PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 2, 1966

# A NOTE ON THE CLASS GROUP

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**The main result yields some information on the class group of a domain $R$ in terms of the class group of $R/xR$. With slightly stronger hypotheses than are strictly necessary, we state the main result: Let $R$ be a regular domain, $x$ a prime element contained in the radical of $R$, and suppose that $R/xR$ is locally a unique factorization domain. Let $\{I_\alpha\}$ be a set of unmixed height 1 ideals of $R$ such that the classes of $\{I_\alpha + xR/xR\}$ generate the class group of $R/xR$; then the classes of $\{I_\alpha\}$ generate the class group of $R$.**

The result of Samuel's and Buchsbaum's stating that if $R$ is a regular U.F.D., then $R[[X]]$ is a regular U.F.D. [4] has been generalized by P. Salmon and the present author in two different directions. Salmon [2, Prop. 3] showed that if $R$ is a regular domain, $x$ is a prime element of $R$ which is contained in the radical of $R$, and $R/xR$ is a U.F.D., then $R$ is a U.F.D. It was shown [1, Cor. 4] that the map of the class group of $R$ into the class group of $R[[X]]$ is onto if $R$ is a regular noetherian domain. We have found a theorem which simultaneously generalizes the last two results, and even allows a little weakening of the hypotheses.

To set the notation and terminology, we will say that a domain $R$ is locally U.F.D. if the quotient ring $R_M$ is a U.F.D. for all maximal ideals $M$ of $R$. For any Krull domain $R$, we will denote the class group (see [3]) of $R$ by $C(R)$. If $I$ is an unmixed height 1 ideal of a Krull domain $R$, we will denote the class of the class group determined by $I$ by $\mathrm{cl}(I)$. Finally, all rings considered will be commutative noetherian domains with identity.

We wish to capitalize on a simple description of the class group valid for domains which are locally U.F.D. We do so and prepare for the main theorem by a sequence of (probably all known) lemmas.

LEMMA 1. *If $R$ is locally U.F.D., then $R$ is a Krull domain.*

*Proof.* Since $R$ is noetherian, it is sufficient to show that $R$ is integrally closed. Since $R = \cap R_M$ as $M$ runs over all maximal ideals of $R$, it will be enough to see that each $R_M$ is integrally closed. But each $R_M$ is a U.F.D., hence integrally closed.

LEMMA 2. *If $R$ is locally U.F.D. and $P$ is a height* 1 *prime of $R$, then $P$ is invertible.*

*Proof.* $P$ is locally principal, hence locally free (as a module), hence projective, hence invertible.

PROPOSITION 3. If $R$ is locally a U.F.D., then the unmixed height 1 ideals of $R$ are precisely all finite products of minimal prime ideals of $R$.

Let $I_1$ and $I_2$ be two unmixed height 1 ideals of $R$, then $\mathrm{cl}(I_1) = \mathrm{cl}(I_2)$ if and only if there are elements $a$ and $b$ in $R$ such that $aI_1 = bI_2$.

*Proof.* From Lemma 2 we know that any product of height 1 prime ideals of $R$ is invertible. Given an unmixed height 1 ideal $I$ determined by the valuation data $I = \{x \mid v_{P_i}(x) \geqq n_i\}$ (almost all $n_i = 0$), we form $J = \prod_{n_i \neq 0} P_i^{n_i}$. Since $J$ is invertible, we have $J = R\colon (R\colon J)$, so $J$ is also unmixed of height 1. Since $I$ and $J$ are determined by same valuation data, this entails $I = J$. If now $I_1$ and $I_2$ are unmixed height 1 ideals such that $\mathrm{cl}(I_1) = \mathrm{cl}(I_2)$, the $I_1 I_2^{-1}$ is invertible and is determined by the same data as some $f \cdot R$, where $f$ is in the quotient field of $R$. We have therefore $I_1 I_2^{-1} = fR$, or $I_1 = fI_2$, which is equivalent to the final assertion.

LEMMA. 4. *Let $R$ be locally U.F.D., and suppose that $R$ is a Macaulay ring. Let $I$ be an unmixed height* 1 *ideal of $R$ and $x$ an element of the radical of $R$ such that $I\colon xR = I$. Then $I + xR$ is unmixed of height* 2.

*Proof.* Word for word the proof of Lemma 2 of [1].

LEMMA 5. *Let the hypotheses be as in Lemma* 4 *and suppose that $x$ is prime and $R/xR$ is a Krull domain. Let $h$ denote the homomorphism of $R$ onto $R/xR$. If $d$ is an element of $R$ such that $dI^{-1} \subseteq R$ (for $I$ an unmixed height* 1 *ideal of $R$), then* $\mathrm{cl}\,(h(dI^{-1})) = \mathrm{cl}(h(I))^{-1}$.

*Proof.* From $II^{-1} = R$, we get $I(dI^{-1}) = dR$. Applying $h$ to both sides of the last equation, we obtain $h(I) \cdot h(dI^{-1}) = h(d) \cdot R/xR$, which yields the result.

THEOREM 6. *Let $R$ be a Macaulay ring which is locally U.F.D. Let $x$ be a prime element of the radical of $R$ such that $R/xR$ is locally U.F.D. Let $h$ denote the natural homomorphism of $R$ onto $R/xR$. If $\{I_\alpha\}$ is a set unmixed height* 1 *ideals of $R$ such that $I_\alpha\colon xR = I_\alpha$ and* $\{\mathrm{cl}\, h(I_\alpha)\}$ *generates $C(R/xR)$, then* $\{\mathrm{cl}\,(I_\alpha)\}$ *generates $C(R)$.*

*Proof.* Let $P$ be a height 1 prime ideal of $R$. *If* $x \varepsilon P$, then

$P = xR$, and $\mathrm{cl}(P)$ is the identity element of $C(R)$. If $x \notin P$, we must have $P: xR = P$ and Lemma 4 shows that $P + xR$ is unmixed of height 2. Thus $h(P)$ is unmixed of height 1 in $R/xR$, so the hypotheses yield that $h(P) = fh(I_1)^{e_1} \cdots h(I_k)^{e_k}$ for some $f$ in the quotient field of $R/xR$ and integers $e_1, \cdots, e_k$. Write $f = h(a)/h(b)$ for $a, b \varepsilon R$. Then $h(b)h(P)h(I_1)^{-e_1} \cdots h(I_k)^{-e_k} = h(a)$. Choose $d_i \varepsilon R$ such that $x$ does not divide $d_i$ and $d_i I_i^{-e_i} \subseteq R$ for $i = 1, \cdots, k$. Form the ideal $I = bP(d_1 I_1^{-e_1}) \cdots (d_k I_k^{-e_k})$. Lemma 5 shows that $h(I)$ is principal; say $h(I) = h(t)R/xR$. We may assume $t \varepsilon I$. From $I \subseteq tR + xR$ and $I: x = I$, we get $I = tR + xI$. Since $x$ is in the radical of $R$, we must have $I = tR$ by Nakayama's lemma. This implies that $P = t/bd_1 \cdots d_k \cdot I_1^{e_1} \cdots I_k^{e_k}$, so $\mathrm{cl}\,(P)$ is in the subgroup of $C(R)$ generated by $\{\mathrm{cl}\,(I_\alpha)\}$. Since $P$ is an arbitrary height 1 prime ideal, the theorem is established.

REMARKS. (1) Salmon's result cited in the introduction is obtained by choosing the set $\{I_\alpha\}$ to consist of all principal ideals of $R$ generated by elements of $R$ which are not divisible by $x$.

(2) If $R$ is a regular domain, then $R[[X]]$ is also, and Theorem 6 may be applied with $x = X$ and the set of ideals $\{P_\alpha R[[X]]\}$ where $P_\alpha$ ranges over the height 1 prime ideals of $R$. We get that $\{\mathrm{cl}(P_\alpha R[[X]])\}$ generate $C(R[[X]])$ which shows that the natural homomorphism $C(R) \rightarrow C(R[[X]])$ is onto (it is easily seen that it is one to one).

(3) Should Samuel's question "Does U.F.D. imply Macaulay?" [4] have an affirmative answer, then the hypotheses of Theorem 6 could be further weakened in the obvious fashion.

## REFERENCES

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Received April 29, 1965.

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