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THE KLEIN GROUP AS AN AUTOMORPHISM GROUP WITHOUT FIXED POINT

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An automorphism group V acting on a group G is said to be without fixed points if for any $g \in G$, $v(g) = g$ for all $v \in V$ implies that $g = 1$. The structure of V in this case has been shown to influence the structure of G. For example if V is cyclic of order p and G finite then John Thompson has shown that G must be nilpotent. Gorenstein and Herstein have shown that if V is cyclic of order 4 then a finite group G must be solvable of p-length 1 for all $p \mid |G|$ and G must possess a nilpotent commutator subgroup.

In this paper we will consider the case where G is finite and V noncyclic of order 4. Since V is a two group all the orbits of G under V save the identity have order a positive power of 2. Thus G is of odd order and by the work of Feit-Thompson G is solvable. We will show that G has p-lengh 1 for all $p \mid |G|$ and G must possess a nilpotent commutator subgroup.

REMARK. It would be interesting to have a direct proof of solvability without resorting to the work of Feit-Thompson.

From now on in this paper G represents a finite group admitting V as a noncyclic four group without fixed points. If X is a group admitting an automorphism group A then $Z(X)$, $\Phi(X)$, $X - A$ will be respectively the center of X, the Frattini subgroup of X and the semi-direct product of S by A in the holomorph of X. All other notations are standard.

Suppose $V = \{v_1, v_2, v_3\}$ where the v_i are the nonidentity elements of V. Denote by G_i the set of elements which are left fixed by v_i. These are easily seen to be V-invariant subgroups of G and by a result of Burnside ([1] p. 90) G_i are Abelian and v_j restricted to G_i is the inverse map if $i \neq j$. These subgroups G_i are in a sense the building blocks of G.

LEMMA 2. ([4] p. 555)
(i) $|G| = |G_1||G_2||G_3|$
(ii) $G = G_1G_2G_3$
(iii) *Every element $g \in G$ has a unique decomposition $g = g_1g_2g_3$, $f_i \in G_i$.*

LEMMA 2. *If $|G| = hm$ where $(h, m) = 1$ then G contains a unique V invariant group H such that $|H| = h$.*

Proof. Since G is solvable by Hall ([5] p. 141) groups of order h exist and are conjugate in G. Thus there exist an odd number of them permuted by V. Since all the orbits have order power of 2 at least one group say H is V invariant. By Lemma 1 $H = H_1H_2H_3$ where clearly $H_i = H \cap G_i$ and $(G : H) = (G_1 : H_1)(G_2 : H_2)(G_3 : H_3)$. Thus the H_i are Hall subgroups of the Abelian G_i and thus uniquely determined by G_i rather than H.

The decomposition of a V invariant group X into $X_1X_2X_3$ will play an important role in what will follow. The $X_i \subseteqq G_i$ are always V invariant and it is clear that if $|X_i| = 1$ for any i then X is Abelian. For example if $X = X_1X_2$ is V invariant and normalized by a subgroup $K_1 \subseteqq G_1$ then $K_1X = K_1X_1X_2$ is Abelian. Thus subgroups of the complex G_1G_2 are centralized by elements in G_1G_2 which normalize them. If $X = X_1$ then even a stronger statement is available.

LEMMA 3. *If* $X \subseteqq G_i$, then $N_G(X) = C_G(X)$.

Proof. Suppose $i = 1$. It is easy to see that $N = N_G(X)$ is V invariant and thus $N = N_1N_2N_3$. By the above remark XN_2 and XN_3 are Abelian. Since XN_1 is Abelian, the result follows.

Before we continue to the main results, we must examine the inheritance properties of groups admitting automorphism groups without fixed points. If G is such a group and H is a V invariant subgroup, then clearly H is also such a group. If K is a normal V invariant subgroup of G, there exists the canonical way of inducing V on G/K. This definition gives rise to an automorphism group $\bar{V}$ acting on G/K.

LEMMA 4. ([4] p. 556) *In above situation*
(i) $\bar{V}$ *is without fixed points on* G/K.
(ii) $(G/K)_i = G_iK/K$.

LEMMA 5. *Suppose V acts on M and A without fixed points. Suppose also that A is an elementary Abelian p-group where $(p, |M|) = 1$ and M is acting faithfully on A. If the complex M_1M_2 is a normal subgroup of M and $A_i \neq \{1\}$ $i = 1, 2, 3$ then A is $M - V$ reducible.*

Proof. By Maschke's theorem it will suffice to show that some proper subgroup of A is $M - V$ invariant. Now $C = C_A(M_1M_2)$ is $M - V$ invariant and $C \neq A$ since M acts faithfully. Hence if $|C| \neq 1$ we are done and so we may assume $|C| = 1$. Now A_2A_3 is the subset of A inverted by v_1 and so is invariant under $M_1 - V$. Set $K = \cap\, m_2(A_2A_3)$ where the intersection is taken over all $m_2 \in M_2$. Since $M_1 - V$ normalizes M_2, K is $M_1M_2 - V$ invariant. Furthermore, $A_2 \subseteqq K$ since M_2 centralizes A_2. If $K = A_2$, then M_1M_2 centralizes K so $K \subseteqq C = \{1\}$ contrary to the fact that $|A_2| \neq 1$. Thus we must have that $|K \cap A_3| \neq 1$. But then if $R = \cap m_3(K)$ where the intersection is taken over all $m_3 \in M_3$ we have that $\{1\} \subsetneqq A \cap K_3 \subseteqq R \subseteqq A_1A_2 \subset A$. Since M_1M_2 is normal in M and M_3 is V invariant it follows that R is $M - V$ invariant and proper in A. This completes the proof of the lemma.

THEOREM 1. *For all $p \mid |G|$ G has p-length* 1.

Proof. We prove the theorem by induction on $|G|$. We may assume G has no normal p'-groups and $P_0 \neq \{1\}$ is the maximal normal p-group of G. By Hall ([5] p. 332) we have $C_G(P_0) \subseteqq P_0$. By Lemma 2, the fact that P_0 is self centralizing and induction, we may assume $G = PQ = QP$ where P and Q are V invariant p and q Sylow groups of G. By induction we also get that $QP_0 \lhd G$, $(P : P_0) = p$ and P_0 is elementary Abelian. By ([2] p. 795) Q possesses a characteristic subgroup C such that class $(C) \leqq 2$, $C/Z(C)$ is elementary Abelian and the only automorphisms of G which become the identity when restricted to C have order a power of q. PC is then a V invariant group and by induction if $C \neq Q$, since P_0 is self centralizing we get $P \lhd PC$. Thus $PC/P_0 = P/P_0 \times CP_0/P_0$. Since P/P_0 does not centralize QP_0/P_0 this contradicts the choice of C. Thus $Q = C$. Since P is normal in any proper V invariant subgroup containing it we get that $(P/P_0 - V)$ is irreducible on $QP_0/\Phi(Q)P_0$. Thus either Q is Abelian or $Z(Q) \subseteqq \Phi(Q)$. Since $Q/Z(Q)$ is elementary we get that $Z(Q) = \Phi(Q)$. Thus either Q is Abelian or nonabelian of class 2 with $Z(Q) = \Phi(Q)$. Since $|P/P_0| = p$ we may suppose $P/P_0 = (P/P_0)_3$. By the irreducibility of $P/P_0 - V$ on $QP_0/\Phi(Q)P_0$ we have that either $Q_1Q_2 \subseteqq \Phi(Q)$ or $Q_3 \subseteqq \Phi(Q)$. The first possibility implies that P/P_0 centralizes $QP_0/\Phi(Q)P_0$ and thus P would be normal in G. Thus we have that $Q_3 \subseteqq Z(Q)$ and since Q/Q_3 is Abelian we have $Q_1Q_2 \lhd Q$ and $Q_2Q_3 \lhd Q$.

Since Q_1Q_2 does not centralize P_0, there exists an irreducible $Q - V$ submodule A of P_0 which is not centralized by Q_1Q_2. Thus $A_3 \neq \{1\}$. Since $QP_0/P_0 \lhd G_0/P_0$ we have that $\cap x(A)$ where x ranges through P/P_0 is a $G/P_0 - V$ subspace of A. Since $P/P_0 = (P/P_0)_3$ and $A_3 \neq \{1\}$ this space is not the identity space. By the irreducibility of A as a $Q - V$ space we get that A is also $G/P_0 - V$ irreducible. If $A_i = \{1\}$ $i = 1$ or $i = 2$ we get that $(P/P_0)_3 \subset \operatorname{Ker} \theta$ where θ maps G/P_0 into Aut (A). Since Q does not centralize A this mapping is not the identity and the result follows by induction. Thus $A = A_1A_2A_3$ where $A_i \neq \{1\}$ $i = 1, 2, 3$.

We have that A admits G/P_0 and thus form the extension $G^* = A \cdot G/P_0$. G^* is V invariant and if $|G^*| < |G|$ we may apply induction to G^*. Let R/P_0 be the maximal normal q-subgroup of G^*. Since Q does not centralize $\varLambda$ we have that R/P_0 is a proper V invariant subgroup of QP_0/P_0. Since G^* has p-length 1, $\varLambda(PR/P_0) \lhd G^*$. Thus $PR/P_0 \lhd G/P_0$ and $PR \neq G$. We are done by induction on PR. We may assume that $\varLambda = P_0$. But since $\varLambda_i \neq \{1\}$ $i = 1, 2, 3$ and $Q - B$ is faithful irreducible on P_0 we have a contradiction to Lemma 5. This completes the proof of Theorem 1.

THEOREM 2. *If G admits V without fixed points then $G' = (G, G)$ is nilpotent.*

Proof. Suppose G contains two distinct minimal normal V invariant subgroups N_1 and N_2. If N_1 is disjoint from G' tden by induction on G/N_1 the theorem is proved. If N_1 and N_2 are in G' then by induction G'/N_1 and G'/N_2 are nilpotent. The minimality of N_i imply that the mapping of G' into $G'/N_1 \times G'/N_2$ is an imbedding and thus again we are done. Therefore G contains a unique minimal normal V invariant group. It is an elementary Abelian p-group P_0 which is characteristic. G must contain no normal p'-groups and by Theorem 1 we have that G has a normal p-Sylow group P. Now $C_G(P) = Z(P) \times K$ where K is a characteristic therefore V invariant p'-group of $C_G(P)$. Since $C_G(P)$ G we get that $C_G(P) = Z(P) \subseteqq P$. Consider $G/\varPhi(P)$. If induction applies $G'\varPhi(P)/\varPhi(P)$ is a nilpotent group and since $C_{G/\varPhi(P)}(P/\varPhi(P)) = P/\varPhi(P)$ we must have that $G'\varPhi(P)/\varPhi(P)$ is a p-group and therefore so is G'. Thus we have that P is elementary Abelian. Let M be a V invariant complement to P in G. By Maschke's theorem and the remark on the number of minimal V invariant normal subgroups of G we have that $P = P_0$ and P is $(M - V)$ irreducible.

Consider any proper V invariant subgroup K of M. Then $PK \subset G$. By induction PK has a nilpotent commutator subgroup. Since $C_{PK}(P) = P$ this must be a p-group and therefore contained in P. Since $PK/P \cong K$ we must have that K is Abelian. Thus every proper V invariant subgroup of M is Abelian. If M is Abelian then $G' \subseteqq P$ and we are done. We assume henceforth that M is not Abelian. Thus $M = M_1M_2M_3$ where $M_i \neq \{1\}$ for any i. Since $C_G(P) = P$, $P = P_1P_2P_3$ where $P_i \neq \{1\}$ for any i.

If M contains two V invariant subgroups K and L of prime index, then since these are both Abelian we get that some M_i say $M_1 \subset Z(M)$. Thus M_1M_2 and M_1M_3 are normal in M. $M - V$ is faithful and irreducible on P. This situation is in contradiction to Lemma 5. Since M is solvable and V invariant, we have a V invariant Sylow system. If more than two primes divide $|M|$ then we would have M Abelian. If M is a q-group for some prime q, we can get an M_iM_j M. Thus to avoid this case we are forced to the following situation. R and S

are V invariant r and s Sylow subgroups, each is Abelian and $M = RS = SR$. We may suppose that M contains a V invariant normal Abelian subgroup K such that $(M: K) = s$. Thus R M and S is cyclic. Thus $S \subseteqq M_i$ for some i. To be specific suppose $S \subseteqq M_1$. Then by Maschke's theorem applied to S_1 acting on $R_1R_2R_3/\Phi(R)$ we get that $R_2R_3 = M_2M_3$ is normalized by S_1 and thus M_2M_3 M. We have $(M - V)$ irreducible and faithful on $P = P_1P_2P_3$, and we again contradict Lemma 5. This completes the proof of Theorem 2.

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