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OPERATORS WITH FINITE ASCENT AND DESCENT

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Let X be a Banach space and T a closed linear operator with range and domain in X. Let $\alpha(T)$ and $\delta(T)$ denote, respectively, the lengths of the chains of null spaces $N(T^K)$ and ranges $R(T^K)$ of the iterates of T. The Riesz region $\mathfrak{R}_T$ of an operator T is defined as the set of λ such that $\alpha(T-\lambda)$ and $\delta(T-\lambda)$ are finite. The Fredholm region $\mathfrak{F}_T$ is defined as the set of λ such that $n(T-\lambda)$ and $d(T-\lambda)$ are finite, $n(T)$ denoting the dimension of $N(T)$ and $d(T)$ the codimension of $R(T)$. It is shown that $\mathfrak{F}_T \cap \mathfrak{I}_T$ is an open set on the components of which $\alpha(T-\lambda)$ and $\delta(T-\lambda)$ are equal, when T is densely defined, with common value constant except at isolated points. Moreover, under certain other conditions, $\mathfrak{R}_T$ is shown to be open. Finally, some information about the nature of these conditions is obtained.

Let X denote an arbitrary Banach space and suppose that T is a linear operator with domain $D(T)$ and range $R(T)$ in X. We shall write $N(T)$ for the nullspace, $N(T) = \{x \in D(T)\colon \; Tx = 0\}$.

Let $D(T^n) = \{x\colon \; x, \; Tx, \; \cdots, \; T^{n-1}x \in D(T)\}$ and define T^n on this domain by the equation $T^n x = T(T^{n-1}x)$ where n is any positive integer and $T^\circ = I$. It is a simple matter to verify that $\{N(T^k)\}$ forms an ascending sequence of subspaces. Suppose that for some k, $N(T^k) = N(T^{k+1})$; we shall then write $\alpha(T)$ for the smallest value of k for which this is true, and call the integer $\alpha(T)$, the *ascent* of T. If no such integer exists, we shall say that T has infinite ascent. In a similar way, $\{R(T^k)\}$ forms a descending sequence; the smallest integer for which $R(T^k) = R(T^{k+1})$ is called the *descent* of T and is denoted by $\delta(T)$. If no such integer exists, we shall say that T has infinite descent.

The quantities $\alpha(T)$ and $\delta(T)$ were first discussed by F. Riesz [4] in his original investigation of compact linear operators. A comprehensive treatment of the properties of $\alpha(T)$ and $\delta(T)$ can be found in [6] pp. 271–284. The purpose of the present work is the consideration of the functions $\alpha(\lambda I - T)$ and $\delta(\lambda I - T)$ for complex λ. When no confusion can arise, we shall write these quantities as $\alpha(\lambda)$ and $\delta(\lambda)$ respectively.

DEFINITION. Let $\mathfrak{R}_T$ denote the set $\{\lambda\colon \; \alpha(\lambda)$ and $\delta(\lambda)$ are finite$\}$. We shall refer to $\mathfrak{R}_T$ as the *Riesz region* of T.

If we write $n(\lambda)$ for the dimension of $N(\lambda I - T)$, i.e., the *nullity* of $\lambda I - T$ and $d(\lambda)$ for the codimension of $R(\lambda I - T)$, i.e., the *defect* of $\lambda I - T$, then it is customary to refer to the set $\{\lambda: n(\lambda)$ and $d(\lambda)$ are finite$\}$ as the *Fredholm region* of T. We shall denote this region by $\mathfrak{F}_T$. It should be observed that the above is a departure from traditional notation where α and β are used for nullity and defect, respectively.

2. Remarks. From this point onwards, we shall assume that all operators are closed, with range and domain in X unless otherwise stated.

1. It is well known that $\mathfrak{F}_T$ is an open set and that $n(\lambda) - d(\lambda)$ is constant on each component of $\mathfrak{F}_T$. These facts and a great many others are proven in papers by Gohberg and Krein [2] and by T. Kato [3]. We shall show below that $\mathfrak{F}_T \cap \mathfrak{R}_T$ is always open and that $\mathfrak{R}_T$ is open when certain other conditions are fulfilled. However the quantity $\delta(\lambda) - \alpha(\lambda)$ need not be constant on the components of $\mathfrak{R}_T$; for consider operator T where $D(T) \neq X$; $D(T) \neq \{0\}$ and $Tx = x$ for $x \in D(T)$. Then $\mathfrak{R}_T$ is the entire complex plane C but $\delta(\lambda) = 1$, $\alpha(\lambda) = 0$, when $\lambda \neq 1$; $\delta(1) = \alpha(1) = 1$. However, if $D(T) = X$, then $\alpha(\lambda) = \delta(\lambda)$ on $\mathfrak{R}_T$ even in the absence of any topology in X. Proof of this fact can be found in [6] Theorem 5.41-E. Another notable difference between $\mathfrak{R}_T$ and $\mathfrak{F}_T$ is seen from the theorem proven in [2]: if $B(X)$ denotes the space of bounded linear operators defined on X and $\mathfrak{F}_T = C$, then X is finite dimensional. It is clear that no such restriction applies to $\mathfrak{R}_T$; indeed $\mathfrak{R}_I = C$.

2. If we adopt the usual notation of $\rho(T)$, $P\sigma(T)$, $C\sigma(T)$ and $R\sigma(T)$ for the resolvent set, point spectrum, continuous spectrum and residual spectrum respectively as given in [6], then it is known that for $T \in B(X)$, $\delta(\lambda) = \infty$ if $\lambda \in C\sigma(T) \cup R\sigma(T)$. This is proven in [1]. Hence $\mathfrak{R}_T$ consists of $\rho(T)$ and possibly some elements of the point spectrum.

3. Some preliminary lemmas.

LEMMA 1. *For any non negative integer* k
(i) $n(T^k) \leqq \alpha(T) n(T)$
(ii) $d(T^k) \leqq \delta(T) d(T)$.

Proof. (i) We firstly observe that $\alpha(T) = 0$ if and only if $n(T) = 0$. Hence the product $\alpha(T)n(T)$ is well defined. We need only consider the case where both $\alpha(T)$ and $n(T)$ are finite. Let $\alpha(T) = p$. Then $n(T^k) \leqq n(T^p)$ for any k and if we show $n(T^k) \leqq kn(T)$ for every nonnegative integer k, the result will follow. We proceed by induction; clearly for $k = 1$, $n(T^k) \leqq kn(T)$. Suppose we have shown its validity for $1 \leqq k \leqq s$. Then we can complete the proof by showing

$$n(T^{s+1}) - n(T^s) \leqq n(T) . \tag{1}$$

Let $N(T^{s+1}) = N(T^s) \oplus Y$. Choose $x_1, x_2, \cdots, x_r$ linearly independent in Y. Then these elements lie in $N(T^{s+1})$ so that $T^s x_i (i = 1, 2, \cdots, r)$ lie in $N(T)$. But $\sum_{i=1}^r c_i T^s x_i = 0$ implies $T^s \sum_{i=1}^r (c_i x_i) = 0$ which would mean that $\sum_{i=1}^r c_i x_i \in N(T^s) \cap Y$. Therefore all c_i must be zero. Hence the elements $\{T^s x_i : i = 1, 2, \cdots, r\}$ are linearly independent in $N(T)$. This implies the validity of (1) and completes the proof.

(ii) Again, since $\delta(T)$ is zero if and only if $d(T)$ is zero, the product $\delta(T)d(T)$ is well defined and we need only consider the case when $\delta(T)$ and $d(T)$ are finite. Again it suffices to prove that for each positive integer k,

$$d(T^k) \leqq kd(T) . \tag{2}$$

Clearly (2) is valid for $k = 1$; suppose we have shown its validity for $1 \leqq k \leqq s$. Let $R(T^{s+1}) \oplus Y = R(T^s)$ and take $y_1, y_2, \cdots, y_r$ linearly independent in Y. Then these element belong to $R(T^s)$ so that there exist $x_1, x_2, \cdots, x_r$ in $D(T^s)$ such that $y_i = T^s x_i$, $i = 1, 2, \cdots, r$.

Suppose now we write $X = R(T) \oplus Z$ so that we can write $x_i = Tx'_i + z_i$ for some $x'_i \in D(T)$ and $z_i \in Z$, $i = 1, 2, \cdots, r$. Then $\{z_i\}$ is a linearly independent set; for if $\sum_{i=1}^r c_i z_i = 0$ then $\sum_{i=1}^r c_i T^s z_i = 0$ so that $\sum_{i=1}^r c_i T^s x_1 = \sum_{i=1}^r c_i T^{s+1} x'_i$ i. e.,

$$\sum_{i=1}^r c_i y_i = \sum_{i=1}^r c_i T^{s+1} x'_i . \tag{3}$$

But the left side of (3) lies in Y, the right side in $R(T^{s+1})$. Hence $\sum_{i=1}^r c_i y_i = 0$. Hence each c_i is zero. This means that dim $Y \leqq$ dim Z so that

$$d(T^{s+1}) - d(T^s) \leqq d(T)$$

and hence (2) is valid for $k = s + 1$. This completes the proof of (ii).

LEMMA 2. *If* $\lambda \in \Re_T \cap \mathfrak{F}_T$ *and* T *is densely defined, then* $n(\lambda) = d(\lambda)$ *and* $\alpha(\lambda) = \delta(\lambda)$.

Proof. Without loss of generality, assume $\lambda = 0$. Then, writing $\kappa(A) = d(A) - n(A)$ for any operator A, we can use Theorem 2.1 of

[2] to write

$$\kappa(AB) = \kappa(A) + \kappa(B) \tag{4}$$

where A, B are operators in X with finite nullities and defects. As remarked at the end of the proof of the theorem cited, (4) is valid in all cases where A, B act from one Banach space to another, the product AB has a sense, and A is densely defined. Moreover AB has finite nullity and defect. In our case, we can write

$$\kappa(T^p) = p\kappa(T) \tag{5}$$

by induction from (4), for any positive integer p. Hence setting $p = k, k+1$ and subtracting we get

$$[n(T^{k+1}) - n(T^k)] - [d(T^{k+1}) - d(T^k)] = n(T) - d(T) . \tag{6}$$

On account of Lemma 1, all quantities involved are finite. Choose k greater than $\alpha(T)$ and $\delta(T)$; then left side of (6) reduces to zero and hence $n(T) = d(T)$. Finally, we can write

$$n(T^{k+1}) - n(T^k) = d(T^{k+1}) - d(T^k) \tag{7}$$

which makes it clear that $\alpha(T) = \delta(T)$.

4. Definitions. Suppose that the norm in X is denoted by $\|\cdot\|$ and that we introduce a new norm into $D(T)$ by setting $|x| = \|x\| + \|Tx\|$. Then, as first shown in [5], $D(T)$ is closed with respect to $|\cdot|$ and can therefore be regarded as a Banach space. T is then a closed operator defined on all of a Banach space so that, by the closed graph theorem, T is bounded i. e., there exists k such that $\|Tx\| \leqq k|x|$ for each $x \in D(T)$. We shall write $|T|$ to denote the infimum of such k. If S is another closed operator with $D(S) \supseteqq D(T)$, then the restriction of S to $D(T)$ can also be regarded as a bounded operator with bound denoted by $|S|$.

Following [3], we define a quantity $\gamma(T)$ as the supremum of all λ which satisfy $\lambda d(x, N(T)) \leqq \|Tx\|$ for all $x \in D(T)$.

5. Consideration of $\mathfrak{R}_T \cap \mathfrak{F}_T$. Let λ_0 be a point in $\mathfrak{R}_T \cap \mathfrak{F}_T$; without loss of generality, we may assume $\lambda_0 = 0$. We define the following positive number:

$$R_p = \begin{cases} \gamma(T) & \text{if } p = 1 \\ 2\left|\sin\dfrac{\Pi}{p}\right|\gamma(T) & \text{if } p > 1 . \end{cases}$$

For each p, we know from [3], Lemma 341, that T^p is a closed operator so that we can make $D(T^p)$ into a Banach space X_p by introducing the norm $|x|_{(p)} = \|x\| + \|T^p x\|$. Then for $i = 0, 1, \cdots, p$, we can consider the restrictions of T^i to X_p. Such restrictions being obviously closed operators, it follows from the closed graph theorem that they are bounded as operators from X_p to X. Write $|T^i|_{(p)}$ to denote the respective bounds of these operators.

Define

$$r_p = \left[1 + \frac{\gamma(T^p)}{[1+\gamma(T^p)] \max\limits_{0 \leqq i \leqq p-1} |T^i|_{(p)}}\right]^{1/p} - 1 .$$

Finally, if $\alpha_0 = \alpha(T)$, $n_0 = n(T)$, $\delta_0 = \delta(T)$ write

$$\Gamma = \min_{1 \leqq p \leqq \alpha_0 n_0 + \delta_0 + 1} \min(r_p, R_p) .$$

THEOREM 1. *$\mathfrak{R}_T \cap \mathfrak{F}_T$ is an open set; indeed, if we take $\lambda = 0$ as a point of $\mathfrak{R}_T \cap \mathfrak{F}_T$, then the interior of the circle $|\lambda| = \Gamma$ lies in $\mathfrak{R}_T \cap \mathfrak{F}_T$.*

Proof. By [3] Theorem 1, inside the circle $|\lambda| = \gamma(T)$, $T - \lambda$ is a closed linear operator, $n(T-\lambda) \leqq n(T)$ and $R(T-\lambda)$ is closed. Moreover, we claim that inside the circle $|\lambda| = R_p$, $(T-\lambda)^p - T^p$ is a closed operator.

$$\text{For} \quad (T-\lambda)^p - T^p = \prod_{K=0}^{p-1} \left[T - \lambda - \left(\exp \frac{2\pi K i}{p} \right) T \right] \tag{8}$$

if $p > 1$, and if we write $T_K = T\left(1 - \exp \dfrac{2\pi K i}{p}\right)$, the T_K is a closed operator with finite nullity.

Also

$$\gamma(T_K) = \inf_{x \notin N(T_K)} \frac{\|T_K x\|}{d(x, N(T_K))} = \left| 1 - \exp \frac{2\pi K i}{p} \right| \inf_{x \notin N(T)} \frac{\|Tx\|}{d(x, N(T))}$$
$$\geqq 2 \left| \sin \frac{\pi}{p} \right| \gamma(T) = R_p .$$

Hence, if $|\lambda| < R_p$, then each factor in (8) is a closed linear operator with finite nullity so that by [3] Lemma 341, $(T-\lambda)^p - T^p$ is closed in this circle. Since the domain of this operator is $D(T^p)$, we can write

$$|(T-\lambda)^p - T^p|_{(p)} \leqq \sum_{i=0}^{p-1} \binom{p}{i} |T^i|_{(p)} |\lambda|^{p-i}$$
$$\leqq [(1+|\lambda|)^p - 1] \max_{0 \leqq i \leqq p-1} |T^i|_{(p)} .$$

If $|\lambda| < r_p$, this shows that $|(T-\lambda)^p - T^p|_{(p)} \leqq \dfrac{\gamma(T^p)}{1+\gamma(T^p)}$.
By [3], Theorem 1a, if $|\lambda| < \min(r_p, R_p)$, then

$$\left.\begin{aligned} n[(T-\lambda)^p] &\leqq n(T^p) \\ d[(T-\lambda)^p] &\leqq d(T^p) \\ \kappa[(T-\lambda)^p] &= \kappa(T^p) \end{aligned}\right\} \tag{9}$$

for $p > 1$.
Observe that (9) also holds for $p = 1$; for we can apply [3] Theorem 1 directly to T and $-\lambda I$.

Now, if $|\lambda| < \Gamma$,

$$\begin{aligned} n[(T-\lambda)^p &\leqq n(T^p) \qquad 1 \leqq p \leqq \alpha_0 n_0 + 1 \\ &\leqq \alpha_0 n_0 \qquad \text{by Lemma 1}\,. \end{aligned}$$

Hence $n[(T-\lambda)^p]$ cannot be strictly increasing for $1 \leqq p \leqq \alpha_0 n_0 + 1$; thus $\alpha(\lambda) \leqq \alpha_0 n_0$.

Finally, from (9), we can write

$$n[(T-\lambda)^K] - d[(T-\lambda)^K] = n(T^K) - d(T)^K$$
$$n[(T-\lambda)^{K+1}] - d[(T-\lambda)^{K+1}] = n(T^{K+1}) - d(T^{K+1})$$

with $K = \alpha_0 n_0 + \delta_0$. Now $\alpha_0 n_0 + \delta_0$ exceeds both α_0 and δ_0 and since all quantities involved in the above equalities are finite by Lemma 1, we get

$$d[(T-\lambda)^{K+1}] = d[(T-\lambda)^K]$$

i.e., $\delta(\lambda) \leqq \alpha_0 n_0 + \delta_0$ in the circle $|\lambda| < \Gamma$.

LEMMA 3. (*This is essentially* [2], *Lemma* 3.1 *in a slightly more general setting.*)

Let T be an operator with $0 \in \mathfrak{F}_T$ and let S be an operator with $D(S) \supseteqq D(T)$. Then if $|S|$ is defined by the norm $\|x\| + \|Tx\|$ on $D(T)$, there exists $\varepsilon > 0$ such that $n(T+S)$ is constant for $0 < |S| < \varepsilon$.

Proof. The original formulation of this Lemma considers A, B operators with domains in Banach space B_1 and ranges in Banach space B_2; $0 \in \mathfrak{F}_A$ and B is a bounded linear operator. The conclusion is that there exists $\varepsilon > 0$ such that $n(A - \lambda B)$ is constant for $0 < |\lambda| < \varepsilon$.

In our case, take B_1 to be $D(T)$ with the norm $|x| = \|x\| + \|Tx\|$ and $B_2 = X$, $A = T$. If S is the restriction of S to B_1, so that S is a bounded operator, take $B = -S/|S|$. Then we can conclude that there exists $\varepsilon > 0$ such that $n(T + \lambda S/|S|)$ is constant for $0 < |\lambda| < \varepsilon$. In particular, if $0 < |S| < \varepsilon$, then $n(T + S)$ is constant.

THEOREM 2. *Let Ω be a component of $\mathfrak{R}_T \cap \mathfrak{F}_T$ where T is densely defined. Then $\alpha(\lambda)$ and $\delta(\lambda)$ will be equal on Ω* (by Lemma 2) *and the common value is constant except at isolated points.*

Proof. Let K be a positive integer. Then by Lemma 1, $n[(T-\lambda)^K]$ is finite in Ω. Let $n_K = \min_\Omega n[(T-\lambda)^K]$ and suppose $n[(T-\lambda_0)^K] = n_K$ and $n[(T-\lambda_1)^K] > n_K$. Join λ_1 to λ_0 by a curve Γ_K lying in Ω. We now apply Lemma 3 to the operators $A = (T-\lambda)^K$ $B = (T-\mu-\lambda)^K - (T-\lambda)^K$ for any point λ on Γ_K. Then $n[(T-\mu-\lambda)^K]$ is constant for $0 < |B| < \varepsilon$ and since $|B|$ is a continuous function of μ, we get a deleted neighbourhood of λ in which $n[(T-\mu)^K]$ is constant. The compactness of Γ_K enables us to deduce in the usual way that there exists an open set U_K containing Γ_K such that $n[(T-\lambda)^K]$ is constant for $\lambda \in U_K$ except at a finite number of points. In particular, relations (9) imply that in some neighbourhood of λ_0, $n[(T-\lambda)^K]$ takes a constant value n_K. Hence in U_K, $n[(T-\lambda)^K] = n_K$ except at a finite number of points. In particular, in some deleted neighbourhood of λ_1, $n[(T-\lambda)^K] = n_K$. Thus on Ω, $n[(T-\lambda)^K] = n_K$ except at isolated points. Let the set of exceptional points be denoted Ω_K. Choose λ^* with the property that $\lambda^* \notin \Omega_K$ for all K. This can be done simply by taking any line segment l in Ω and choosing λ^* to be any points of $l - \bigcup_1^\infty \Omega_K$. Let $\alpha(\lambda^*) = \alpha^*$ and $\delta(\lambda^*) = \delta^*$. By Lemma 2, $\alpha^* = \delta^*$. Consider $\lambda \in \Omega - \bigcup_1^{1+\alpha^*} \Omega_K$. Then $n[(T-\lambda)^K] = n[(T-\lambda^*)^K]$ for each k, $1 \leqq k \leqq 1 + \alpha^*$. Hence $\alpha(\lambda) = \alpha^*$ and by Lemma 2, $\delta(\lambda) = \delta^*$ for $\lambda \in \Omega - \bigcup_1^{1+\alpha^*} \Omega_K$.

COROLLARY. *If $\Omega \cap \rho(T) \neq \emptyset$, then $\Omega \cap \sigma(T)$ consists of poles of the resolvent $R_\lambda(T)$.*

Proof. Since $\rho(T)$ is an open set in which $\alpha(\lambda) = \delta(\lambda) = 0$, $\alpha(\lambda)$ and $\delta(\lambda)$ must be zero on Ω except at isolated points. It is known that such a point λ_0 is a pole of $R_\lambda(T)$ if $R[(T-\lambda_0)^{\alpha(\lambda_0)}]$ is closed. But $(T-\lambda_0)^{\alpha(\lambda_0)}$ has finite codimension by Lemma 1 and hence, by [3] Lemma 332, closed range.

6. Consideration of $\mathfrak{R}_T$.

THEOREM 3. *Let T be a closed linear operator such that $\alpha(T) = p < \infty$. Suppose that there exists $\varepsilon > 0$ such that if $|\lambda| < \varepsilon$, then it is possible to write*

$$X = N[(T-\lambda)^p] \oplus S(\lambda) \tag{10}$$

in such a manner that

$$S(\lambda) \cap D(T^{p+1}) = S(0) \cap D(T^{p+1}) . \tag{11}$$

Then if $R(T^{p+1})$ *is closed, there exists* $\rho > 0$ *such that* $\alpha(\lambda) \leqq \alpha(T)$ *for* $|\lambda| < \rho$.

Proof. Write $S(0) = S$ and define $D = S \cap D(T^{p+1})$. Let T_p be the restriction of T^{p+1} to D. We first show that

$$N(T^{p+1}) = N(T^p) \oplus N(T_p) .$$

Suppose $x \in N(T^p) \cap N(T_p)$; then

$$x \in N(T^p) \cap D(T_p) = N(T^p) \cap S \cap D(T^{p+1}) = \{0\}$$

by (10). Hence $N(T^p) \oplus N(T_p)$ is well defined. Now let $x \in N(T^{p+1})$. By (10), we can write $x = x_1 + x_2$ with $x_1 \in N(T^p)$ and $x_2 \in S$. Now $x_2 = x - x_1 \in N(T^{p+1}) \cap S \subseteqq D$, and $T_p x_2 = T^{p+1}x_2 = 0$. Hence $N(T^{p+1}) = N(T^p) \oplus N(T_p)$.

We next verify that $R(T_p) = R(T^{p+1})$. It is obvious that $R(T_p) \subseteqq R(T^{p+1})$. Suppose then that $x \in R(T^{p+1})$; then $x = T^{p+1}y$ for some $y \in D(T^{p+1})$. Use (10) again to write $y = y_1 + y_2$ with $y_1 \in N(T^p)$, $y_2 \in S$. Then $T^{p+1}y = T^{p+1}y_2$ and since $y_2 \in S \cap D(T^{p+1})$, we have $x = T^{p+1}y_2 = T_p y_2$. Hence $R(T_p) = R(T^{p+1})$.

If we now repeat the same arguments replacing T by $T - \lambda$ we obtain an operator $T_p(\lambda)$ with domain $S(\lambda) \cap D[(T-\lambda)]$, range equal to $R[(T-\lambda)^{p+1}]$ such that

$$N[(T-\lambda)^{p+1}] = N[(T-\lambda)^p] \oplus N[T_p(\lambda)].$$

Now by assumption, $N(T_p) = \{0\}$ and T_p has closed range. Hence T_p^{-1} can be considered as a bounded linear operator on $R(T_p)$; hence there exists $m > 0$ such that $\| T_p x \| \geqq m\,| x |$ for all $x \in D(T_p)$ where $|x|$ is defined, as in § 4, by $|x| = \|x\| + \|T_p x\|$. For $|\lambda| < \varepsilon$, $D[T_p(\lambda)] = D(T_p)$ so that $T_p(\lambda) - T_p$ is defined on $D(T_p)$ and has bound $|T_p(\lambda) - T_p|$ where

$$\begin{aligned}
| T_p(\lambda) - T_p | &= \sup\left\{\frac{\| (T_p(\lambda) - T_p)x \|}{\|x\| + \|T_p x\|} :\ x \in D(T_p),\, x \neq 0\right\} \\
&\leqq \sup\left\{\frac{\| [(T-\lambda)^{p+1} - T^{p+1}]x \|}{\|x\| + \|T^{p+1}x\|} :\ x \in D(T^{p+1})x \neq 0\right\} \\
&= |(T-\lambda)^{p+1} - T^{p+1}|_{(p+1)} \quad \text{where } |\cdot|_{(p+1)} \text{ is defined in the proof of Theorem 1} \\
&\leqq (|T|_{(p+1)} + |\lambda|)^{p+1} - |T|_{(p+1)}^{p+1} .
\end{aligned}$$

Let λ be chosen such that $|\lambda| < \varepsilon$ and $(|T|_{(p+1)} + |\lambda|)^{p+1} - |T|_{(p+1)}^{p+1} < m/3$. Then

$$\|T_p(\lambda)x\| = \|T_px + [T_p(\lambda) - T_p]x\| \geqq \|T_px\| - \|[T_p(\lambda) - T_p]x\|$$
$$\geqq m|x| - \frac{m}{3}|x| = \frac{2m}{3}|x| \text{ for } x \in D[T_p(\lambda)] .$$

Hence $N[T_p(\lambda)] = \{0\}$ so that $\alpha(\lambda) \leqq \alpha(T)$ if $|\lambda|$ is suitably chosen; in fact, if $|\lambda| < \varepsilon$ and $|\lambda| < [|T|_{(p+1)}^{p+1} + m/3]^{1/(p+1)} - |T|_{(p+1)}$. This concludes the proof.

6.1 We shall assume from now on that T and all its iterates are densely defined. Then T has an adjoint T^* defined in the space X^* of bounded linear functionals on X. We shall write $\langle x, x^*\rangle$ to denote the value of functional x^* at x.

DEFINITION. Operator A is said to be an *extension* of operator B if $D(A) \supseteqq D(B)$ and $Ax = Bx$ for $x \in D(B)$. If $D(A)$ can be written as $D(A) = D(B) \oplus Y$ where Y is a subspace of dimension k, then we call A a *k-dimensional extension* of B and write $[A : B] = k$.

LEMMA 4. *$(T^K)^*$ is an extension of $(T^*)^K$ for any positive integer K.*

Proof. The lemma is trivial for $K = 1$; suppose it has been verified for $K \leqq p$. Let $x^* \in D[(T^*)^{p+1}]$. Then $x^* \in D[(T^*)^p]$ and $(T^*)^p x^* \in D(T^*)$. Hence for any $x \in D(T^{p+1})$, we can write

$$\begin{aligned}\langle T^{p+1}x, x^*\rangle &= \langle Tx, (T^p)^*x^*\rangle \\ &= \langle Tx, (T^*)^p x^*\rangle \quad \text{by assumption} \\ &= \langle x, (T^*)^{p+1}x^*\rangle .\end{aligned}$$

Hence $x^* \in D[(T^{p+1})^*]$ and $(T^*)^{p+1}x^* = (T^{p+1})^*$. This completes the proof.

DEFINITION. We shall say that T is of *finite type* if, for each K, $(T^K)^*$ is a finite dimensional extension of $(T^*)^K$. If, in addition, $[(T^K)^* : (T^*)^K]$ is a bounded sequence, we shall say that T is of *bounded type.*

EXAMPLE. Every $T \in B(X)$ is of bounded type since $(T^K)^* = (T^*)^K$ for all K.

LEMMA 5. *Suppose that T is of finite type and that $R(T^K)$ is closed for each positive integer K. Then*

(a) $\alpha(T^*)$ *is finite if* $\delta(T)$ *is finite*

(b) $\alpha(T)$ *is finite if* $\delta(T^*)$ *is finite.*

If, in addition, T is of bounded type, then we also have

(c) $\delta(T)$ *is finite if* $\alpha(T^*)$ *is finite*

(d) $\delta(T^*)$ *is finite if* $\alpha(T)$ *is finite.*

Proof. By [4], Lemma 335, since T is a closed operator with closed range

$$\left.\begin{aligned} N(T^*) &= R(T)^\perp \\ R(T^*) &= N(T)^\perp \end{aligned}\right\} \tag{12}$$

where for any $Y \subseteqq X$, $Y^\perp = \{x^* \in X^*: \langle y, x^*\rangle = 0\ \forall y \in Y\}$.

For each positive integer K, we can write, by assumption

$$[R(T^K)]^\perp = N[(T^K)^*] = N[(T^*)^K] \oplus Y_K \tag{13}$$

where clearly Y_K must be of finite dimension. Now for $K > \delta(T)$, it is clear from (13) that $N[(T^*)^K] \oplus Y_K$ must be independent of K. But if $\alpha(T^*)$ is infinite, $\{N[T^*)^K]\}$ is a strictly increasing sequence of subspaces so that $\{Y_K\}$ would need to be strictly decreasing. This is not possible for finite dimensional subspaces. Hence (a) is verified. Conversely, if $\alpha(T^*)$ is finite, then $\delta(T)$ must also be finite when T is of bounded type. For were $\delta(T)$ infinite, $\{[R(T^K)]^\perp\}$ would be strictly increasing and for $K > \alpha(T^*)$, $\{N[(T^*)^K]\}$ is independent of K. By (13), this would imply that $\{Y_K\}$ is strictly increasing. For T of bounded type, this is not possible. This proves (c).

Next, we write, for each nonnegative integer K,

$$R[(T^K)^*] = R[(T^*)^K] \oplus Z_K \tag{14}$$

and again we can deduce from our assumptions that each Z_K is finite dimensional. But, from (12),

$$\begin{aligned} R[(T^K)^*] &= [N(T^K)]^\perp \\ &\cong [X/N(T^K)]^* \quad \text{by } [6] \text{ p. } 227\,, \end{aligned} \tag{15}$$

where $\cong$ indicates linear homeomorphism.

Now suppose $X = N(T^K) \oplus W_K$. Then W_K is isomorphic to $X/N(T^K)$.

Using $\approxeq$ to denote isomorphism, we obtain

$$\begin{aligned} R[(T^K)^*] &\approxeq W_K^* \\ &\approxeq X^*/W_K^\perp \quad \text{by } [2],\ \text{p. } 188\,. \end{aligned} \tag{16}$$

Let $\alpha(T)$ be infinite; then $\{W_K\}$ is strictly descending; $\{W_K^\perp\}$ strictly ascending. By (16), $\{R[(T^K)^*]\}$ is strictly descending. Now, if $\delta(T^*)$

is finite, then by (14), $\{Z_K\}$ must be strictly descending. But this is not possible. Hence (b) is proved.

Finally, suppose $\delta(T^*)$ infinite and $\alpha(T)$ is finite. Then $\{W_K\}$ is independent of K for $K > \alpha(T)$. From (16) and (14), we deduce that $\{Z_K\}$ must be strictly increasing, contrary to assumption. This verifies (d) and completes the proof.

THEOREM 4. *Suppose T is a closed linear operator such that $\delta(T) = q < \infty$. Let T be of bounded type. Then $\alpha(T^*) < \infty$. Suppose that T^* satisfies the assumptions of Theorem 3 and that there exists $\eta > 0$ such that $(T - \lambda)^*$ is of bounded type for $|\lambda| < \eta$. Then there exists $\sigma > 0$ such that $\delta(\lambda)$ is finite in the circle $|\lambda| < \sigma$.*

Proof. The assertion that $\alpha(T^*)$ is finite follows directly from Lemma 5. Moreover since $R(T^K)$ is closed for $K = 1 + \alpha(T^*)$, then by [4] Lemma 324, $R[(T^K)^*]$ is closed for $K = 1 + \alpha(T^*)$. By assumption $(T^K)^*$ is a finite dimensional extension of $(T^*)^K$ so that by [3] Lemma 333, $(T^*)^K$ has closed range. We now apply Theorem 3 to T^* and deduce that $T^* - \lambda$ has finite ascent for $|\lambda| < \rho^*$ for some $\rho^* > 0$. Now $(T - \lambda)^* = T^* - \lambda$ so that by Lemma 5, we can conclude that if $\sigma = \min(\rho^*, \eta)$, then $\delta(\lambda)$ is finite in the circle $|\lambda| < \sigma$. This concludes the proof.

In view of the additional hypothesis regarding the nature of $(T - \lambda)^*$, it is of some interest to examine the relationship between extensions and their adjoints. The following lemmas shed some light on the situation.

LEMMA 6. *Suppose A_1 is an extension of A_2 and $[A_1 : A_2] = k$. Then A_2^* is an extension of A_1^* and if $\overline{D(A_1)} = \overline{D(A_2)}$, then $[A_2^* : A_1^*] = k$.*

Proof. It is well known that A_2^* is an extension of A_1^* and this fact is trivial to verify. Let $\overline{D(A_1)} = \overline{D(A_2)} = X_0$ and define a mapping E

$$E\colon \quad X^* \times X_0^* \rightarrow (X_0 \times X)^*$$

by means of

$$E(f, g) - (x, y) \rightarrow f(y)\, g(x) \; .$$

If the usual norm topology is introduced into the Cartesian products, then we can show that E established a linear homeomorphism between $X^* \times X_0^*$ and $(X_0 \times X)^*$. It is easy to see that E is a linear map; moreover E is surjective, for if $F \in (X_0 \times X)^*$, we have $g \in X_0^*$ defined by $x \rightarrow F(x, 0)$ and $f \in X^*$ defined by $y \rightarrow F(0, y)$ so that

$$E(f, g) : (x, y) \rightarrow f(y) + g(x) = F(x, y) \; .$$

E is also injective, for if $E(f, g) = 0$, then $f(y) + g(x) = 0$ for all $x \in X$, $y \in X_0$. This is possible if and only if $f = g = 0$. Finally, we can see that E is continuous; for

$$| E(f, g)(x, y) | \leqq \| f \| \| y \| + \| g \| \| x \| \leqq (\| f \| + \| g \|)(\| x \| + \| y \|) .$$

By the closed graph theorem, E^{-1} is also continuous. Hence we have shown that E is a linear homeomorphism.

We next observe that if we write $G(T)$ to denote the graph of T, then

$$E\{G(A_i^*)\} = \{G(-A_i)\}^\perp \qquad i = 1, 2 \tag{17}$$

where $\{G(-A_i)\}^\perp$ denotes the elements F in $(X_0 \times X)^*$ such that $F(x, y) = 0$ for all $(x, y) \in G(-A_i)$.

For, if $x \in D(A_i)$ and $f \in D(A_i^*)$, then

$$E(f, A_i^* f)(x, -A_i x) = A_i^* f(x) - f(A_i x) = 0$$

so that $E\{G(A_i^*)\} \subseteqq \{G(-A_i)\}^\perp$.

On the other hand, if $E(f, g) \in \{G(-A_i)\}^\perp$, then $E(f, g)(x, -A_i x) = 0$ for all $x \in D(A_i)$. Then $f(A_i x) = g(x)$ for all $x \in D(A_i)$ so that $f \in D(A_i^*)$ and $g = T^* f$. Hence any $E(f, g)$ in $\{G(-A_i)\}^\perp$ is of the form $E(f, T^* f)$. This proves the validity of (17).

Now

$$\begin{aligned} E\{G(A_i^*)\} &= \{G(-A_i)\}^\perp = \{X_0 \times X / G(-A_i)\}^* \text{ by [6] p. 227} \\ &\approxeq \{(X_0 \times X) \ominus G(-A_i)\}^* . \end{aligned} \tag{18}$$

Now suppose $(X_0 \times X) \ominus G(-A_i) = X_i$. Then by [6] p. 188,

$$X_i^* = (X_0 \times X)^* / X_i^\perp \tag{19}$$

where $X_i^\perp = \{F : F \in (X_0 \times X)^*;\ F(x, y) = 0 \text{ for all } (x, y) \in X_i\}$.

It is easy to verify that $D(A_i)$ is isomorphic to $G(-A_i)$ by means of the natural mapping $x \to (x, -A_i x)$. Hence, $X_2 \ominus X_1$ is a k dimensional subspace and from (19), $X_1^* \ominus X_2^*$ is also k dimensional. Finally from (18), we see that $E(G(A_2^*) \ominus G(A_1^*))$ is k-dimensional from which we easily deduce that

$$[A_2^* : A_1^*] = k .$$

LEMMA 7. *Suppose T is of finite, resp. bounded type and $\overline{D[(T^K)^*)} = \overline{D[(T^*)^K]}$ for each positive integer K. Moreover, let either of the following conditions hold:*

(i) *$[(T^K)^{**} : T^K]$ is a sequence of finite terms, resp. bounded sequence*

(ii) *X is reflexive.*

Then T^* is of finite, resp. bounded, type.

Proof. To begin with, it is well known that if X is reflexive, then $T^{**} = T$ for any closed linear operator T. Hence condition (ii) implies condition (i). Suppose condition (i) holds. Then we have

$$[(T^K)^* : (T^*)^K] = m_K < \infty \tag{20}$$

and

$$[(T^K)^{**} : T^K] = n_K < \infty . \tag{21}$$

By Lemma 6, (20) yields

$$[((T^*)^K)^* : (T^K)^{**}] = m_K$$

and this together with (21) gives

$$[((T^*)^K)^* : T^K] = m_K + n_K . \tag{22}$$

But applying Lemma 4 to T^* we get

$$((T^*)^K)^* \supseteqq (T^{**})^K \supseteqq T^K \tag{23}$$

and from (22) and (23) we deduce

$$[((T^*)^K)^* : (T^{**})^K] \leqq m_K + n_K .$$

But this gives exactly the required conclusion.

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