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THE LACK OF SELF-ADJOINTNESS IN THREE-POINT BOUNDARY VALUE PROBLEMS

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Suppose that $a < c < b$, $C_{[a,b]}$ is the set of all real-valued continuous functions on $[a, b]$, each of p and q is in $C_{[a,b]}$, $p(x) > 0$ for all x in $[a, b]$ and each of P, Q and S is a real 2×2 matrix. The assumption is made that the only member f of $C_{[a,b]}$ so that $(pf')' - qf = 0$ and

$$(\Delta) \quad P\begin{bmatrix} f(a) \\ p(a)f'(a) \end{bmatrix} + Q\begin{bmatrix} f(c) \\ p(c)f'(c) \end{bmatrix} + S\begin{bmatrix} f(b) \\ p(b)f'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the zero function. It follows that there is a real-valued continuous function K_{12} on $[a, b] \times [a, b]$ such that if g is in $C_{[a,b]}$, then the only element f of $C_{[a,b]}$ so that $(pf')' - qf = g$ and (Δ) holds is given by

$$f(x) = \int_a^b K_{12}(x, t)g(t)dt \qquad \text{for all } x \text{ in } [a, b] .$$

In this note it is shown that if in addition it is specified that Q is not the zero 2×2 matrix, then K_{12} is not symmetric, i.e., it is not true that $K_{12}(x, t) = K_{12}(t, x)$ for all x, t in $[a, b]$.

The union of (a, c) and (c, b) is denoted by R. The symbol j denotes the identity function on $[a, b]$, i.e., $j(x) = x$ for all x in $[a, b]$. If V is a function from $[a, b] \times [a, b]$ and x is in $[a, b]$, then $V(j, x)$ is the function h such that $h(t) = V(t, x)$ for all t in $[a, b]$. If each of f and $(pf')' - qf$ is in $C_{[a,b]}$, then $(pf')' - qf$ is denoted by Lf.

Given an element g of $C_{[a,b]}$, one has the problem of determining a function f so that

$$(*) \qquad \begin{cases} Lf = g \quad \text{and} \\ (\Delta) \quad \text{holds} . \end{cases}$$

Denote $\begin{bmatrix} 0 & \int_a^t 1/p \\ \int_a^t q & 0 \end{bmatrix}$ by $F(t)$ and $\begin{bmatrix} 0 \\ \int_a^t g \end{bmatrix}$ by $G(t)$ for all t in $[a, b]$.

Then problem $(*)$ may be reformulated as follows: find a function Y from $[a, b]$ to E_2 such that

$$(**) \qquad Y(t) = Y(x) + G(t) - G(x) + \int_x^t dF \cdot Y \quad \text{for all } t, x \text{ in } [a, b] \text{ and}$$

$\int_a^b dH \cdot Y = N$ where

$$H(x) = \begin{cases} 0 & \text{if } x = a \\ P & \text{if } a < x \leqq c \\ P + Q & \text{if } c < x < b \\ P + Q + S & \text{if } x = b\,. \end{cases}$$

The assumption is made for the rest of this paper that only the function Y which is constant at N satisfies $(**)$ if G is constant at N. It follows that for each continuous function G from $[a, b]$ to E_2, $(**)$ has exactly one solution.

Consider the function M from $[a, b] \times [a, b]$ to the set of 2×2 matricies which has the following property:

$$M(t, x) = I + \int_x^t dF \cdot M(j, x) \text{ for all } t, x \text{ in } [a, b]$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Using Theorem B of [2], one has that the unique solution Y of $(**)$ is given by

$$Y(t) = \int_a^b K(t, j) dG \text{ for all } t \text{ in } [a, b] \text{ where}$$

$$K(t, x) = \begin{cases} -\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_x^b dH \cdot M(j, x) + M(t, x) & \text{if } a \leqq x \leqq t \\ -\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_x^b dH \cdot M(j, x) & \text{if } t < x \leqq b\,. \end{cases}$$

That $\left[\int_a^b dH \cdot M(j, t)\right]^{-1}$ exists for all t in $[a, b]$ follows from the assumption that was made above.

Some straightforward calculation gives that

$$K(t, x) = \begin{cases} M(t, b)U(x)M(b, x) + M(t, x) & \text{if } a \leqq x \leqq t \\ M(t, b)U(x)M(b, x) & \text{if } t < x \leqq b \end{cases}$$

where

$$U(x) = \begin{bmatrix} u_{11}(x) & u_{12}(x) \\ u_{21}(x) & u_{22}(x) \end{bmatrix} = -\left[\int_a^b dH \cdot M(j\ b)\right]^{-1} \int_x^b dH \cdot (j, b)$$

for all x in $[a, b]$.

Note that $Y = \begin{bmatrix} f \\ pf' \end{bmatrix}$ where f is the unique solution to $(*)$. Denote K by $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$. It follows that

$$f(t) = \int_a^b K_{12}(t, j) g dj \quad \text{for all } t \text{ in } [a, b] .$$

THEOREM A. *If Q is not the 0-matrix (i.e., (*) is r three-point problem) then it is not true that $K_{12}(t, x) = K_{12}(x, t)$ for all x and t in R.*

Proof. Denote M by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. From [2] one has the following identities:

$$B(t, x) = A(t, b)B(b, x) + B(t, b)D(b, x) \quad \text{if } x \text{ and } t \text{ are in } [a, b]$$
$$(\text{since } M(t, b)M(b,x) = M(t, x) \text{ for all } t, x \text{ in } [a, b]) ,$$

$$A(t, x)D(t, x) - B(t, x)C(t, x) = 1 \quad (\text{i.e., } \det M(t, x) = 1) ,$$
$$A(t, x) = D(x, t) ,$$
$$B(t, x) = -B(x, t) , \quad \text{and}$$
$$C(t, x) = -C(x, t) \quad \text{if } x \text{ and } t \text{ are in } [a, b] .$$

Note that $LA(j, x) = LB(j, x) = 0$ if x is in $[a, b]$.

Suppose that

$$K_{12}(t, x) = K_{12}(x, t) \quad \text{for all } x \text{ and } t \text{ in } R .$$

If $a < x < t < b$, then

$$K_{12}(t, x) = [A(t, b)u_{11}(x) + B(t, b)u_{21}(x)]B(b, x) + [A(t, b)u_{12}(x) + B(t, b)u_{22}(x)]D(b, x) + B(t, x)$$

and

$$K_{12}(x, t) = [A(x, b)u_{11}(t) + B(x, b)u_{21}(t)]B(b, t) + [A(x, b)u_{12}(t) + B(x, b)u_{22}(t)]D(b, t) .$$

Using the identities listed above,

$$\begin{aligned} A(t, b)[&- u_{11}(x)B(x, b) + u_{12}(x)A(x, b) - B(x, b)] \\ &+ B(t, b) [- u_{21}(x)B(x, b) + u_{22}(x)A(x, b) + A(x, b)] \\ &= A(t, b)[u_{12}(t)A(x, b) + u_{22}(t)B(x, b)] \\ &- B(t, b)[u_{11}(t)A(x, b) + u_{21}(t)B(x, b)] . \end{aligned}$$

An examination of this expression yields the fact that it remains true if x and t are interchanged or x is set equal to t.

Denote by x a number in R. Since $u_{11}, u_{11}, u_{22}, u_{22}$ are constant on (a, c) and (c, b) and $A(j, b)$ and $B(j, c)$ are independent solutions v of $Lv = 0$, it follows that

$$- u_{11}(x)B(x, b) + u_{12}(x)A(x, b) - B(x, b) = u_{12}(t)A(x, b) + u_{22}(t)B(x, b)$$

and

$$- u_{21}(x)B(x, b) + u_{22}(x)A(x, b) + A(x, b) = -u_{11}(t)A(x, b) - u_{21}(t)B(x, b)] \quad \text{for all } x \text{ and } t \text{ in } R .$$

Similarly, it follows that

(i) $- u_{11}(x) - 1 = u_{22}(t)$,
(ii) $u_{12}(x) = u_{12}(t)$,
(iii) $u_{21}(x) = u_{21}(t)$ and
(iv) $u_{22}(x) + 1 = -u_{11}(t)$ for all x and t in R.

(ii) and (iii) imply that u_{12} and u_{21} are constant on R. (i) and (iv) give the same information so that only (i) need be considered. Denote $u_{11}(c-)$ by c_1, $u_{22}(c-)$ by c_2, $u_{11}(c+)$ by c_3 and $u_{22}(c+)$ by c_4. Hence (i) gives that $c_1 + c_2 = -1$, $c_1 + c_4 = -1$, $c_3 + c_4 = -1$ and $c_3 + c_2 = -1$. But these equations imply that $c_2 = c_4$ and $c_1 = c_3$, i.e., that u_{11} and u_{22} are constant on R. Hence, U is constant on R. If t is in (a, c) and x is in (c, b), then

$$\left[\int_a^b dH \cdot M(j, b)\right]^{-1} \int_t^x dH \cdot M(j, b) = U(x) - U(t) = 0$$

so that

$$QM(c, b) = \int_t^x dH \cdot M(j, b) = 0 ,$$

i.e., $Q = 0$, a contradiction. Hence the theorem is established.

If n is an integer greater than 3, this theorem can be extended to n point boundary value problems. This is the case in which H is a step function with n discontinuities (with one at a and another at b). What happens when H has points of change other than discontinuities is not at all clear to this author.

References

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