MINIMAL GERSCHGORIN SETS II

B. W. LEVINGER AND R. S. VARGA

The Gerschgorin Circle Theorem, which yields \( n \) disks whose union contains all the eigenvalues of a given \( n \times n \) matrix \( A = (a_{i,j}) \), applies equally well to any matrix \( B = (b_{i,j}) \) of the set \( \Omega_A \) of \( n \times n \) matrices with \( b_{i,j} = a_{i,j} \) and \( |b_{i,j}| = |a_{i,j}| \), \( 1 \leq i, j \leq n \). This union of \( n \) disks thus bounds the entire spectrum \( S(\Omega_A) \) of the matrices in \( \Omega_A \). The main result of this paper is a precise characterization of \( S(\Omega_A) \), which can be determined by extensions of the Gerschgorin Circle Theorem based only on the use of positive diagonal similarity transformations, permutation matrices, and their intersections.

Given any \( n \times n \) complex matrix \( A = (a_{i,j}) \), it is well known that the simplest of Gerschgorin arguments, which depends upon row sums of the moduli of off-diagonal entries of the matrix \( X^{-1}AX \), \( X \) a positive diagonal matrix, yields the union of \( n \) disks which contains all the eigenvalues of \( A \). It is clear that this union of \( n \) disks necessarily contains all the eigenvalues of any \( n \times n \) matrix in the set \( \Omega_A \) defined as follows: \( B = (b_{i,j}) \in \Omega_A \) if \( b_{i,j} = a_{i,j} \), \( 1 \leq i \leq n \), and \( |b_{i,j}| = |a_{i,j}| \) for all \( 1 \leq i, j \leq n, i \neq j \). Hence, this union of \( n \) Gerschgorin disks can be viewed as giving bounds for the entire spectrum \( S(\Omega_A) = \{z | \det (zI - B) = 0 \text{ for some } B \in \Omega_A\} \) of the set \( \Omega_A \).

It is logical to ask to what extent the spectrum \( S(\Omega_A) \) can be more precisely determined by extensions of Gerschgorin's original argument [3]. In the previous paper [6], it was shown that

\[
\partial G(\Omega_A) \subset S(\Omega_A) \subset G(\Omega_A),
\]

where \( G(\Omega_A) \) is the minimal Gerschgorin set deduced from \( A \) and \( \partial G(\Omega_A) \) is its boundary. The first inclusion of (1.1) states that every point of the boundary \( \partial G(\Omega_A) \) of the minimal Gerschgorin set is then an eigenvalue of some \( B \in \Omega_A \). We now extend the results of [6] by making use of results of Schneider [4], and Camion and Hoffman [1]. In so doing, we shall precisely determine \( S(\Omega_A) \).

To begin, let \( P_\phi = (\delta_{i,\phi(j)}) \) be an \( n \times n \) permutation matrix, where \( \phi \) is a permutation of the integers \( 1 \leq i \leq n \) and \( \delta_{i,j} \) is the Kronecker delta function, and let \( X = \text{diag}(x_1, x_2, \ldots, x_n) \), where \( x > 0 \). Given \( B \in \Omega_A \), we define the \( n \times n \) matrix \( M_\phi(x) \) by

\[
M_\phi(x) = (X^{-1}BX - xI)P_\phi = (m_{i,j}),
\]

so that

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(1.3) \[ m_{i,j} = b_{i,\phi(j)}x_{\phi(j)}x_i - \lambda \delta_{i,\phi(j)}, \quad 1 \leq i, j \leq n. \]

Following Schneider [4], if \( \lambda \) is an eigenvalue of \( B \), then \( M^\phi(x) \) is surely singular and thus not strictly diagonally dominant. Hence,

(1.4) \[ |m_{i,i}| \leq \sum_{j \neq i} |m_{i,j}| \]

must be true for at least one \( i, 1 \leq i \leq n \). Defining first

(1.5) \[ A_i(x) \equiv \left( \sum_{j \neq i} |a_{i,j}|x_j \right)x_i, \quad 1 \leq i \leq n, \]

then (1.4) implies that either

(1.6) \[ |\lambda - a_{i,i}| \leq A_i(x) \quad \text{if} \quad \phi(i) = i, \]
or

(1.6') \[ 2|x_i|a_{i,\phi(i)}|x_i| \leq |\lambda - a_{i,i}| + A_i(x) \quad \text{if} \quad \phi(i) \neq i. \]

For any complex number \( \sigma \), we consequently define

(1.7) \[ r_\phi^i(\sigma; x) \equiv A_i(x) - |\sigma - a_{i,i}| \quad \text{if} \quad \phi(i) = i, \]

and let

(1.7') \[ r_\phi^i(\sigma; x) \equiv |\sigma - a_{i,i}| + A_i(x) - 2 |a_{i,\phi(i)}|x_{\phi(i)}/x_i \quad \text{if} \quad \phi(i) \neq i. \]

With this, we next define the set \( G_\phi^i(x) \) as

(1.8) \[ G_\phi^i(x) \equiv \{ \sigma | r_\phi^i(\sigma; x) \geq 0 \}, \quad 1 \leq i \leq n. \]

If \( \phi(i) = i \), then \( G_\phi^i(x) \) reduces to the familiar Gerschgorin disk \( |z - a_{i,i}| \leq A_i(x) \). If \( \phi(i) \neq i \), we observe from (1.7') that \( G_\phi^i(x) \) is the closed exterior of a disk, and is thus an unbounded set.

Defining \( G_\phi(x) \) to be the union of the sets \( G_\phi^i(x) \):

(1.9) \[ G_\phi(x) \equiv \bigcup_{i=1}^n G_\phi^i(x), \]

the inequalities of (1.6) and (1.6') show that if \( \lambda \in S(\Omega_A) \), then \( \lambda \in G_\phi(x) \) for some \( i \), and hence \( \lambda \in G_\phi(x) \). Thus, \( S(\Omega_A) \subset G_\phi(x) \) for every \( x > 0 \), and we then have that

(1.10) \[ G_\phi(\Omega_A) \equiv \bigcap_{x > 0} G_\phi(x), \]

called the minimal Gerschgorin set relative to the permutation \( \phi \), is such that

(1.11) \[ S(\Omega_A) \subset G_\phi(\Omega_A) \]

for every permutation \( \phi \). It is clear that \( G_\phi(\Omega_A) \) is a closed set for
any permutation \( \phi \). Since \( G^\phi(x) \) is a bounded set only when \( \phi(i) = i \), it follows that \( G^\phi(Q_A) \) is a bounded set only when \( \phi \) is the identity permutation. We remark that the results of [6] are for the special case when \( \phi \) is the identity permutation.

Since (1.11) is valid for any permutation \( \phi \), it then follows that

\[
S(\Omega_A) \subset H(\Omega_A) ,
\]

where

\[
H(\Omega_A) \equiv \bigcap_\phi G^\phi(\Omega_A) .
\]

In § 2, we first characterize (Theorem 1) the minimal Gerschgorin sets \( G^\phi(Q_A) \), and then show (Theorem 2) that their boundaries \( \partial G^\phi(\Omega_A) \) are subsets of \( S(\Omega_A) \). Finally, using a result of Camion and Hoffman [1], we prove (Theorem 3) in § 3 our main result that

\[
S(\Omega_A) = H(\Omega_A) .
\]

Summarizing, the now elementary Gerschgorin Circle Theorem [3], applied to a particular matrix \( A \), actually gives eigenvalue bounds for a set \( \Omega_A \) of related matrices. Our main result is that the exact spectrum \( S(\Omega_A) \) of \( \Omega_A \) can be determined from extensions of the Gerschgorin Circle Theorem based only on positive diagonal similarity transformations, permutation matrices, and intersections.

In § 4, we include an extension of a result of [6] concerning the number of eigenvalues of any \( B \in \Omega_A \) in a bounded component of \( G^\phi(Q_A) \). Finally, in § 5 we include several examples to show how \( S(\Omega_A) \) can be determined.

2. The Function \( \nu_\phi(\sigma) \). In order to determine \( G^\phi(\Omega_A) \), let \( \sigma \) be any complex number, and consider the real \( n \times n \) matrix \( Q^\phi(\sigma) = (q_{i,j}) \) whose entries are defined by

\[
q_{i,j} = (-1)^{\delta_{i,j}} |a_{i,\phi(j)} - \sigma \delta_{i,\phi(j)}| , \quad 1 \leq i, j \leq n .
\]

Since the off-diagonal entries of \( Q^\phi(\sigma) \) are nonnegative, then \( Q^\phi(\sigma) \) is essentially nonnegative [2; 5, p. 260], and hence we can associate with the matrix \( Q^\phi(\sigma) \) the real number \( \nu_\phi(\sigma) \), where \( \nu_\phi(\sigma) \) is the (possibly multiple) eigenvalue of \( Q^\phi(\sigma) \) with largest real part. From the Perron-Frobenius theory of nonnegative matrices [5, pp. 46-47], \( \nu_\phi(\sigma) \) corresponds to a nonnegative eigenvector \( y \geq 0 \), i.e., \( Q^\phi(\sigma)y = \nu_\phi(\sigma)y \), and it is further known that

\[
\nu_\phi(\sigma) = \inf_{u > 0} \max_{1 \leq i \leq n} \left\{ \frac{(Q^\phi(\sigma)u)_i}{u_i} \right\} .
\]
We remark that \( v_\phi(\sigma) \) is a continuous function of \( \sigma \).

**Theorem 1.** Let \( A = (a_{i,j}) \) be an \( n \times n \) complex matrix, let \( \phi \) be any permutation, and let \( \sigma \) be a complex number. Then, \( \sigma \in G^\phi(\Omega_A) \) if and only if \( v_\phi(\sigma) \geq 0 \).

**Proof.** From the definitions of \( Q^\phi(\sigma) \) in (2.1) and \( r^\phi_i(\sigma; x) \) in (1.7)-(1.7'), it follows that

\[
(2.3) \quad r^\phi_i(\sigma; x) = \left( \frac{x_{\phi(i)}}{x_i} \right) \left[ \frac{(Q^\phi(\sigma)z)_i}{z_i} \right], \quad \text{where} \quad z_i = x_{\phi(i)}.
\]

Now, if \( \sigma \in G^\phi(\Omega_A) \), then \( \sigma \in G^\phi(x) \) for every \( x > 0 \). But for every \( x > 0 \), there is an \( i \) such that \( \sigma \in G^\phi_\phi(x) \), so that \( r^\phi_i(\sigma; x) \geq 0 \). Since \( x > 0 \), then \( (x_{\phi(i)}/x_i) \) is positive for all \( 1 \leq i \leq n \), and it therefore follows from (2.2) that

\[
\max_{1 \leq i \leq n} |(Q^\phi(\sigma)z)_i/z_i| \geq 0 \quad \text{for every} \quad x > 0.
\]

Clearly, as \( x > 0 \) runs over all positive vectors, so does the corresponding vector \( z > 0 \). Hence, \( v_\phi(\sigma) \geq 0 \) from (2.2). Conversely, assume that \( v_\phi(\sigma) \geq 0 \). From (2.2) and (2.3), it follows that \( r^\phi_i(\sigma; x) \geq 0 \) for some \( i \) for every \( x > 0 \). Hence, \( \sigma \in G^\phi(x) \) for every \( x > 0 \), and thus \( \sigma \in G^\phi(\Omega_A) \), which completes the proof.

Our interest turns now to the boundary \( \partial G^\phi(\Omega_A) \) of the minimal Gerschgorin set \( G^\phi(\Omega_A) \). As usual, it is defined by

\[
(2.4) \quad \partial G^\phi(\Omega_A) = G^\phi(\Omega_A) \cap \overline{G^\phi(\Omega_A)}',
\]

where \( \overline{G^\phi(\Omega_A)}' \) is the closure of the complement \( G^\phi(\Omega_A)' \) of \( G^\phi(\Omega_A) \). It follows from Theorem 1 that \( G^\phi(\Omega_A)' \) is the set of all \( \sigma \) which satisfy \( v_\phi(\sigma) < 0 \). Similarly, the boundary \( \partial G^\phi(\Omega_A) \) of the minimal Gerschgorin set is the set of all \( \sigma \) for which \( v_\phi(\sigma) = 0 \), and to which there exists a sequence of complex numbers \( \{z_j\}_{j=1}^\infty \) with \( \lim_{j \to \infty} z_j = \sigma \) such that \( v_\phi(z_j) < 0 \).

As in [6], we now show that every point of the boundary \( \partial G^\phi(\Omega_A) \) is an eigenvalue of some matrix \( B \in \Omega_A \).

**Theorem 2.** Let \( A = (a_{i,j}) \) be an \( n \times n \) complex matrix, and let \( \phi \) be any permutation. If \( v_\phi(\sigma) = 0 \), then \( \sigma \) is an eigenvalue of some matrix \( B \in \Omega_A \), and thus \( \sigma \in S(\Omega_A) \).

**Proof.** If \( v_\phi(\sigma) = 0 \), then there exists a vector \( y \geq 0 \) with \( y \neq 0 \) such that \( Q^\phi(\sigma)y = 0 \). Writing \( (\sigma - a_{i,k}) = |\sigma - a_{i,k}| \exp(i\psi_k) \), \( 1 \leq k \leq n \), let the \( n \times n \) matrix \( B = (b_{k,i}) \) be defined by
It is evident that $B \in \Omega_A$, and if $y_j = z_{\phi(j)}$, it can be verified (upon considering separately the cases when $\phi(i) = i$ and $\phi(i) \neq i$) that $Q^\phi(\sigma)y = 0$ is equivalent to

$$\sum_{j=1}^n b_{k,j}z_j = \sigma z_k, \quad 1 \leq k \leq n. \quad (2.6)$$

Since $y \neq 0$, then $z \neq 0$, and we conclude from (2.6) that $\sigma$ is an eigenvalue of $B$, which completes the proof.

In order to prove a somewhat stronger result, let $\sigma \in \partial G^\phi(\Omega_A)$. Then, $\nu_\phi(\sigma) = 0$ and $\sigma \in S(\Omega_A)$. But as $S(\Omega_A) \subset G^\phi(\Omega_A)$ from (1.11), we have the

**Corollary 1.** Let $A$ be an $n \times n$ complex matrix. Then, for any permutation $\phi$,

$$\partial G^\phi(\Omega_A) \subset \partial S(\Omega_A). \quad (2.7)$$

In [6], an interesting geometrical property of the boundary $\partial G^\phi(\Omega_A)$ was given when $\phi$ was the identity permutation, and $A$ was assumed to be irreducible. In that case, each boundary point of $G^\phi(\Omega_A)$ was shown to be the intersection of $n$ Gerschgorin circles. An analogous result is true for an arbitrary permutation $\phi$, under slightly stronger hypotheses.

**Corollary 2.** Let $A$ be an $n \times n$ complex matrix, let $\phi$ be any permutation, and let $\sigma \in \partial G^\phi(\Omega_A)$. If $Q^\phi(\sigma)$ is irreducible, then there exists a vector $x > 0$ such that $\sigma \in \partial G^\phi_{\min}(x)$ for all $1 \leq i \leq n$.

**Proof.** If $Q^\phi(\sigma)$ is irreducible, then $Q^\phi(\sigma)$ is essentially positive [5, p. 257]. Thus, there exists a vector $z > 0$ such that $Q^\phi(\sigma)z = \nu_\phi(\sigma)z$. But, if $\sigma \in \partial G^\phi(\Omega_A)$, then $\nu_\phi(\sigma) = 0$, and $Q^\phi(\sigma)z = 0$. Letting $x > 0$ be defined component-wise by $z_i = x_{\phi(i)}$, it then follows from (2.3) that $r^\phi_i(\sigma; x) = 0$ for all $1 \leq i \leq n$. Now, $r^\phi_i(\sigma; x)$ is obviously a continuous function of $\sigma$ from (1.7)-(1.7'), and from (1.8) we deduce that $\partial G^\phi_{\min}(x) = \{\mu \mid r^\phi_i(\mu; x) = 0\}$. Hence, $\sigma \in \partial G^\phi_{\min}(x)$ for all $1 \leq i \leq n$, which completes the proof.

We remark that if $\phi$ is the identity permutation, then $Q^\phi(\sigma)$ is irreducible for any $\sigma$ if and only if $A$ is irreducible. For general $\phi$, it is not difficult to show that $A$ irreducible implies that $Q^\phi(\sigma)$ is irreducible when $\sigma \neq a_{i,i}$ for any $i$.

3. Main Result. We shall now show that $S(\Omega_A) = H(\Omega_A) \equiv \bigcap_{\phi} G^\phi(\Omega_A)$. Since $S(\Omega_A) \subset H(\Omega_A)$ by (1.12), it suffices to prove that
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$S(\Omega_{A})' \subset H(\Omega_{A})'$, where $S(\Omega_{A})'$ denotes the complement of $S(\Omega_{A})$. This last inclusion will follow quite easily from the following theorem of Camion and Hoffman [1]:

Given an arbitrary $n \times n$ complex matrix $B = (b_{i,j})$, let $\hat{\Omega}_{B}$ be the set of all matrices $C = (c_{i,j})$ with $|c_{i,j}| = |b_{i,j}|$ for all $1 \leq i, j \leq n$. Then, if all matrices $C \in \hat{\Omega}_{B}$ are nonsingular, there exists a positive diagonal matrix $X = \text{diag}(x_{1}, \ldots, x_{n})$, $x_{i} > 0$, and a permutation matrix $P_{\phi} = (\delta_{i,\phi(j)})$ such that the matrix $M = BXP_{\phi} = (m_{i,j})$ is strictly diagonally dominant, i.e.,

$$m_{i,i} > \sum_{j \neq i} |m_{i,j}| \quad \text{for all} \quad 1 \leq i \leq n.$$  \hfill (3.1)

We first prove

**Lemma 1.** $\sigma \in S(\Omega_{A})'$ if and only if each $R \in \hat{\Omega}_{A-\sigma}$ is nonsingular.

Proof. It is clear that each $R \in \hat{\Omega}_{A-\sigma}$ can be uniquely expressed as $R = D(B - \sigma I)$, where $D = \text{diag}(e^{i\psi_{1}}, \ldots, e^{i\psi_{n}})$, $\psi_{j}$ is real, and $B \in \Omega_{A}$. Then, $\sigma \in S(\Omega_{A})'$ implies that $\det(B - \sigma I) \neq 0$ for any $B \in \Omega_{A}$. But as $|\det D| = 1$, then $\det R = \det D \cdot \det(B - \sigma I) \neq 0$ for any $R \in \hat{\Omega}_{A}$. The converse follows similarly.

Now, suppose $\sigma \in S(\Omega_{A})'$. From Lemma 1 and the result of Camion and Hoffman applied to $B = A - \sigma I$, there exists a positive diagonal matrix $X = \text{diag}(x_{1}, \ldots, x_{n})$ and a permutation matrix $P_{\phi} = (\delta_{i,\phi(j)})$ such that the matrix $M = (A - \sigma I)XP_{\phi} = (m_{i,j})$ is strictly diagonally dominant, where

$$m_{i,j} = (a_{i,\phi(j)} - \sigma \delta_{i,\phi(j)})x_{\phi(j)}.$$

Comparing (3.2) with the definition of $Q^{\phi}(\sigma)$ in (2.1) and setting $z_{j} = x_{\phi(j)}$, $1 \leq j \leq n$, (3.1) can be equivalently expressed as

$$0 > \sum_{j \neq i} |m_{i,j}| - |m_{i,i}| = (Q^{\phi}(\sigma)z)_{i}, \quad 1 \leq i \leq n.$$  \hfill (3.3)

Since $z > 0$, it follows from (2.2) that $\nu_{\phi}(\sigma) < 0$, and hence from Theorem 1 we deduce that $\sigma \notin G^{\phi}(\Omega_{A})$. Consequently, $\sigma \notin S(\Omega_{A})$ implies that $\sigma \in G^{\phi}(\Omega_{A})$, which in turn implies that $\sigma \notin H(\Omega_{A})$, or

$$S(\Omega_{A})' \subset H(\Omega_{A})'.$$  \hfill (3.4)

This, coupled with the result that $S(\Omega_{A}) \subset H(\Omega_{A})$, gives us

**Theorem 3.** Let $A = (a_{i,j})$ be any $n \times n$ complex matrix. Then
4. Disconnected minimal gerschgorin sets. A familiar result of Gerschgorin [3] states that if \( k \) disks of the Gerschgorin set \( G^i(x) \) (where \( I \) is the identity permutation) are disjoint from the remaining \( n - k \) disks, then these \( k \) disks contain exactly \( k \) eigenvalues of any matrix \( B \in \Omega_A \). In this section, we give a generalization of this result (cf. Theorem 5 of [6]). For a given \( n \times n \) matrix \( A = (a_{i,j}) \) and an arbitrary permutation \( \phi \), let \( G^\phi(\Omega_A) \) denote the nonempty disjoint closed connected components of the minimal Gerschgorin set \( G^\phi(\Omega_A) \):

\[
G^\phi(\Omega_A) = \bigcup_{j=1}^{m} G_j^\phi(\Omega_A) , \quad 1 \leq m \leq n .
\]

For each bounded component \( G_j^\phi(\Omega_A) \), let the order \( s^\phi_j \) be defined as the number of diagonal elements \( a_{i,i} \) of \( A \) contained in \( G_j^\phi(\Omega_A) \) for which \( \phi(i) = i \). We shall show that each matrix \( B \in \Omega_A \) contains exactly \( s^\phi_j \) eigenvalues in each bounded component \( G_j^\phi(\Omega_A) \) of the minimal Gerschgorin set \( G^\phi(\Omega_A) \).

To begin, we enlarge the set \( \Omega_A \). An \( n \times n \) matrix \( B = (b_{i,j}) \) is defined to be an element of the extended set \( \Omega_A^\phi \) if

\[
\begin{align*}
|b_{i,i} - a_{i,i}|, & \quad 1 \leq i \leq n ; \\
|b_{i,j}| & \leq |a_{i,j}|, \quad \phi(i) = i , \\
|b_{i,j}| & \leq |a_{i,j}|, \quad 1 \leq i, j \leq n, \quad \text{for which \( j \neq i \) and \( j \neq \phi(i) \)}.
\end{align*}
\]

Clearly, \( \Omega_A \subseteq \Omega_A^\phi \).

LEMMA 2. Given \( B \in \Omega_A^\phi \), then \( G^\phi(\Omega_B) \subseteq G^\phi(\Omega_A) \).

Proof. For any vector \( u > 0 \) and any complex number \( \sigma \), consider the vector \( Q^\phi_B(\sigma)u \), where we are using an obvious subscript notation. With \( B \in \Omega_A^\phi \), one verifies from (4.2) and (2.1) that \( Q^\phi_B(\sigma)u \leq Q^\phi_A(\sigma)u \) for any \( u > 0 \) and any \( \sigma \), from which it follows that

\[
\max_{1 \leq i \leq n} \left\{ \frac{(Q^\phi_B(\sigma)u)_i}{u_i} \right\} \leq \max_{1 \leq i \leq n} \left\{ \frac{(Q^\phi_A(\sigma)u)_i}{u_i} \right\} .
\]

Thus, from (2.2), \( \nu_{\phi,B}(\sigma) \leq \nu_{\phi,A}(\sigma) \). Hence, by Theorem 1, \( \sigma \in G^\phi(\Omega_B) \) implies that \( \sigma \in G^\phi(\Omega_A) \), which completes the proof.

For this extended set \( \Omega_A^\phi \), we remark that it can be further shown that \( S(\Omega_A^\phi) = G^\phi(\Omega_A) \) for any permutation \( \phi \). This generalizes another result (Theorem 6) of [6].

In the spirit of Gerschgorin's original continuity argument [3], we prove
THEOREM 4. Let \( A = (a_{i,j}) \) be any \( n \times n \) complex matrix, and let \( \phi \) be any permutation. If \( G^s(\Omega_A) \) has a bounded component \( G^s(\Omega_A) \) of order \( s^2 \), then, for any matrix \( B \in \Omega_A \), \( B \) contains exactly \( s^2 \) eigenvalues in \( G^s(\Omega_A) \).

Proof. For any \( B = (b_{i,j}) \in \Omega_A \), consider the family of matrices
\[
B_{m}(\alpha) = (b_{i,j}(\alpha))
\]
deﬁned by
\[
\begin{cases}
  b_{i,i}(\alpha) = b_{i,i}, & 1 \leq i \leq n; \\
  b_{i,\phi(i)}(\alpha) = b_{i,\phi(i)}[m(1 - \alpha) + \alpha] \text{ when } \phi(i) \neq i; \\
  b_{i,j}(\alpha) = \alpha b_{i,j} \text{ for any } 1 \leq i, j \leq n \text{ for which } j \neq i \text{ and } j \neq \phi(i).
\end{cases}
\]

By deﬁnition, \( B_{m}(\alpha) \in \Omega^2_A \) for all \( 0 \leq \alpha \leq 1 \) and all \( m \geq 1 \), and \( B_{m}(1) = B \). Moreover, \( B_{m}(\alpha) \in \Omega^2_{m'(\alpha)} \) for all \( 0 \leq \alpha \leq \alpha' \leq 1 \). Thus, from Lemma 2, \( G^s(\Omega_{m'(\alpha)}) \subset G^s(\Omega_A) \) for all \( 0 \leq \alpha \leq 1 \) and all \( m \geq 1 \), and it is clear that the set \( G^s(\Omega_{m'(\alpha)}) \) increases monotonically with \( \alpha \). We shall show that \( B_{m}(0) \) has exactly \( s^2 \) eigenvalues in the bounded component \( G^s(\Omega_A) \), and the theorem will follow by continuously increasing \( \alpha \) from zero to unity.

From (4.4), the only possibly nonzero entries of the matrix \( B_{m}(0) \) are \( b_{i,i}(0) \) and \( b_{i,\phi(i)}(0) \) where \( \phi(i) \neq i \). Hence, by considering the disjoint cycles of the permutation \( \phi \), we can ﬁnd an \( n \times n \) permutation matrix \( P \) such that

\[
P B_{m}(0) P^T = \begin{bmatrix}
B_{1,1} & 0 \\
0 & B_{N,N}
\end{bmatrix}, \quad 1 \leq N < n.
\]

Here, \( B_{1,1} \) is a diagonal matrix corresponding to all disjoint cycles with \( \phi(i) = i \). The other matrices \( B_{j,j} \) have the cyclic form

\[
B_{j,j} = \begin{bmatrix}
b_{j,j} & b_{j,j}^{(j)} & 0 \\
0 & b_{j,j}^{(j)} & \cdots & \cdots \\
b_{j,j}^{(j)} & \cdots & b_{j,j}^{(j)}
\end{bmatrix}, \quad 2 \leq j \leq N,
\]

where the off-diagonal entries of \( B_{j,j} \) are, from (4.4), given by \( mb_{i,\phi(i)}, \phi(i) \neq i \). Obviously, the eigenvalues of all the \( B_{j,j} \) are the eigenvalues of \( B_{m}(0) \).

The spectrum of matrices of the form (4.6) is discussed in Example 1 of the next section, and in § 6 of [6]. We now assert that

\[
|b_{j,j}^{(j)}b_{j,j}^{(j)} \cdots b_{j,j}^{(j)}| \neq 0 \text{ for any } 2 \leq j \leq N.
\]

Otherwise, \( b_{k,\phi(k)} = 0 \) for some integer \( k \), where \( \phi(k) \neq k \), and, as shown...
in the next section, this implies that $G^\phi(\Omega)$ is the entire complex plane. This contradicts the hypothesis that $G^\phi(\Omega)$ has a bounded component. From (4.4), we can write the product in (4.7) as $m^\sigma j \cdot K_j$, where $K_j$ is independent of $m$ and $\alpha$. Then, it is readily verified that the eigenvalues $\lambda$ of $B_{j,j}$ satisfy

$$
(4.8) \quad \prod_{k=1}^{r_j} |b_{k,k}^{j} - \lambda| = m^\sigma j \cdot K_j, \quad 2 \leq j \leq N,
$$

for any $B_m(0)$ derived from $B \in \Omega_A$. Since $B_m(0) \in \Omega_A^\phi$ for all $m \geq 1$, we may choose $m$ to be arbitrarily large, and it is clear from (4.8) that the eigenvalues of $B_{j,j}$ must lie in an unbounded component of $G^\phi(\Omega)$ for any $2 \leq j \leq N$. Hence, the number of eigenvalues of $B_m(0)$ which lie in the bounded component $G^\phi(\Omega)$ is just the number of diagonal entries of $B_{m}$ in $G^\phi(\Omega)$, which by definition is precisely $s^\phi_j$. Now, increasing $\alpha$ continuously from zero to unity, it follows that $B$ has exactly $s^\phi_j$ eigenvalues in $G^\phi(\Omega)$, which completes the proof.

We remark that the order $s^\phi_j$ of a bounded component $G^\phi(\Omega)$ is a positive integer. For, if $s^\phi_j$ were zero, no $B_m(0)$ would have an eigenvalue in $G^\phi(\Omega)$, so that $S(\Omega) \cap G^\phi(\Omega)$ would be empty, which is a contradiction.

5. Some examples. We now give three examples to illustrate our results concerning the sets $S(\Omega), G^\phi(\Omega)$, and $H(\Omega)$.

Example 1. It was previously shown [6] for the matrix

$$
(5.1) \quad A = \begin{bmatrix}
    a_{1,1} & a_{1,2} & 0 \\
    0 & a_{2,2} & a_{2,3} \\
    0 & a_{n-1,n} & a_{n,n}
\end{bmatrix},
$$

where

$$
(5.2) \quad |a_{1,1}a_{2,3} \cdots a_{n,1}| = 1,
$$

that $\partial G^\psi(\Omega) = S(\Omega)$, $I$ being the identity permutation. Let $\psi$ be the permutation$^1$ $(1\ 2\ 3 \cdots n)$. If $\phi$ is any permutation other than $\psi$ or $I$, there is a positive integer $k$, $1 \leq k \leq n$, such that $\phi(k) \neq k$, and $\phi(k) \neq \psi(k)$, so that $a_{k,\phi(k)} = 0$. Thus, from (1.7'),

$$
(5.2) \quad \gamma^\psi(\sigma; x) = |\sigma - a_{k,k}| + |a_{k,\phi(k)}| x_{\phi(k)} x_k > 0
$$

for all $x > 0$, and for all complex numbers $\sigma$. Hence, we deduce from

$^1$ That is, in this section we are describing a permutation by its disjoint cycles.
(2.2), (2.3), and Theorem 1 that $G^\phi(\Omega_A)$ is the entire complex plane. This argument shows more generally for an arbitrary matrix $A$ that any permutation $\phi$ which places a zero on the diagonal of $Q^\phi(\sigma)$ yields a minimal Gerschgorin set $G^\phi(\Omega_A)$ which is the entire complex plane.

For $\phi = I$, it was shown [6] for the matrix of (5.1) that

\begin{equation}
G^I(\Omega_A) = \left\{ \sigma \left| \prod_{i=1}^{n} |\sigma - a_{i,i}| \leq 1 \right. \right\}
\end{equation}

and in an identical fashion, we can show that

\begin{equation}
G^\phi(\Omega_A) = \left\{ \sigma \left| \prod_{i=1}^{n} |\sigma - a_{i,i}| \geq 1 \right. \right\}.
\end{equation}

Hence, it follows that

\begin{equation}
S(\Omega_A) = H(\Omega_A) = G^I(\Omega_A) \cap G^\phi(\Omega_A) = \partial G^I(\Omega_A).
\end{equation}

**EXAMPLE 2.** Consider the matrix

\begin{equation}
A = \begin{bmatrix}
2 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}
\end{equation}

In this case, there are only three permutations, corresponding to $\phi = I$, $\phi = (13)$, and $\phi = (23)$, for which $G^\phi(\Omega_A)$ is not the entire complex plane, and it is readily verified that

\begin{equation}
\begin{cases}
G^I(\Omega_A) = \{ \sigma \left| 2 - \sigma \cdot |1 - \sigma| \leq |1 - \sigma| + |2 - \sigma| \right. \}, \\
G^{(13)}(\Omega_A) = \{ \sigma \left| 2 - \sigma \cdot |1 - \sigma| \geq |1 - \sigma| - |2 - \sigma| \right. \}, \\
G^{(23)}(\Omega_A) = \{ \sigma \left| 2 - \sigma \cdot |1 - \sigma| \geq - |1 - \sigma| + |2 - \sigma| \right. \}.
\end{cases}
\end{equation}

The boundaries $\partial G^\phi(\Omega_A)$ are obviously determined by choosing the equality signs in (5.7). The spectrum $S(\Omega_A)$ in this case is a multiply connected region and is illustrated in Figure 1.
Example 3. Consider the matrix

\[(5.8) \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -5 & -1 & -1 \end{bmatrix},\]

which is the companion matrix of the polynomial

\[p(z) = z^4 + z^3 + z^2 + 5z + 1.\]

As previously shown, any permutation \(\phi\) which places a zero on the diagonal of \(Q^{\phi}(\sigma)\) yields a minimal Gerschgorin set \(G^{\phi}(\Omega_A)\) which is the entire complex plane. Consequently, we need consider only the permutations \(I, (1234), (234),\) and \((34)\). The associated minimal Gerschgorin sets are given by

\[(5.9) \quad \begin{align*}
G^I(\Omega_A) &= \{\sigma \mid |\sigma|^4 \cdot 1 + \sigma| \leq 1 + 5|\sigma| + |\sigma|^3\}, \\
G^{(1234)}(\Omega_A) &= \{\sigma \mid |\sigma|^4 \cdot 1 + \sigma| \geq 1 - 5|\sigma| - |\sigma|^3\}, \\
G^{(234)}(\Omega_A) &= \{\sigma \mid |\sigma|^4 \cdot 1 + \sigma| \geq -1 + 5|\sigma| - |\sigma|^3\}, \\
G^{(34)}(\Omega_A) &= \{\sigma \mid |\sigma|^4 \cdot 1 + \sigma| \geq -1 - 5|\sigma| + |\sigma|^3\}.
\end{align*}\]

The last minimal Gerschgorin set \(G^{(34)}(\Omega_A)\) is the entire complex plane, and thus yields no boundary components of \(S(\Omega_A)\). The set \(G^{(1234)}(\Omega_A)\) yields, however, two separate boundaries, and \(G^{(234)}(\Omega_A)\) has a bounded component. Applying Theorem 4, we can assert that each matrix of the set \(\Omega_A\) has exactly one eigenvalue in this component, and hence each matrix of \(\Omega_A\) has exactly one eigenvalue in the inner annular region of Figure 2.

These examples have interesting common features. In each ex-
ample, the minimum number of permutations necessary to define all the boundary components of $S(\mathcal{O}_A)$ does not exceed the order $n$ of the matrix $A$. Similarly, the total number of boundary components of $S(\mathcal{O}_A)$ does not exceed $2n$. We conjecture this to be true in general. We do point out that examples can be constructed where these upper bounds are attained.

**BIBLIOGRAPHY**


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