle}$, $(i = 0, \cdots, m-1)$, is defined as follows:

$$g_0^{\langle n,\beta\rangle}(x, t) = \sum_{j=0}^{n} p_{mj}(x)g_0^{\langle j,\beta\rangle}(x, t) \ ;$$

if $g^{\langle n-1+i,\beta\rangle}$ is a.c. in its first argument, then

$$g_0^{\langle n+i,\beta\rangle}(x, t) = \sum_{j=1}^{n} p_{m-i\,j}(x)g_0^{\langle j,\beta\rangle}(x, t) - \partial/\partial x\, g_0^{\langle n-1+i,\beta\rangle}(x, t) \ , \quad (i = 1, \cdots, m-1) \ .$$

THEOREM 5.3. *If $\alpha + \beta \leqq m + n - 2$, and g_0 is the function of Theorem 5.2, then $g_0^{\langle\alpha,\beta\rangle}$ exists and is continuous on $\Box$.*

This result clearly holds for $g_{0,2}$, hence one need only consider specifically $g_{0,1}$. Let $\alpha + \beta \leqq m + n - 2$, and suppose first that $\alpha < n$. If $\beta < m$, then the theorem follows from the fact that $\hat{Y}(x)\check{Z}^*(x) \equiv 0$. If $\beta = m - 1 + j$, $(j = 1, \cdots, n - \alpha - 1)$, then use the identity $\hat{Y}(x)\tilde{Z}^*(x) \equiv E_m$. On the other hand, if $\beta < m$ and $\alpha = n - 1 + i$, $(i = 1, \cdots, m - \beta - 1)$, then use the identity $\tilde{Y}(x)\check{Z}^*(x) \equiv -E_m$.

THEOREM 5.4. *The generalized Green's function for the system* (5.2) *is not unique. If $u_1, \cdots, u_r$ form a basis for the solutions of* (5.2), *$v_1, \cdots, v_\rho$ form a basis for the solutions of* (5.3), *and g_0 is one generalized Green's function for* (5.2) *then a function g on $\Box$ is also a generalized Green's function for* (5.2) *if and only if there exist essentially bounded and measurable functions $\psi_1, \cdots, \psi_r$, $\varphi_1, \cdots, \varphi_\rho$ such that if $(x, t) \in \Box$, then*

$$g(x, t) = g_0(x, t) + \sum_{i=1}^{r} u_i(x)\psi_i(t) + \sum_{j=1}^{\rho} \varphi_j(x)\bar{v}_j(t) \ . \tag{5.10}$$

If g is a function on $\Box$ satisfying (5.10), then in view of Theorem

4.3 it follows that g is a generalized Green's function for (5.2).

To establish the converse we may assume without loss of generality that $(u_i, u_j) = \delta_{ij}$, $(i, j = 1, \cdots, r)$, and $(v_\alpha, v_\beta) = \delta_{\alpha\beta}$, $(\alpha, \beta = 1, \cdots, \rho)$. If $w \in \mathfrak{A}_0$ and $f(x) = w(x) - \sum_{j=1}^{\rho} (w, v_j)v_j(x)$, then $(f, v_\alpha) = 0$, $(\alpha = 1, \cdots, \rho)$. Thus for this choice of f it follows from Theorem 4.3 that (5.1) has a solution. Suppose that g is a second generalized Green's function for (5.2) and let $d(x, t) = g(x, t) - g_0(x, t)$. Then there are constants $\xi_1, \cdots, \xi_r$ such that

$$\int_a^b d(x, t)f(t)dt = \sum_{i=1}^{r} u_i(x)\xi_i ,$$

and if $\Phi(x, t) = d(x, t) - \sum_{j=1}^{\rho} \bar{v}_j(t) \int_a^b d(x, s)v_j(s)ds$, then

$$\int_a^b \Phi(x, t)f(t)dt = \sum_{i=1}^{r} u_i(x)\xi_i . \tag{5.11}$$

Multiplying (5.11) by $\bar{u}_i(x)$, and integrating with respect to x, we have

$$\int_a^b \int_a^b \bar{u}_i(x)\Phi(x, t)f(t)dtdx = \xi_i , \qquad (i = 1, \cdots, r) ,$$

and consequently

$$\int_a^b \left[\Phi(x, t) - \sum_{i=1}^{r} u_i(x) \int_a^b \bar{u}_i(s)\Phi(s, t)ds\right]w(t)dt = 0 .$$

But w is an arbitrary integrable function, and hence

$$\Phi(x, t) - \sum_{i=1}^{r} u_i(x) \int_a^b \bar{u}_i(s)\Phi(s, t)ds = 0 \quad \text{on } \square ,$$

and

$$d(x, t) = \sum_{i=1}^{r} u_i(x) \int_a^b \bar{u}_i(s)\Phi(s, t)ds + \sum_{j=1}^{\rho} \bar{v}_j(t) \int_a^b d(x, s)v_j(s)ds .$$

Hence (5.10) holds with ψ_i and φ_j defined by $\psi_i(t) = \int_a^b u_i(s)\Phi(s, t)\, ds$. and $\varphi_j(x) = \int_a^b d(x, s)v_j(s)ds$, $(i = 1, \cdots, r; j = 1, \cdots, \rho)$, and clearly these functions are essentially bounded and measurable.

We now show that a generalized Green's function g for (5.2) has the property that the function h defined by $h(x, t) = \bar{g}(t, x)$ is a generalized Green's function for the adjoint system (5.3). Preliminary to this result we shall prove the following theorem.

THEOREM 5.5. *Suppose that $u_1, \cdots, u_r$ form a basis for the solutions of* (5.2), *$v_1, \cdots, v_\rho$ from a basis for the solutions of* (5.3), *and $\Theta = \{\theta_1, \cdots, \theta_r\}$, $\Omega = \{\omega_1, \cdots, \omega_\rho\}$ are sets of integrable functions* *with the property that the matrices* $[(u_i, \theta_j)]$, $(i, j = 1, \cdots, r)$, *and* $[(v_\alpha, \omega_\beta)]$, $(\alpha, \beta = 1, \cdots, \rho)$, *are nonsingular. Then there exists a unique generalized Green's function* $g_L(, ; \Theta, \Omega)$ *for* (5.2) *satisfying the conditions*

$$\begin{aligned} &\int_a^b g_L(x, t; \Theta, \Omega)\omega_\alpha(t)dt = 0 , &&(\alpha = 1, \cdots, \rho) , \\ &\int_a^b \bar{\theta}_i(x) g_L(x, t; \Theta, \Omega)dx = 0 , &&(i = 1, \cdots, r) . \end{aligned} \tag{5.12}$$

Without any loss of generality we can assume that $[(u_i, \theta_j)] = E_r$ and $[(v_\alpha, \omega_\beta)] = E_\rho$. Let g_0 be the generalized Green's function for (5.2) described in Theorem 5.2. We now determine functions $\psi_1, \cdots, \psi_r$ and functions $\varphi_1, \cdots, \varphi_\rho$ such that the generalized Green's function given by (5.10) satisfies conditions (5.12). Such a generalized Green's function g will satisfy the conditions (5.12) if and only if the functions ψ_i, $(i = 1, \cdots, r)$, and φ_α, $(\alpha = 1, \cdots, \rho)$, satisfy the equations

$$\begin{aligned} &\psi_i(x) + \int_a^b \sum_{\beta=1}^{\rho} \bar{\theta}_i(s)\varphi_\beta(s)\bar{v}_\beta(x)ds + \int_a^b \bar{\theta}_i(s) g_0(s, x)ds = 0 , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \cdots, r) , \\ &\varphi_\alpha(x) + \int_a^b \sum_{j=1}^{r} u_j(x)\psi_j(s)\omega_\alpha(s)ds + \int_a^b g_0(x, s)\omega_\alpha(s)ds = 0 , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha = 1, \cdots, \rho) . \end{aligned} \tag{5.13}$$

A particular set of solutions for equations (5.13) is

$$\begin{aligned} \varphi_\alpha(x) &= -\int_a^b g_0(x, s)\omega_\alpha(s)ds , &&(\alpha = 1, \cdots, \rho) , \\ \psi_i(x) &= \int_a^b \int_a^b \sum_{\beta=1}^{\rho} \bar{\theta}_i(t) g_0(t, s)\omega_\beta(s)\bar{v}_\beta(x)dsdt \\ &\quad -\int_a^b \bar{\theta}_i(t) g_0(t, x)dt , &&(i = 1, \cdots, r) . \end{aligned} \tag{5.14}$$

Moreover, if ψ_i and φ_α, $(i = 1, \cdots, r;\ \alpha = 1, \cdots, \rho)$, is any collection of solutions of (5.13), then after substituting the value of $\psi_i(x)$ given by the first equation into the second equation of (5.13) it can be shown by straightforward computation that the value of

$$\sum_{i=1}^{r} u_i(x)\psi_i(t) + \sum_{\alpha=1}^{\rho} \varphi_\alpha(x)\bar{v}_\alpha(t)$$

is independent of the particular ψ_i and φ_α. Hence there is a unique generalized Green's function for (5.2) satisfying (5.12).

The conditions of Theorem 5.5 are clearly satisfied by the sets $\theta_i = u_i$, $(i = 1, \cdots, r)$, and $\omega_\alpha = v_\alpha$, $(\alpha = 1, \cdots, \rho)$; in particular, for linear homogeneous differential operators whose coefficients satisfy suitable differentiability conditions, the treatment of Greub and Rheinboldt [4] is limited to this specification.

It is to be remarked that, in view of the definition of g_0, if ψ_i and φ_α, $(i = 1, \cdots, r;\ \alpha = 1, \cdots, \rho)$, is any collection of solutions of (5.13), then $\varphi_\alpha \in \tilde{\mathfrak{A}}_n$, $(\alpha = 1, \cdots, \rho)$, and $\bar{\psi}_i \in \tilde{\mathfrak{A}}_m$, $(i = 1, \cdots, r)$.

Correspondingly, there exists a unique generalized Green's function $g_{L^\star}(\ ,\ ; \Omega, \Theta)$ for the system (5.3) which satisfies the conditions

$$(5.15)\qquad \begin{aligned} &\int_a^b \bar{\omega}_\alpha(x) g_{L^\star}(x, t; \Omega, \Theta)dx = 0\ , &&(\alpha = 1, \cdots, \rho)\ ,\\ &\int_a^b g_{L^\star}(x, t; \Omega, \Theta)\theta_i(t)dt = 0\ , &&(i = 1, \cdots, r)\ . \end{aligned}$$

For brevity, denote by b_Ω and b_Θ the functions defined on $\square$ by the formulas

$$b_\Omega(x, t) = \sum_{j=1}^{\rho} \omega_j(x)\bar{v}_j(t)\ , \qquad b_\Theta(x, t) = \sum_{i=1}^{r} \theta_i(x)\bar{u}_i(t)\ .$$

THEOREM 5.6. *If $g_L(\ ,\ ; \Theta, \Omega)$ is the unique generalized Green's function satisfying* (5.12), *then the following conditions* (5.16)–(5.20) *are satisfied:*

(5.16) $g_L^{\langle j,0\rangle}(\ ,\ ; \Theta, \Omega)$, $(j = 0, \cdots, m + n - 2)$, *exists and is continuous on $\square$ while $g_L^{\langle m+n-1,0\rangle}(x, t; \Theta, \Omega)$ and $\partial/\partial x\ g_L^{\langle m+n-1,0\rangle}(x, t; \Theta, \Omega)$ exist on the individual domains $a \leqq t < x$, $a < x < b$ and $a \leqq x < b$, $x < t \leqq b$;*

(5.17) *if $t \in [a, b]$, then the function whose value at $x \neq t$ is $g_L^{\langle m+n-1,0\rangle}(x, t; \Theta, \Omega)$ has a right and a left limit at t, denoted by $g_L^{\langle m+n-1,0\rangle}(t^+, t; \Theta, \Omega)$ and $g_L^{\langle m+n-1,0\rangle}(t^-, t; \Theta, \Omega)$, respectively, and*

$$g_L^{\langle m+n-1,0\rangle}(t^-, t; \Theta, \Omega) - g_L^{\langle m+n-1,0\rangle}(t^+, t; \Theta, \Omega) = 1\ ;$$

(5.18) *if $t \in [a, b]$, then $L[g_L(\ , t; \Theta, \Omega)] = b\Omega(\ , t)$ on $[a, t)$ and $(t, b]$;*

(5.19) *if $t \in (a, b)$, then the function whose value at x is $g_L(x, t; \Theta, \Omega)$ satisfies the boundary conditions which characterize the set $\mathscr{D}$;*

$$(5.20)\qquad \int_a^b \bar{\theta}_i(x) g_L(x, t; \Theta, \Omega)dx = 0\ , \qquad (i = 1, \cdots, r;\ t \in [a, b])\ .$$

Conditions (5.16)–(5.18) may be verified directly using the properties of g_0 and the remark following the proof of Theorem 5.5. Condition (5.20) is merely one of the conditions in (5.12). If $\mathscr{D} = \tilde{\mathfrak{A}}_n$, then (5.19) is trivially satisfied. Otherwise, let w be any integrable function, and define f by

$$f(x) = w(x) - \sum_{\alpha=1}^{\rho} \omega_\alpha(x)(w, v_\alpha) = w(x) - \int_a^b b_\Omega(x, t)w(t)dt\ .$$

In view of the assumption that $[(v_\alpha, \omega_\beta)] = E_\rho$, it follows that $(f, v_\alpha) = 0$, $(\alpha = 1, \cdots, \rho)$, and therefore the function u defined by

$$u(x) = \int_a^b g_L(x, t; \Theta, \Omega) f(t) dt$$

is a solution of (5.1). But it follows from (5.12) that

$$\int_a^b g_L(x, t, \Theta, \Omega) f(t) dt = \int_a^b g_L(x, t; \Theta, \Omega) w(t) dt .$$

Therefore,

$$\begin{aligned} 0 &= M\hat{u}(a) + N\hat{u}(b) \\ &= \int_a^b (M\hat{g}_L(a, t; \Theta, \Omega) + N\hat{g}_L(b, t; \Theta, \Omega)) w(t) dt , \end{aligned}$$

from which (5.19) follows in view of the arbitrariness of the function w.

COROLLARY. *If $w \in \mathfrak{A}_0$ and y is defined by*

$$y(x) = \int_a^b g_L(x, t; \Theta, \Omega) w(t) dt ,$$

then

$$\begin{aligned} &L[y] = w - \int_a^b b_\Omega(\ , t) w(t) dt , \\ &y \in \mathscr{D} , \qquad (y, \theta_i) = 0 , \qquad\qquad (i = 1, \cdots, r) . \end{aligned}$$

It should be noted that the unique generalized Green's function $g_{L^\star}(\ , \ ; \Omega, \Theta)$ for (5.3) which satisfies (5.15) also satisfies conditions analogous to (5.16)–(5.20).

THEOREM 5.7. *If $x, t \in [a, b]$, then $g_{L^\star}(x, t; \Omega, \Theta) = \bar{g}_L(t, x; \Theta, \Omega)$.*

Let w and h be arbitrary integrable functions and define y and z by

$$\begin{aligned} y(x) &= \int_a^b g_L(x, t; \Theta, \Omega) w(t) dt , \\ z(x) &= \int_a^b g_{L^\star}(x, t; \Omega, \Theta) h(t) dt , \end{aligned}$$

respectively. Then it follows from the corollary to Theorem 5.6 and its analogue that $y \in \mathscr{D}$, $z \in \mathscr{D}^*$, and therefore

$$(L[y], z) - (y, L^\star[z]) = 0 .$$

But it also follows from the corollary to Theorem 5.6 that $L[y] =$

$w - \int_a^b b_\Omega(\ ,t)w(t)dt$, $L^\star[z] = h - \int_a^b b_\Theta(\ ,t)h(t)dt$, and therefore in view of (5.12), (5.15), and the definition of b_Ω and b_Θ, we have

$$\int_a^b \int_a^b \bar{h}(x)[\bar{g}_{L^\star}(t, x; \Omega, \Theta) - g_L(x, t; \Theta, \Omega]w(t)dtdx = 0 ,$$

from which the theorem follows since w and h are arbitrary integrable functions.

COROLLARY I. *The function $g_L(\ ,\ ; \Theta, \Omega)$ is characterized by conditions* (5.16)–(5.20), *and the function $g_{L^\star}(\ ,\ ; \Theta, \Omega)$ is characterized by analogous conditions.*

As a consequence of Theorems 5.4 and 5.7 one has the following result:

COROLLARY II. *If g is a generalized Green's function for* (5.2), *then the function h defined by $h(x, t) = \bar{g}(t, x)$ is a generalized Green's function for* (5.3).

6. A canonical form for boundary conditions. Let $[f_{ij}]$ and $[g_{ij}]$, $(i = 0, \cdots, m \geqq 1;\ j = 0, \cdots, n \geqq 1)$, be $(m+1) \times (n+1)$ integrable matrix functions. Suppose that the matrix function $[f_{ij}]$, $(i = 0, \cdots, m;\ j = 0, \cdots, n)$, satisfies hypothesis (H), and $g_{mj}(x) \equiv g_{in}(x) \equiv 0$, $(i = 0, \cdots, m;\ j = 0, \cdots, n)$.

For a complex number λ let $p_{ij}(\ ; \lambda)$ be the function defined on $[a, b]$ by

$$p_{ij}(x; \lambda) = f_{ij}(x) + \lambda g_{ij}(x) , \qquad (i = 0, \cdots, m;\ j = 0, \cdots, n) .$$

It follows that for each number λ hypothesis (H) holds for the matrix function $[p_{ij}(\ ; \lambda)]$. For suitable y in $\mathfrak{A}_n$ let $\tilde{y}_1(\ ; \lambda), \cdots, \tilde{y}_m(\ ; \lambda)$ be defined on $[a, b]$ as follows:

$$\tilde{y}_m(x; \lambda) = \sum_{j=0}^{n} p_{mj}(x; \lambda)y^{(j)}(x) = \sum_{j=0}^{n} f_{mj}(x)y^{(j)}(x) ;$$

$$\text{(6.1)} \qquad \textit{if } \tilde{y}_{i+1}(\ ; \lambda) \in \mathfrak{A}_1, \textit{ then } \tilde{y}_i(x; \lambda) = \sum_{j=0}^{n} p_{ij}(x; \lambda)y^{(j)}(x) - \tilde{y}'_{i+1}(x; \lambda) ,$$

$$(i = m-1, \cdots, 1) .$$

The class of functions y in $\mathfrak{A}_n$ for which $\tilde{y}_1(\ , \lambda), \cdots, \tilde{y}_m(\ ; \lambda)$ are a.c. will be denoted by $\tilde{\mathfrak{A}}_n(\lambda)$, and $L[\ ; \lambda]$ will be the operator with domain $\tilde{\mathfrak{A}}_n(\lambda)$, and defined by

$$\text{(6.2)} \qquad L[y; \lambda] = \sum_{j=0}^{n} p_{0j}(\ ; \lambda)y^{(j)} - \tilde{y}'_1(\ ; \lambda) .$$

The vector function $(\tilde{y}_i(\ ;\lambda))$, $(i=1,\cdots,m)$, will be represented by $\tilde{y}(\ ;\lambda)$, and $\hat{y}(\ ;\lambda)$ will signify the $(n+m)$-vector function $(y,\cdots,y^{(n-1)},\tilde{y}_1(\ ;\lambda),\cdots,\tilde{y}_m(\ ;\lambda))$. For a complex number ν let $p^{\star}_{ji}(\ ;\nu)$ be the function on $[a,b]$ defined by

$$p^{\star}_{ji}(x;\nu)=\bar{f}_{ij}(x)+\nu\bar{g}_{ij}(x)\ ,\qquad (i=0,\cdots,m;j=0,\cdots,n)\ .$$

For suitable z in $\mathfrak{A}_m$ define $\tilde{z}_1(\ ;\nu),\cdots,\tilde{z}_n(\ ;\nu)$ by

$$\tilde{z}_n(x;\nu)=\sum_{i=0}^{m}p^{\star}_{ni}(x;\nu)z^{(i)}(x)=\sum_{i=1}^{m}\bar{f}_{in}(x)z^{(i)}(x)\ ;$$

$$\text{(6.3)}\qquad \textit{if}\ \tilde{z}_{j+1}(\ ;\nu)\in\mathfrak{A}_1,\ \textit{then}\ \tilde{z}_j(x;\nu)=\sum_{i=1}^{m}p^{\star}_{ji}(x;\nu)z^{(i)}(x)-\tilde{z}'_{j+1}(x;\nu)\ ;$$

$$(j=n-1,\cdots,1)\ .$$

The class of functions z in $\mathfrak{A}_m$ for which $\tilde{z}_1(\ ;\nu),\cdots,\tilde{z}_n(\ ;\nu)$ are a.c. will be denoted by $\tilde{\mathfrak{A}}_m(\nu)$ and $L^{\star}[\ ;\nu]$ will be operator with domain $\tilde{\mathfrak{A}}_m(\nu)$, and defined by

$$\text{(6.4)}\qquad L^{\star}[z;\nu]=\sum_{i=1}^{m}p^{\star}_{0i}(\ ;\nu)z^{(i)}-\tilde{z}'_1(\ ;\nu)\ .$$

The vector function $(\tilde{z}_j(\ ;\nu))$, $(j=1,\cdots,n)$, will be represented by $\tilde{z}(\ ;\nu)$, and $\hat{z}(\ ;\nu)$ will denote the vector function $(z,\cdots,z^{(m-1)},\tilde{z}_1(\ ,\nu),\cdots,\tilde{z}_n(\ ;\nu))$. Let A_{10}, A_{11}, A_{20}, and A_{21} be $k\times n$ matrices, and let B_1 and B_2 be $k\times m$ matrices, $(1\leqq k\leqq 2m+2n-1)$, such that for each number λ the $k\times 2(m+n)$ matrix

$$[A_1(\lambda)\ -B_1\ A_2(\lambda)\ B_2]$$

has rank k, where $A_1(\lambda)=A_{10}+\lambda A_{11}$ and $A_2(\lambda)=A_{20}+\lambda A_{21}$. Let $\mathscr{D}(\lambda)$ be the collection of functions y in $\tilde{\mathfrak{A}}_n(\lambda)$ for which

$$\text{(6.5)}\qquad A_1(\lambda)\hat{y}(a)-B_1\tilde{y}(a;\lambda)+A_2(\lambda)\hat{y}(b)+B_2\tilde{y}(b;\lambda)=0\ .$$

This section is concerned with the particular Euler type quasi-differential system

$$\text{(6.6)}\qquad L[y;\lambda]=0\ ,\qquad y\in\mathscr{D}(\lambda)\ .$$

It follows from Theorem 3.2 that the system adjoint to (6.6) is

$$\text{(6.7)}\qquad L^{\star}[z;\bar{\lambda}]=0\ ,\qquad z\in\mathscr{D}^*(\bar{\lambda})\ ,$$

where $\mathscr{D}^*(\bar{\lambda})\subset\tilde{\mathfrak{A}}_m(\bar{\lambda})$. The following assumption is made about $\mathscr{D}^*(\bar{\lambda})$:

HYPOTHESIS (H_1). *There exist* $(2m+2n-k)\times m$ *matrices* $A_3(\nu)=A_{30}+\nu A_{31}$, $A_4(\nu)=A_{40}+\nu A_{41}$ *and* $(2m+2n-k)\times n$ *matrices* B_3, B_4 *such that for arbitrary* λ *the set* $\mathscr{D}^*(\bar{\lambda})$ *is the collection of function* z *in* $\tilde{\mathfrak{A}}_m(\bar{\lambda})$ *for which*

$$A_3(\bar{\lambda})\check{z}(a) - B_3\tilde{z}(a;\bar{\lambda}) + A_4(\bar{\lambda})\check{z}(b) + B_4\tilde{z}(b;\bar{\lambda}) = 0\,. \tag{6.8}$$

It shoud be noted that the assumption used by Zimmerberg to obtain Theorem 2.1 of [13] does not imply that hypothesis (H_1) holds. For if $m = n = 1$ and $k = 2n$, then let the matrices A_{10}, A_{11}, B_1, A_{20}, A_{21}, B_2 be defined as

$$\begin{aligned} A_{10}^* &= [1\,1]\,, & A_{11}^* &= [0\,1]\,, & B_1^* &= [2\,1]\,, \\ A_{20}^* &= [1\,0]\,, & A_{21}^* &= [0\,1]\,, & B_2^* &= [0\,1]\,. \end{aligned}$$

Then the hypothesis of Theorem 2.1 of [13] is satisfied, but hypothesis (H_1) does not hold.

If hypothesis (H_1) holds then for each complex number ν the $(2m + 2n - k) \times 2(m + n)$ matrix

$$[A_3(\nu)\,B_3\,A_4(\nu)\,B_4] \tag{6.9}$$

has rank $2m + 2n - k$. Moreover, by a proof quite analogous to that used by Reid to obtain (11.11′) of [6] one may establish the following result.

LEMMA 6.1. *If hypothesis (H_1) holds, then $\mathscr{D}(\lambda)$ is the collection of functions y in $\tilde{\mathfrak{A}}_n(\lambda)$ for which there is a $(2m + 2n - k)$-vector e_0 such that*

$$\begin{aligned} \hat{y}(a) &= B_3^* e_0\,, & \tilde{y}(a;\lambda) &= A_3^*(\bar{\lambda})e_0\,, \\ \hat{y}(b) &= B_4^* e_0\,, & \tilde{y}(b;\lambda) &= -A_4^*(\bar{\lambda})e_0\,, \end{aligned} \tag{6.10}$$

and $\mathscr{D}^*(\bar{\lambda})$ is the collection of functions z in $\tilde{\mathfrak{A}}_m(\bar{\lambda})$ for which there is a k-vector e_1 such that

$$\begin{aligned} \check{z}(a) &= B_1^* e_1\,, & \tilde{z}(a;\bar{\lambda}) &= A_1^*(\lambda)e_1\,, \\ \check{z}(b) &= B_2^* e_1\,, & \tilde{z}(b;\bar{\lambda}) &= -A_2^*(\lambda)e_1\,, \end{aligned} \tag{6.11}$$

where $A_i^*(\nu) = (A_i(\nu))^*$, $(i = 1, 2, 3, 4)$.

Now let $K_{10} = A_{10}B_3^* + A_{20}B_4^*$, $K_{11} = A_{11}B_3^* + A_{21}B_4^*$, $K_1(\lambda) = K_{10} + \lambda K_{11}$, $K_{20} = A_{30}B_1^* + A_{40}B_2^*$, $K_{21} = A_{31}B_1^* + A_{41}B_2^*$, and $K_2(\lambda) = K_{20} + \lambda K_{21}$. Then the next result follows from Lemma 6.1 and Lemma 3.1.

LEMMA 6.2. *If hypothesis (H_1) holds, then $K_2^*(\bar{\lambda}) = K_1(\lambda)$.*

LEMMA 6.3. *Suppose that hypothesis (H_1) holds, the $k \times 2m$ matrix $[B_1\,B_2]$ has rank $k - p$, and the $(2m + 2n - k) \times 2n$ matrix $[B_3\,B_4]$ has rank $2m + 2n - k - q$. Then there exist $p \times n$ matrices ψ_1, ψ_2 and $q \times m$ matrices ψ_3, ψ_4 such that the $p \times 2n$ matrix $[\psi_1\,\psi_2]$ has rank p, the $q \times 2m$ matrix $[\psi_3\,\psi_4]$ has rank q, and*

$$\psi_1\hat{y}(a) + \psi_2\hat{y}(b) = 0\,, \qquad \textit{for } y \in \mathscr{D}(\lambda)\,, \tag{6.12}$$

$$\psi_3\check{z}(a) + \psi_4\check{z}(b) = 0\,, \qquad \textit{for } z \in \mathscr{D}^*(\bar{\lambda})\,. \tag{6.13}$$

Suppose that R is a $p \times k$ matrix of rank p such that $R[B_1\,B_2] = 0$, and define ψ_1 and ψ_2 as $\psi_1 = RA_{10}$, $\psi_2 = RA_{20}$. In view of Lemma 6.2 and the fact that for arbitrary complex λ the $k \times 2(m+n)$ matrix $[A_1(\lambda)\,B_1\,A_2(\lambda)\,B_2]$ has rank k it follows that there exists a $p \times p$ matrix V such that

$$[RA_1(\lambda)\,RA_2(\lambda)] = (E_p + \lambda V)R[A_{10}\,A_{20}]\,.$$

Hence $E_p + \lambda V$ is nonsingular and the equation (6.12) is equivalent to

$$RA_1(\lambda)\hat{y}(a) + RA_2(\lambda)\hat{y}(b) = 0\,.$$

If R_0 is a $q \times (2m + 2n - k)$ matrix of rank q such that $R_0[B_3\,B_4] = 0$, and ψ_3, ψ_4 are defined as $\psi_3 = R_0A_{30}$, $\psi_4 = R_0A_{40}$, then equation (6.13) may be verified in a similar fashion. The conclusion concerning the ranks of $[\psi_1\,\psi_2]$ and $[\psi_3\,\psi_4]$ is clear.

From Lemma 6.2 it then follows that $[B_1\,B_2][\psi_3\,\psi_4]^* = 0$ and $[B_3\,B_4][\psi_1\,\psi_2]^* = 0$, so that $q \leqq 2m - (k - p)$ and $p \leqq 2n - [2m + 2n - k - q] = k + q - 2m$, from which one has the following result.

LEMMA 6.4. *If hypothesis* (H_1) *holds, then the columns of* $[\psi_3\,\psi_4]^*$ *form a basis for the null space of* $[B_1\,B_2]$ *and the columns of* $[\psi_1\,\psi_2]^*$ *form a basis for the null space of* $[B_3\,B_4]$.

The following theorem gives a simultaneous canonical representation of the boundary conditions for (6.6) and (6.7) in terms of parameter matrices ψ_i, Q_i, G_i, $(i = 1, 2, 3, 4)$, and is the central result of this section.

THEOREM 6.1. *Suppose that hypothesis* (H_1) *holds. Then there exist* $m \times n$ *matrices* Q_i *and* G_i, $(i = 1, 2, 3, 4)$, *such that* $y \in \mathscr{D}(\lambda)$ *if and only if there exists a* q*-vector* η_1 *such that*

$$\begin{aligned} &\psi_1\hat{y}(a) + \psi_2\hat{y}(b) = 0\,, \\ &(Q_1 - \lambda G_1)\hat{y}(a) + (Q_2 - \lambda G_2)\hat{y}(b) + \psi_3^*\eta_1 - \tilde{y}(a;\lambda) = 0\,, \\ &(Q_3 - \lambda G_3)\hat{y}(a) + (Q_4 - \lambda G_4)\hat{y}(b) + \psi_4^*\eta_1 + \tilde{y}(b;\lambda) = 0\,. \end{aligned} \tag{6.14}$$

Moreover, $z \in \mathscr{D}^*(\bar{\lambda})$ *if and only if there exists a* p*-vector* η_2 *such that*

$$\begin{aligned} &\psi_3\check{z}(a) + \psi_4\check{z}(b) = 0\,, \\ &(Q_1^* - \bar{\lambda}G_1^*)\check{z}(a) + (Q_3^* - \bar{\lambda}G_3^*)\check{z}(b) + \psi_1^*\eta_2 - \tilde{z}(a;\bar{\lambda}) = 0\,, \\ &(Q_2^* - \bar{\lambda}G_2^*)\check{z}(a) + (Q_4^* - \bar{\lambda}G_4^*)\check{z}(b) + \psi_2^*\eta_2 + \tilde{z}(b;\bar{\lambda}) = 0\,. \end{aligned} \tag{6.15}$$

Suppose that the matrices K_{10} and K_{11} have ranks q_0 and q_1, respectively. Let D_{10} and D_{11} be $(2m + 2n - k) \times (2m + 2n - k - q_0)$ and $(2m + 2n - k) \times (2m + 2n - k - q_1)$ matrices, respectively, whose individual column vectors form orthonormal bases for the null spaces of K_{10} and K_{11}, that is, $K_{10}D_{10} = 0$ and $K_{11}D_{11} = 0$. As $K_{20} = K_{10}^*$ and $K_{21} = K_{11}^*$ by Lemma 6.2, there exist matrices D_{20} and D_{21} of respective orders $k \times (k - q_0)$ and $k \times (k - q_1)$ whose individual column vectors form orthonormal bases for the null spaces of K_{20} and K_{21}. Then

$$(6.16) \qquad \begin{bmatrix} K_{10} & D_{20} \\ D_{20}^* & 0 \end{bmatrix}, \quad \begin{bmatrix} K_{11} & D_{21} \\ D_{11}^* & 0 \end{bmatrix}, \quad \begin{bmatrix} K_{20} & D_{10} \\ D_{20}^* & 0 \end{bmatrix}, \quad \begin{bmatrix} K_{21} & D_{11} \\ D_{21}^* & 0 \end{bmatrix}$$

are nonsingular and have inverses of the form

$$(6.17) \qquad \begin{bmatrix} H_{10} & D_{10} \\ D_{20}^* & 0 \end{bmatrix}, \quad \begin{bmatrix} H_{11} & D_{11} \\ D_{21}^* & 0 \end{bmatrix}, \quad \begin{bmatrix} H_{10}^* & D_{20} \\ D_{10}^* & 0 \end{bmatrix}, \quad \begin{bmatrix} H_{11}^* & D_{21} \\ D_{11}^* & 0 \end{bmatrix},$$

respectively. The matrices H_{10}, H_{11}, H_{10}^*, and H_{11}^* are generalized reciprocals of the respective matrices K_{10}, K_{11}, $K_{20} = K_{10}^*$, and $K_{21} = K_{11}^*$. Let Q_i and G_i, $(i = 1, 2, 3, 4)$, be defined as $Q_1 = A_{30}^*H_{10}A_{10}$, $Q_2 = A_{30}^*H_{10}A_{20}$, $Q_3 = A_{40}^*H_{10}A_{10}$, $Q_4 = A_{40}^*H_{10}A_{20}$, $G_1 = -A_{31}^*H_{11}A_{11}$, $G_2 = -A_{31}^*H_{11}A_{21}$, $G_3 = -A_{41}^*H_{11}A_{11}$, and $G_4 = -A_{41}^*H_{11}A_{21}$.

Now if $y \in \mathscr{D}(\lambda)$ then in view of Lemma 6.3 we need only verify the last two equations of (6.14). Suppose that e_0 is determined by (6.10). Then it follows from (6.10) and the fact that the matrices (6.17) are the inverses of the matrices (6.16) that

$$(6.18) \qquad \begin{aligned} e_0 &= H_{10}A_{10}\hat{y}(a) + H_{10}A_{20}\hat{y}(b) + D_{10}D_{10}^*e_0 \,, \\ e_0 &= H_{11}A_{11}\hat{y}(a) + H_{11}A_{21}\hat{y}(b) + D_{11}D_{11}^*e_0 \,. \end{aligned}$$

Now it follows from (6.10) and (6.18) that

$$(6.19) \qquad \begin{aligned} &(Q_1 - \lambda G_1)\hat{y}(a) + (Q_2 - \lambda G_2)\hat{y}(b) + (A_{30}^*D_{10}D_{10}^* + \lambda A_{31}^*D_{11}D_{11}^*)e_0 \\ &\qquad\qquad - \tilde{y}(a; \lambda) = 0 \,, \\ &(Q_3 - \lambda G_3)\hat{y}(a) + (Q_4 - \lambda G_4)\hat{y}(b) + (A_{40}^*D_{10}D_{10}^* + \lambda A_{41}^*D_{11}D_{11}^*)e_0 \\ &\qquad\qquad + \tilde{y}(b; \lambda) = 0 \,. \end{aligned}$$

But $B_1(A_{30}^*D_{10}D_{10}^* + \lambda A_{31}^*D_{11}D_{11}^*) + B_2(A_{40}^*D_{10}D_{10}^* + \lambda A_{41}^*D_{11}D_{11}^*) = K_{20}^*D_{10}D_{10}^* + \lambda K_{21}^*D_{11}D_{11}^* = 0$, and consequently the two equations of (6.19) may be written in the form of the last two equations of (6.14) involving the parameter vector η_1.

On the other hand, suppose that $y \in \tilde{\mathfrak{A}}_n(\lambda)$ and (6.14) holds. Now the first equation of (6.14) implies that there is a $(2m + 2n - k)$-vector e_0 such that $\hat{y}(a) = B_3^*e_0$ and $\hat{y}(b) = B_4^*e_0$. Hence it follows from (6.16) and (6.17) that (6.18) holds for this value of e_0. Solving the equations

(6.18) for $H_{10}A_{10}\hat{y}(a) + H_{10}A_{20}\hat{y}(b)$ and $H_{11}A_{11}\hat{y}(a) + H_{11}A_{21}\hat{y}(b)$, multiplying the first equation on the left by A_{30}^* and A_{40}^*, and the second equation on the left by λA_{31}^* and λA_{41}^*, respectively, and adding it can be shown that the last two equations of (6.14) may be written as

$$(6.20)\quad \begin{aligned} &A_{30}^*(e_0 - D_{10}D_{10}^*e_0) + \lambda A_{31}^*(e_0 - D_{11}D_{11}^*e_0) + \psi_3^*\eta_1 - \tilde{y}(a;\lambda) = 0\ , \\ &A_{40}^*(e_0 - D_{10}D_{10}^*e_0) + \lambda A_{41}^*(e_0 - D_{11}D_{11}^*e_0) + \psi_4^*\eta_1 + \tilde{y}(b;\lambda) = 0\ . \end{aligned}$$

In view of Lemma 6.2, the definition of the matrices D_{10}, D_{11}, and the choice of the vector e_0, one sees after multiplying the first equation of (6.20) by B_1, the second equation by B_2, and adding the two equations, that y satisfies the boundary conditions of (6.6). The conclusion concerning $D^*(\bar{\lambda})$ may be established in a similar manner.

The next theorem is an application of Theorem 6.1, where it is to be noticed that if $m = n$ and $[f_{ij}(x)]$, $[g_{ij}(x)]$ are Hermitian, then $\tilde{\mathfrak{A}}_n(\lambda) = \tilde{\mathfrak{A}}_n(\lambda)$; in particular, if $z \in \tilde{\mathfrak{A}}_n(\lambda)$, then $\breve{z}(\ ;\lambda) = \tilde{z}(\ ;\lambda)$.

THEOREM 6.2. *Suppose that* $m = n$, $[f_{ij}(x)]$ *and* $[g_{ij}(x)]$ *are Hermitian on* $[a, b]$, $k = 2n$, *and* $\mathscr{D}^*(\bar{\lambda}) = \mathscr{D}(\bar{\lambda})$. *Then the system* (6.6) *is equivalent to the Euler-Lagrange equations and transversality conditions for minimizing the functional*

$$\hat{y}^*(a)[Q_1\hat{y}(a) + Q_2\hat{y}(b)] + \hat{y}^*(b)[Q_2^*\hat{y}(a) + Q_4\hat{y}(b)] + \int_a^b \sum_{\alpha\,\beta=0}^{n} \bar{y}^{(\alpha)} f_{\alpha\beta} y^{(\beta)}\ ,$$

subject to the restraints

$$\psi_1\hat{y}(a) + \psi_2\hat{y}(b) = 0\ ,$$

$$\hat{y}^*(a)[G_1\hat{y}(a) + G_2\hat{y}(b)] + \hat{y}^*(b)[G_2^*\hat{y}(a) + G_4\hat{y}(b)] + \int_a^b \sum_{\alpha,\beta=0}^{n-1} \bar{y}^{(\alpha)} g_{\alpha\beta} y^{(\beta)} = \text{const}\ .$$

If $m = n$, the problem is restricted to the field of real numbers, $g_{ij}(x) \equiv f_{ij}(x) \equiv 0$ for $i \neq j$, and if $f_{ii}, g_{ii} \in \mathfrak{C}_i$, $(i, j = 0, \cdots, n)$, then the results of this section are the same as obtained by Zimmerberg [12], provided that the formula (2.4) of that paper is corrected by replacing $f_i, f_{i+1}, \cdots, f_{n-1}$ by $f_i - \lambda g_i$, $f_{i+1} - \lambda g_{i+1}, \cdots, f_{n-1} - \lambda g_{n-1}$, respectively. If, moreover, $g_{ii}(x) \equiv 0$ for $i \geqq 1$, then these are the same results as obtained by Reid [6, Section 11].

7. **An application.** In this section the results of Section 6 and a theorem of Reid [7] will be used to show that the boundary conditions for a rather large class of linear νth order differential operators may be written in the form given by Theorem 6.1.

Reid [7] has considered νth order linear differential operators L of the form

$$L[y] = \sum_{j=0}^{\nu} q_j(x) y^{(j)} , \qquad \nu \geqq 1 , \tag{7.1}$$

with integrable coefficients. Functions $\Lambda_i(y; p)$, $(i = 0, 1, 2, \cdots)$, were defined as

$$\Lambda_0(y; p) \equiv p(x)y , \qquad \Lambda_{2r}(y; p) \equiv (p(x)y^{(r)})^{(r)} ,$$

$$\Lambda_{2r-1}(y; p) \equiv \frac{1}{2}[(p(x)y^{(r-1)})^{(r)} + (p(x)y^{(r)})^{(r-1)}] , \qquad (r = 1, 2, \cdots) ,$$

with the understanding that $p \in \mathfrak{A}_r$ in the definition of Λ_{2r} and Λ_{2r-1}. The primary result of that paper, and the one of most interest here, is Theorem 3.2, to the effect that if the polynomials 1, $x, \cdots, x^n/n!$, where $n = \nu/2$ or $n = (\nu + 1)/2$ according as ν is even or odd, belong to the domain of the adjoint operator T_0^*, then there exist functions π_j, $(j = 0, \cdots, \nu)$, with $\pi_0 \in \mathfrak{A}_0$, $\pi_{2\alpha-1} \in \mathfrak{A}_\alpha$, $\pi_{2\alpha} \in \mathfrak{A}_\alpha$ such that $L[y]$ is given by

$$L[y] = \sum_{j=0}^{\nu} \Lambda_j(y; \pi_j) , \tag{7.2}$$

while $\mathfrak{A}_\nu$ is contained in the domain of the adjoint operator T_0^* and

$$T_0^*(z) = L^{\star}[z] \equiv \sum_{j=0}^{\nu} \Lambda_j(z;\ (-1)^j \bar{\pi}_j) \qquad \textit{for} \quad z \in \mathfrak{A}_\nu . \tag{7.3}$$

In view of the differentiability properties of π_j, $(j = 1, \cdots, \nu)$, it follows that (7.2) and (7.3) are of the form (6.2) and (6.4), respectively, which in turn reduce to (2.2) and (2.4), respectively, provided that $m = n$, $g_{ij}(x) \equiv 0$ when $i \geqq 1$ or $j \geqq 1$, and for $i, j = 0, \cdots, n$ one defines $f_{ij}(x)$ as follows: $f_{ii}(x) = (-1)^i \pi_{2i}(x)$; $f_{ii-1}(x) = (-1)^i (1/2)\pi_{2i-1}(x)$, $(i = 1, \cdots, n)$; $f_{ii+1}(x) = (-1)^i (1/2)\pi_{2i+1}(x)$, $(i = 0, \cdots, n-1)$; $f_{ij}(x) \equiv 0$, $(j < i - 1$ and $j > i + 1)$.

In particular, if $\nu = 2n$ and $\pi_{2n}(x) \not\equiv 0$, then the vector $\hat{y}(x)$ consists of $y(x)$ and its first $n - 1$ derivatives. Similarly, $\check{z}(x)$ consists of $z(x)$ and its first $n - 1$ derivatives. The coordinates $\tilde{y}_i(x)$ of the n-vector $\tilde{y}(x)$ are defined by (2.1), and may be expressed in terms of $y(x)$ and its first $2n - j$ derivatives, $(j = 1, \cdots, n - 1)$, and similarly for the coordinates of $\tilde{z}(x)$, defined by (2.3). Consequently, $L[y]$ and $L^{\star}[z]$ are defined for $y, z \in \mathfrak{A}_\nu$.

If $\nu = 2n - 1$, and $\pi_\nu(x) \not\equiv 0$, then L is an operator of odd order and we modify the above defined matrix $[f_{ij}(x)]$ in the following way: delete the last row, replace $f_{n-1\,n}(x)$ with $(-1)^{n-1}\pi_{2n-1}(x)$, and replace $f_{n-1\,n-1}(x)$ with $(-1)^{n-1}(\pi_{2n-2}(x) + (1/2)\pi'_{2n-1}(x))$. This change from an $(n + 1) \times (n + 1)$ matrix $[f_{ij}(x)]$ to the $n \times (n + 1)$ matrix $[f_{ij}^0]$ changes neither the value of $L[y]$ nor the value of $L^{\star}[z]$. Now if $\pi_{2n-1} \in \mathfrak{A}_n$, then $\pi'_{2n-1} \in \mathfrak{A}_{n-1}$ so that $\tilde{y}_j(x)$ may still be differentiated out and written in terms of y and its first $2n - j$ derivatives, $(j = 1, \cdots, n-2)$, and similarly $\tilde{z}_i(x)$, $(i = 1, \cdots, n-1)$, may be written in terms of $z(x)$ and its first $2n - i$ derivatives. Consequently we still have that L and $L^{\star}$ have the common domain $\mathfrak{A}_\nu$.

If now it is assumed that there is an $\varepsilon > 0$ such that $|q_\nu(x)| \geqq \varepsilon$ almost everywhere, then it follows from Theorem 3.2, or Theorem 4.1 of [7], that the domain of the adjoint operator T_0^* is $\mathfrak{A}_\nu$. Moreover, in view of the formulas which give the canonical variables $\tilde{y}_j(x)$ and $\tilde{z}_i(x)$ in terms of $y(x), \cdots, y^{(n-1)}(x)$ and $z(x), \cdots, z^{(m-1)}(x)$, respectively, we see that there exist nonsingular linear transformations T and T_1 which transform the vector functions $(y, y', \cdots, y^{(\nu-1)})$ and $(z, z', \cdots, z^{(\nu-1)})$ into the vector functions $(y, y', \cdots, y^{(n-1)}, \tilde{y}_1, \cdots, \tilde{y}_m)$ and $(z, z', \cdots, z^{(m-1)}, \tilde{z}_1, \cdots, \tilde{z}_n)$, respectively. Therefore, in view of Theorem 3.2 of Reid [7] and Theorem 6.1, it follows that boundary conditions for a νth order differential operator of the type described above which involve linearly y and its first $\nu - 1$ derivatives at two points may be written as (6.14), and the adjoint boundary conditions may be written as (6.15).

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