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COMMUTATIVE RINGS WHOSE HOMOMORPHIC IMAGES ARE SELF-INJECTIVE

LAWRENCE S. LEVY

Dedekind domains are characterized among integral domains by the property that every ideal be a projective module. The most naive dual characterization—that every homomorphic image of R be an injective module—is false. In fact, a domain with this property would have to be a field. An injectivity property that works, in the noetherian case, is the property that every proper homomorphic image be a self-injective *ring*. The main result of this note is:

THEOREM. *Let R be a commutative, noetherian ring (with identity). Every proper homomorphic image of R is a self-injective ring if and only if*

(1) *R is a Dedekind domain, or*

(2) *R is a principal ideal ring with descending chain condition, or*

(3) *R is a local ring whose maximal ideal M has composition length 2 and satisfies $M^2 = 0$.*

A more complete description of the rings of type (3) is given in Remark (ii) below.

LEMMA. *Suppose that $M^2 = 0$ for some maximal ideal M of a self-injective, commutative ring R. Then $0 \subsetneqq M \subsetneqq R$ are all of the ideals of R.*

Proof. For x in R let $\operatorname{ann} x$ (the *annihilator* of x) be the ideal of all b in R such that $bx = 0$. If x is a nonzero element of M, then $M \subseteqq \operatorname{ann} x$ since $M^2 = 0$; and since R has an identity, $R \neq \operatorname{ann} x$. Maximality of M therefore shows that $M = \operatorname{ann} x$ so that $Rx \cong R/\operatorname{ann} x = R/M$ (as modules). Hence, for any two nonzero elements x, y of M we have $Rx \cong Ry$.

Injectivity of the R-module R implies that any isomorphism of Rx onto Ry can be extended to a homomorphism of R into itself. Suppose that, in such an extended map, $1 \rightarrow u$. Then $Rxu = Ry$ so that $Rx \supseteqq Ry$ for every nonzero x and y in M. This shows that M has no proper submodules.

Now let N be any ideal $\neq 0$ or M. Then since there are no ideals between 0 and M, and since M is maximal, we have $R = M + N$. Multiplying by M and recalling that $M^2 = 0$ we get $M = MN \subseteqq N$ so that $N = R$.

Proof of the theorem. Suppose that all of the proper homomorphic images of R are self-injective rings, and consider first the case that R is a domain. If M is a maximal ideal of R, the ring R/M^2 must be self-injective and the image $\bar{M}$ of M in R/M^2 satisfies $\bar{M}^2 = 0$; so by our lemma, R has no ideals between M and M^2. But a theorem of I. S. Cohen [2, Theorem 8] states that a noetherian domain with this last property for every maximal ideal M must be a Dedekind domain. Thus we have obtained rings of type(1).

Suppose next that R is *not* a domain. Then 0 is not a prime ideal. Observe that every prime ideal P must be maximal: The integral domain R/P is, by hypothesis, an injective, hence divisible R/P-module [1, Chap. 7, Prop. 12]; that is, a field. But a noetherian commutative ring in which every prime ideal is maximal must satisfy the DCC (descending chain condition) [2, Theorem 1], and hence be the direct sum $R = R_1 \oplus \cdots \oplus R_n$ of local rings with DCC [6, Chap. 4, Theorem 3.3].

Let M_i be the maximal ideal of R_i. Again we consider two cases. Suppose first that either $n > 1$, or $n = 1$ but $M_1^2 \neq 0$. Then for each i, $R/(M_i^2 + \sum_{j\neq i} R_j) \cong R_i/M_i^2$ is a self-injective ring. The lemma (applied to R_i/M_i^2) tells us that for any m_i in M_i but not in M_i^2 we have $M_i = R_i m_i + M_i^2 = R_i m_i + (\operatorname{rad} R_i)M_i$; and Nakayama's lemma then shows that $M_i = R_i m_i$. Thus R_i is a noetherian commutative ring in which every maximal ideal is principal, and a theorem of Kaplansky asserts that any such ring must be a PIR (principal ideal ring) [3, Theorem 12,3]. Thus each R_i and hence R itself is of type (2).

Finally, suppose that R is a local ring with DCC whose maximal ideal M satisfies $M^2 = 0$. We can suppose that M has composition length at least 2, since otherwise R would be a PIR. By the DCC, M contains a minimal ideal N of R. By the lemma, the nonzero ideal M/N of the self-injective ring R/N has composition length 1. Hence M has composition length 2. Thus R is of type (3) and the proof of half of the theorem is done.

Conversely, suppose that R is of type (1), (2), or (3). Observe that the proper homomorphic images of all three types of rings are all PIR's with DCC, and that every ring of this last type is the direct sum of rings R which have exactly one composition series [3, Theorem 13,3]:

$$R \supset Rm \supset Rm^2 \supset Rm^3 \supset \cdots \supset Rm^t = 0 \tag{4}$$

It is therefore sufficient to show that the ring in (4) is self-injective.

To do this let f be a homomorphism of Rm^s into R. Then the composition length of $f(Rm^s)$ cannot exceed that of Rm^s, so that the fact that (4) contains all the ideals of R shows that $f(Rm^s) \subseteqq Rm^s$. Consequently, $f(m^s) = m^s x$ for some x in R. The map $r \rightarrow rx$ (r in R) is then an extension of f to a homomorphism of R into itself, showing that R is an injective R-module and completing the proof of the theorem.

The following consequence (actually, restatement) of the theorem is the converse of an old lemma which is the starting point for some accounts of the theory of finite abelian groups. We define the *order ideal* of an element m of an R-module M to be $\{r \in R\colon rm = 0\}$, and we say that M has *bounded order* if $rM = 0$ for some $r \neq 0$.

COROLLARY. *Let R be a commutative noetherian ring and suppose whenever m is an element with minimal order ideal in an R-module M of bounded order, that Rm is a direct summand of M. Then R is of type* (1), (2), *or* (3).

For a proof, let the ring $\bar{R}$ be a proper homomorphic image of R, and let M be any $\bar{R}$-module which contains $\bar{R}$. Then M has bounded order *as an R-module*, and the identity of $\bar{R}$ is an element with minimal order ideal in the R-module M. Hence, by hypothesis, $\bar{R}$ is a direct summand of M. We have thus shown that $\bar{R}$ is a direct summand of every $\bar{R}$-module which contains it, and hence is self-injective. The theorem now completes the proof of the corollary.

REMARKS. (i) The hypothesis that R be noetherian cannot be dropped from the theorem. For an example, let F be a field and x an indeterminate; and let W be the family of all well-ordered sets {i} of nonnegative real numbers, the order relation being the natural order of the real numbers. Then let R be the set of all "formal power series"

$$\sum_{i \in \{i\}} a_i x^i$$

with a_i in F and $\{i\}$ in W. Then R *is a nonnoetherian domain whose finitely generated ideals are principal and whose proper homomorphic images are self-injective.*

We will need the facts that R is actually a ring and that every element of R whose constant term is nonzero is invertible in R [4, part I]. It follows that every nonzero element of R has the form $x^b u$ where u is invertible in R. This implies that R has only two types of nonzero ideals: The principal ideals (x^b), and those of the form $(x^{>b}) = \{x^c u\colon c > b \text{ and } u \text{ is invertible or zero}\}$.

Let $S = R/J$ where $J \neq 0$ and let $y = x + J$ so that S can be considered at the collection of "formal power series"

$$\sum_{i \in \{i\}} a_i y^i$$

with a_i in F, $\{i\}$ in W, and (I) $y^b = 0$ if $J = (x^b)$, or (II) $y^c = 0$ for $c > b$ if $J = (x^{>b})$. Observe that for $c \leqq b$

$$(5) \qquad (\text{if } J = (x^b)) \qquad \operatorname{ann}(y^c) = (y^{b-c}) \qquad \operatorname{ann}(y^{>c}) = (y^{b-c})$$

$$(6) \qquad (\text{if } J = (x^{>b})) \qquad \operatorname{ann}(y^c) = (y^{>b-c}) \qquad \operatorname{ann}(y^{>c}) = (y^{b-c}) \,.$$

From (5) and (6) it follows that whether S is of type (I) or (II), the *principal ideals of* S satisfy $\operatorname{ann}\operatorname{ann}(y^c) = (y^c)$.

To see that S is self-injective, let f be an S-homomorphism of an ideal K of S into S. If $K = (y^c)$, then $(\operatorname{ann} K)f(K) = f(0) = 0$ so that $f(K) \subseteqq \operatorname{ann}\operatorname{ann} K = K$ so that $f(y^c) = y^c p$ for some p in S. Thus f can be extended to the endomorphism $s \to sp$ of S.

Next, let $K = (y^{>c})$, and choose an infinite sequence $c(1) > c(2) > c(3) > \cdots$ such that $\lim_{i \to \infty} c(i) = c$. Then $K = \cup_{i=1}^{\infty} (y^{c(i)})$. For each i the previous paragraph shows that we can choose a "power series" p_i such that $f(y^{c(i)}) = y^{c(i)} p_i$. If $j > i$ so that $y^{c(i)} = y^{c(i)-c(j)} y^{c(j)}$ the fact that f is an S-homomorphism shows that $f(y^{c(i)}) = y^{c(i)} p_j$ so that (since also $f(y^{c(i)}) = y^{c(i)} p_i$) we have $p_j - p_i \in \operatorname{ann}(y^{c(i)})$. If (5) holds, this means that all the terms of p_i of degree $< b - c(i)$ are equal to the terms of the same degree of p_j while the terms of higher degree do not affect the products $y^{c(i)} p_i$ and $y^{c(i)} p_j$. A similar statement is true if (6) holds. Thus we can assemble a single "power series" p such that $(p - p_i) y^{c(i)} = 0$ for all i (It must be verified that the collection of exponents appearing in p is well-ordered so that p is in S). Then the map $s \to sp$ extends f to an endomorphism of S. We have now shown S to be self-injective.

(ii) The rings of type (3) in the theorem can be characterized as the following type of combination of PIR's: Let $\bar{R}$ be a field, and let R_i be a local PIR with maximal ideal $M_i \neq 0$ such that $R_i/M_i \cong \bar{R}$ and $M_i^2 = 0$ $(i = 1, 2)$. Note that M_i is the only proper ideal of R_i. Choose pair of homomorphisms k_i: R_i onto $\bar{R}$ and let $R = \{(r_1, r_2)\colon k_1(r_1) = k_2(r_2)\}$.

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ R_1 \arrow[dr, "k_1"'] & & R_2 \arrow[dl, "k_2"] \\ & \bar{R} & \end{tikzcd}$$

To see that R is of type (3), note that an element (r_1, r_2) of R is a nonunit if and only if $k_1(r_1) = 0$ $(= k_2(r_2))$. Hence R is a local ring. Its maximal ideal is readily seen to be $M = M_1 \oplus M_2$ and hence has composition length 2 and satisfies $M^2 = 0$.

Conversely, let R be of type (3). Then since $M^2 = 0$, M is a vector space over the field $\bar{R} = R/M$, and this space has dimension 2. Hence M is the direct sum of two minimal ideals M_1 and M_2 of R. The map $f\colon r \to (r + M_1, r + M_2)$ of R into R_1 $(= R/M_1) \oplus R_2$ $(= R/M_2)$ is a monomorphism (see the diagram). Finally, define $k_i\colon R_i$ onto $\bar{R}$ by $k_i(r + M_i) = r + M$. Then it is straightforward to verify that

$$R \cong f(R) = \{(r + M_1, s + M_2)\colon r + M = s + M\} = \{(r_1, r_2)\colon k_1(r_1) = k_2(r_2)\} .$$

(iii) Finally, it seems fitting to mention a related theorem by Barbara Osofsky [5] which states that the only rings R (commutative or not) having the property that all of their (left R-) homomorphic images are injective R-modules are the semi-simple rings with DCC.

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UNIVERSITY OF WISCONSIN
UNIVERSITY OF CHICAGO