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THE INVERSION OF A CLASS OF LINER OPERATORS

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Let $\bar{Q}_L$ denote the set of all quasi-continuous number valued functions on a number interval $[a, b]$ which vanish at a and are left continuous at each point of $(a, b]$. Every linear operator, $\mathscr{L}$, on $\bar{Q}_L$ which is continuous relative to the sup norm topology for $\bar{Q}_L$ has a unique representation of the form $\mathscr{L} f(s) = \int_a^b f(t)dL(t, s)$, $f \in \bar{Q}_L$, $a \leqq s \leqq b$, where all integrals are taken in the σ-mean Stieltjes sense, and L is a function on the square $a \leqq {t \atop s} \leqq b$, satisfying the conditions of Definition 1.2. This paper is concerned primarily with those linear operators, the P-operators, which are abstractions from that class of linear physical systems whose output signals at a given time do not depend on their input signals at a later time; and with a sub-family of the P-operators, the P_1-operators which include all stationary linear operators. The P-operators are the Volterra operators on $\bar{Q}_L$. Necessary conditions and sufficient conditions for a P-operator to have an inverse which is a P-operator are found; and a necessary and sufficient condition for a P_1-operator to have an inverse which is a P-operator is given in Theorem 3.1. In addition it is shown that if $\mathscr{L}$ is a P_1-operator and $\mathscr{L}^{-1}$ is a P-operator then $\mathscr{L}^{-1}$ may be written as the product of two operators whose generating functions may be found by successive approximation techniques. An analogue of Lane's inversion theorem for stationary operators on QC_{0L} is found as a special case of these results.

In [1] subspaces of the space of functions which are quasi-continuous on an interval $[a, b]$ for which every linear operator $\mathscr{L}$ may be written as a σ-mean Stieltjes integral of the form $\mathscr{L}f(s) = \int_a^b f(t)dL(t, s)$ are investigated. In this paper we will be concerned with one such subspace, $\bar{Q}_L$, and with a class of linear operators on $\bar{Q}_L$, the P-operators, which are essentially the abstractions from that class of linear physical systems whose output signals at a given time do not depend on their input signals at a later time. In particular we shall be concerned with determining conditions which will guarantee that a P-operator has an inverse which is a P-operator.

In § 2 some of the basic properties of P-operators are developed and in § 3 a subfamily of these operators, the P_1-operators, are introduced. The P_1-operators have the property that if a P_1-operator, $\mathscr{K}$, has an inverse which is a P-operator then the generating function for $\mathscr{K}^{-1}$ may be determined by successive approximation techniques. In

Theorem 3.1 necessary and sufficient conditions for a P_1-operator to have an inverse which is a P-operator are given. The inversion theorems obtained in § 2 and § 3 are related to some of Lane's results on stationary linear operators ([3] and [4]), and this relationship is also discussed in § 3. The author is grateful to the referee for his comments and suggestions in connection with this paper.

1. Preliminary theorems. In the main body of this paper it will be assumed that $[a, b]$ is a given number interval and that the statement "f is a left-continuous function on $[a, b]$" means that f is a quasi-continuous function on $[a, b]$; i.e. f is the limit of a uniformly convergent sequence of step functions on $[a, b]$; and f is left-continuous at each point of $(a, b]$. All integrals referred to will be σ-mean Stieltjes integrals and the reader is referred to [2] or [5] for a definition. We will need the following lemma which is a trivial consequence of Corollaries 1.1 and 1.2 of [5] and the definition of the σ-mean Stieltjes integral.

LEMMA 1.1. *Suppose f is a bounded function on $[a, b]$ and g a function of bounded variation on $[a, b]$. If $\int_a^b fdg$ exists then*

$$\int_a^b |f(t)|\, d[V_{\xi=a}^t g(\xi)]$$

exists and

$$\left|\int_a^b fdg\right| \leqq \int_a^b |f(t)|\, d[V_{\xi=a}^t g(\xi)]\ .$$

The symbol $\bar{Q}_L$ will denote the set of functions on $[a, b]$ to which a function f belongs if, and only if, f is left-continuous on $[a, b]$ and $f(a) = 0$. If f is in $\bar{Q}_L$ then the norm of f is taken to be $\sup |f(s)|$ for s in $[a, b]$. It follows immediately from the properties of quasi-continuous functions that $\bar{Q}_L$ is a Banach space. The following additional definitions will also be used.

DEFINITION 1.1. Suppose $\bar{t}$ is a number, $a \leqq \bar{t} < b$. The statement that $\tau_{\bar{t}}$ is a test function means that $\tau_{\bar{t}}$ is that function in $\bar{Q}_L$ defined by $\tau_{\bar{t}}(s) = -J_L(s - \bar{t})$, $a \leqq s \leqq b$, where J_L denotes the function defined by $J_L(s) = 0$, $s \leqq 0$, $J_L(s) = 1$, $s > 0$.

It is clear that the span of the set of all test functions is dense in $\bar{Q}_L$.

DEFINITION 1.2. The statement that A is a generating function means that A is a function on the square $a \leqq {t \atop s} \leqq b$ and that

(i) $A(b, s) = 0,\ a \leqq s \leqq b$,

(ii) for each number $\bar{s},\ a \leqq \bar{s} \leqq b,\ A(t, \bar{s})$ is of bounded variation on $[a, b]$,

(iii) for each number $\bar{t}, a \leqq \bar{t} < b,\ 1/2[A(\bar{t}, s) + A(\bar{t}+, s)]$ is in $\bar{Q}_L$, and

(iv) there exists a positive number M, such that if $\bar{s}$ is in $[a, b]$, $V_{t=a}^{b} A(t, \bar{s}) \leqq M$. The smallest such number M will be denoted by V_A. It is clear from this definition that any finite linear combination of generating functions is also a generating function.

If A is a generating function, then $\bar{A}$ will denote the function defined by

$$\bar{A}(t, s) = \begin{cases} A(b, s) & t = b, \quad a \leqq s \leqq b \\ \frac{1}{2}[A(t, s) + A(t+, s)], & a \leqq t < b, \quad a \leqq s \leqq b\,. \end{cases}$$

While the space $\bar{Q}_L$ introduced here differs from the space Q_L studied in [1], it is easy to show that the basic results of [1] can also be developed for the space $\bar{Q}_L$. In particular, Theorem 1.1, stated here without proof, can be established by the same techniques used for the analogous theorem in [1].

For the purposes of this paper, it will be assumed that operator means a continuous mapping whose domain is $\bar{Q}_L$ and whose range is a subset of $\bar{Q}_L$. The statement that an operator, $\mathscr{K}$, has an inverse will mean that the mapping inverse to $\mathscr{K}$ is an operator.

THEOREM 1.1. *If A is a generating function then there exists a linear operator, $\mathscr{A}$, on $\bar{Q}_L$ such that if s is in $[a, b]$ and f is in $\bar{Q}_L$ then $\mathscr{A}f(s) = \int_a^b f(t)dA(t, s)$, with $\|\mathscr{A}\| \leqq V_A$. Conversely if $\mathscr{B}$ is a linear operator on $\bar{Q}_L$ then $\mathscr{B}$ admits a representation of this type for some generating function, B, with $V_B \leqq 3\ \|\mathscr{B}\|$. Furthermore B is unique.*

COROLLARY 1.11. *Suppose $\mathscr{A}$ is a linear operator on $\bar{Q}_L$, and that A is the generating function for $\mathscr{A}$. If $\bar{t}$ is a number such that $a \leqq \bar{t} < b$, and $\tau_{\bar{t}}$ is a test function then $\mathscr{A}\tau_{\bar{t}}(s) = \bar{A}(\bar{t}, s), a \leqq s \leqq b$.*

Proof. By Theorem 1.1, $\mathscr{A}\tau_{\bar{t}}(s) = \int_a^b \tau_{\bar{t}}(\xi)dA(\xi, s) = \bar{A}(\bar{t}, s)$.

It follows from this corollary that the generating function, I, for the identity operator, $\mathscr{I}$, on $\bar{Q}_L$ is given by $I(t, s) = -J_L(s-t)$, $a \leqq {t \atop s} \leqq b$. Also, if each of $\mathscr{K}, \mathscr{L}$, and $\mathscr{M}$ is a linear operator on $\bar{Q}_L$, with generating functions K, L, and M respectively, and $\mathscr{K}\mathscr{L} = \mathscr{M}$

then $\bar{M}(t, s) = \int_a^b \bar{L}(t, \xi) dK(\xi, s),\ a \leqq {t \atop s} \leqq b.$

2. P-operators.

DEFINITION 2.1. Suppose $\mathscr{A}$ is a linear operator on $\bar{Q}_L$. The statement that $\mathscr{A}$ is a P-operator means that if f is in $\bar{Q}_L$, and has the property that for some number $c, a < c < b, f(t) = 0, t \leqq c$ then $\mathscr{A} f(s) = 0, s \leqq c$.

It follows immediately from the definition that the identity operator, $\mathscr{I}$, is a P-operator; and that sums and products of P-operators are P-operators. A more interesting result however is:

THEOREM 2.1. *Suppose $\mathscr{K}$ is a linear operator with generating function K. A necessary and sufficient condition that $\mathscr{K}$ be a P-operator is that $K(t, s) = 0,\ a \leqq t \leqq b,\ a \leqq s \leqq t$.*

Proof. Since $K(t, s) = 2\bar{K}(t, s) - \bar{K}(t+, s),\ a \leqq t < b,\ a \leqq s \leqq b$, the necessity of this condition follows Corollary 1.11 and the definition of a test function.

Conversely, if $K(t, s) = 0,\ a \leqq t \leqq b,\ a \leqq s \leqq t$ then $\bar{K}(t, s) = 0$, $a \leqq t \leqq b,\ a \leqq s \leqq t$, and by Corollary 1.11 $\mathscr{K} \tau_{\bar{t}}(s) = 0,\ a \leqq s \leqq \bar{t}$. If g is in $\bar{Q}_L$, and for some number $k,\ a < k < b,\ g(t) = 0,\ t \leqq k$, then any sequence of linear combinations of test functions which converges to g need contain only test functions $\tau_{\bar{t}}$ for which $\bar{t} \geqq k$, therefore $\mathscr{K} g(s) = 0,\ s \leqq k$, and $\mathscr{K}$ is a P-operator.

From Theorem 2.1 and the properties of the mean integral, it is clear that if $\mathscr{K}$ is a P-operator with generating function K and f is in $\bar{Q}_L$ then $\mathscr{K} f(s) = \int_a^s f(t) dK(t, s),\ a \leqq s \leqq b$; or in other words, a P-operator is an operator of Volterra type.

Throughout the remainder of this section it will be assumed that $\mathscr{K}$ denotes a P-operator with generating function K. If $\mathscr{K}$ has an inverse then the generating function for $\mathscr{K}^{-1}$ will be denoted by $K^{(-1)}$.

The remaining theorems of this section are concerned with necessary conditions and sufficient conditions for $\mathscr{K}$ to have an inverse which is a P-operator.

THEOREM 2.2. *Suppose there exists a number $\bar{t}, a \leqq \bar{t} < b$, such that $\mathscr{K} \tau_{\bar{t}}(\bar{t}+) = \bar{K}(\bar{t}, \bar{t}+) = 0$. Then $\mathscr{K}$ does not have an inverse which is a P-operator.*

Proof. If $\mathscr{K}$ has an inverse then $\mathscr{K}^{-1} \mathscr{K} \tau_{\bar{t}}(\bar{t}+) = -1$. If $\mathscr{L}$ is a P-operator then $\mathscr{L} \mathscr{K} \tau_{\bar{t}}(s) = \int_{\bar{t}}^s \mathscr{K} \tau_{\bar{t}}(\xi) dL(\xi, s)$. Hence from

Lemma 1.1, $|\mathscr{L}\mathscr{K}\tau_{\bar t}(s)| \leqq \sup_{\xi\in[\bar t, s]} |\mathscr{K}\tau_{\bar t}(\xi)| V_L$, and $\mathscr{L}\mathscr{K}\tau_{\bar t}(\bar t+) = 0$.

If $\bar t$ in Theorem 2.2 is a, then $\mathscr{K}$ not only does not have an inverse which is a P-operator, $\mathscr{K}$ does not have an inverse. This follows because $\mathscr{K}\tau_a(a+) = 0$ implies that for any $f \in Q_L$, $\mathscr{K}f(a+) = 0$, since for any P-operator $\mathscr{K}$, $\mathscr{K}\tau_{\bar t}(a+) = 0$, $a < \bar t < b$. If $\bar t$ is not a then $\mathscr{K}$ may have an inverse. As an example, suppose that $[a, b]$ is $[0, 1]$ and let $\mathscr{L}$ be the P-operator defined by:

$$\mathscr{L}\tau_t = \tau_{(3/2)t},\ 0 \leqq t \leqq 1/2;\ \mathscr{L}\tau_t = \tau_{(1/2)(t+1)},\ 1/2 < t < 1\,.$$

Here, $\mathscr{L}\tau_t(t+) = 0$, $0 < t < 1$. However, $\mathscr{L}$ has an inverse, $\mathscr{L}^{-1}$ being the linear operator defined by:

$$\mathscr{L}^{-1}\tau_t = \tau_{(2/3)t},\ 0 \leqq t \leqq \frac{3}{4};\ \mathscr{L}^{-1}\tau_t = \tau_{2t-1},\ \frac{3}{4} < t < 1\,.$$

$\mathscr{L}^{-1}$ is clearly not a P-operator.

If $\mathscr{K}$ is a P-operator whose generating function has the property that for some number $\bar t$, $a \leqq \bar t < b$ and every positive number c, there exists a positive number d such that if s is in $(\bar t, \bar t + d)$, $V_{\xi=\bar t}^{s} K(\xi, s) < c$, then $\mathscr{K}\tau_{\bar t}(\bar t+) = 0$. This follows from Lemma 1.1 since

$$|\mathscr{K}\tau_{\bar t}(s)| = \left|\int_{\bar t}^{s} \tau_{\bar t}(\xi) dK(\xi, s)\right|,\ \bar t \leqq s \leqq b\,.$$

This condition will be needed in § 3.

THEOREM 2.3. *Suppose that $\mathscr{U}$ is an operator on $\bar Q_L$ whose generating function has the property that for some number h,*

$$o < h < b - a,\ U(t, s) = 0,\ a \leqq s \leqq b,\ s - h \leqq t \leqq b\,.$$

Then $\mathscr{L} = \mathscr{I} - \mathscr{U}$ has an inverse and $\mathscr{L}^{-1} = \sum_{n=0}^{p-1} \mathscr{U}^n$, where p is the smallest integer such that $a + ph \geqq b$.

Proof. Suppose that g is in $\bar Q_L$. Since $U(t, s) = 0$, $a \leqq s \leqq b$, $s - h \leqq t \leqq b$, it follows that

$$\mathscr{U}g(s) = \begin{cases} 0, & a \leqq s \leqq a + h \\ \displaystyle\int_a^{s-h} g(t) dU(t, s), & a + h < s \leqq b\,. \end{cases} \tag{1}$$

By successive applications of equation (1) it can be shown that if q is an integer, $q \geqq p$, then $\mathscr{U}^q g(s) = 0$, $a \leqq s \leqq b$. The theorem then follows.

It should be noted that the hypothesis on $\mathscr{U}$ implies that $\mathscr{U}$ is a P-operator. Hence $\mathscr{L}$ and $\mathscr{L}^{-1}$ are P-operators.

THEOREM 2.4. *Suppose that K has the property that there exists a number k, $0 < k < b - a$, such that*

(i) *$K(s - k, s)$ is a left-continuous function on $[a + k, b]$, and*

(ii) *the P-operator, $\mathscr{A}$, generated by*

$$A(t, s) = \begin{cases} K(s - k, s), & a + k < s \leqq b,\ a \leqq t \leqq s - k \\ K(t, s), & a \leqq s \leqq a + k,\ a \leqq t \leqq b \text{ or} \\ & a + k < s \leqq b,\ s - k \leqq t \leqq b \end{cases}$$

has an inverse.

Then $\mathscr{K}$ has an inverse, and $\mathscr{K}^{-1}$ is a P-operator if, and only if, $\mathscr{A}^{-1}$ is a P-operator.

Proof. There exists a unique operator $\mathscr{L}$ such that $\mathscr{A}\mathscr{L} = \mathscr{K}$. From Theorem 1.1 and its corollary it then follows that if $\bar{t}$ is a number in $[a, b]$ then $\bar{L}(\bar{t}, s)$ is the unique solution in $\bar{Q}_L$ to the integral equation

$$\bar{K}(\bar{t}, s) = \int_a^s f(\bar{t}, \xi) dA(\xi, s),\ a \leqq s \leqq b\ .$$

By direct computation it can be seen that

$$\bar{L}(\bar{t}, s) = -J_L(s - \bar{t}) = L(\bar{t}, s),\ \text{if } a \leqq s \leqq a + k,\ a \leqq \bar{t} \leqq b\ ,$$

or $a + k < s \leqq b$, $s - k \leqq \bar{t} \leqq b$. Hence, $\mathscr{I} - \mathscr{L}$ satisfies the hypotheses of Theorem 2.3 and $\mathscr{L}$ has an inverse. Therefore $\mathscr{K}^{-1} = \mathscr{L}^{-1}\mathscr{A}^{-1}$. The remaining assertions follow immediately since $\mathscr{L}$ and $\mathscr{L}^{-1}$ are P-operators. This completes the proof.

If there exists an integer q such that $(\mathscr{I} - \mathscr{A})^q \| < 1$, then K satisfies the hypotheses of Theorem 2.4, and in this case, for each number t in $[a, b]$ $\bar{A}^{(-1)}$ is the successive approximations solution to the integral equation

$$-J_L(s - t) = g(t, s) - \int_a^s g(t, \xi) d[-J_L(s - \xi) - A(\xi, s)],\ a \leqq {t \atop s} \leqq b\ ,$$

and $\bar{L}$ is the successive approximations solution to

$$\bar{K}(t, s) = f(t, s) - \int_a^s f(t, \xi) d[-J_L(s - \xi) - A(\xi, s)],\ a \leqq {t \atop s} \leqq b\ ;$$

the approximating sequences converging in both cases uniformly in s on $[a, b]$ for each number t in $[a, b]$. Consideration of the approximating sequence for $\bar{A}^{(-1)}$ shows that in this case $\mathscr{A}^{-1}$, and hence $\mathscr{K}^{-1}$, is a P-operator. In particular, if it is true that

$$(2) \qquad V_{t=a}^{b}[-J_L(s-t)-A(t,s)] \leqq M < 1, \ a \leqq s \leqq b ,$$

it can be shown that these two approximating sequences are in fact uniformly convergent on the square $a \leqq {t \atop s} \leqq b$.

If K is right continuous in t for each number s in $[a, b]$ and A satisfies equation (2) one would have usable computational techniques for the determination of $\bar{K}^{(-1)}$, since if K has this right continuity property then so does A. It can then be shown from the approximating sequences for $\bar{A}^{(-1)}$ and $\bar{L}$ that they are also right continuous in t for each number s in $[a, b]$. Hence, in this case, $\bar{A}^{(-1)} = A^{(-1)}$ and $\bar{L} = L$. One would still have the problem of obtaining $K^{(-1)}$ from $\bar{K}^{(-1)}$, but for the solution of many operator problems knowledge of $\bar{K}^{(-1)}$ would suffice. The P_1-operators to be considered in § 3 will have this right continuity property.

3. P_1-operators. In this section we will consider a class of P-operators that are of interest in the study of electrical networks.

DEFINITION 3.1. The statement that a linear operator, $\mathscr{S}$, on $\bar{Q}_L$ is a stationary operator means that if each of f and g is in $\bar{Q}_L$ and for some number k, $0 < k < b - a$,

$$g(t) = \begin{cases} 0, & a \leqq t \leqq a + k \\ f(t-k), & a + k < t \leqq b, \end{cases}$$

then

$$\mathscr{S}g(s) = \begin{cases} 0 & a \leqq s \leqq a + k \\ \mathscr{S}f(s-k), & a + k < s \leqq b . \end{cases}$$

It follows from this definition, Definition 1.1, and Theorem 1.1, that a linear operator, $\mathscr{S}$, is stationary if, and only if, $\mathscr{S}$ has a representation of the form, $\mathscr{S}f(s) = \int_a^s f(\xi)d[u(s-\xi)]$, where

$$u(t) = \begin{cases} 0, & a - b \leqq t \leqq 0 \\ \mathscr{S}\tau_a(t+a), & 0 < t \leqq b - a . \end{cases}$$

Hence, every stationary operator is a P-operator. It is also a trivial consequence of Definition 3.1 and this representation that $\mathscr{I}$ is a stationary operator and that sums and products of stationary operators are stationary.

It may also be concluded from this representation that the generating function for a stationary operator is right continuous in t for each s in $[a, b]$, and that u is of bounded variation on $[a - b, b - a]$. Furthermore if u^* denotes the function defined by $u^*(t) = V_{\xi=a-b}^{t} u(t)$,

$a - b \leqq t \leqq b - a$, then the mapping, $\mathscr{S}^*$, given by

$$\mathscr{S}^* f(s) = \int_a^s f(\xi) du^*(s - \xi), \ a \leqq s \leqq b$$

is a stationary operator on $\bar{Q}_L$.

In [3] and [4], Lane has developed the theory of a class of linear operators, T_{OL}, on the set, QC_{OL}, of functions on the real line, which are quasi-continuous on each closed bounded interval, are everywhere left continuous, and vanish for negative values of their argument. A linear operator, $\mathscr{L}$, is in T_{OL} if, and only if, there exists a function, u, in QC_{OL}, of bounded variation on each closed bounded interval, such that if f is in QC_{OL} and s is a number then $\mathscr{L} f(s) = \int_0^\infty f(s - t) du(t)$. Using the properties of the integral this condition can be rewritten

$$\mathscr{L} f(s) = \begin{cases} 0, & s \leqq 0 \\ \int_0^s f(\xi) d[-u(s - \xi)], & s > 0 \ . \end{cases}$$

Thus, the stationary operators on $\bar{Q}_L$ are analogous to Lane's T_{OL} operators.

DEFINITION 3.2. The statement that a bounded linear operator, $\mathscr{K}$, on $\bar{Q}_L$ is a P_1-operator means that there exists a stationary operator, $\mathscr{S}$, such that if each of τ_p and τ_q is a test function then

$$| \mathscr{K}(\tau_p - \tau_q)(s) | \leqq | \mathscr{S}(\tau_p - \tau_q)(s) |, \ a \leqq s \leqq b \ .$$

From Corollary 1.11, an equivalent form of Definition 3.2 is

$$| \bar{K}(p, s) - \bar{K}(q, s) | \leqq | u(s - p) - u(s - q) | \leqq | u^*(s - p) - u^*(s - q) | \ .$$

Therefore if $\mathscr{K}$ is a P_1-operator, $\bar{K} = K$. Also if $\mathscr{S}$ dominates $\mathscr{K}$ then so does $\mathscr{S}^*$. From this it may be shown by direct computation that if each of $\mathscr{K}_1$ and $\mathscr{K}_2$ is a P_1-operator with dominating stationary operators $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively then $\mathscr{S}_1^* + \mathscr{S}_2^*$ dominates $\mathscr{K}_1 + \mathscr{K}_2$ and $\mathscr{S}_1^* \mathscr{S}_2^*$ dominates $\mathscr{K}_1 \mathscr{K}_2$.

In the remainder of this section it will be assumed that $\mathscr{K}$ denotes a P_1-operator, with generating function K, and that $\mathscr{S}$ is a dominating stationary operator for $\mathscr{K}$.

THEOREM 3.1. *$\mathscr{K}$ has an inverse which is a P-operator if, and only if,* $\inf | K(s-, s) |$ *for s in $(a, b]$ is not zero. Furthermore if* $\lim_{s \to a+} K(s-, s) = 0$ *then $\mathscr{K}$ has no inverse.*

Proof. It can be shown from Definition 3.2 that if h, $0 < h < b - a$, is a point of continuity of u, then $K(s - h, s)$ is a left continuous function on $[a+h, b]$. It then follows that if f is a function on $[a, b]$ such that $f(s) = K(s-, s)$, $a < s \leqq b$ then f is a left continuous function on $[a, b]$. Suppose that $f(a) \neq 0$ and $\inf_{s\in(a,b]} |K(s-, s)| = L > 0$. If M is a number, $0 < M < 1$, and h is a point of continuity of u such that $u^*(h) - u^*(0+) \leqq ML$, then it follows from Definition 3.2 that for every number s in $[a, b]$,

$$V_{t=\varepsilon}^{b}\{-J_L(s-t) - [-f(s)]^{-1}K(t, s)\} \leqq M\,,$$

where ε is the larger of a and $s-h$. Let $\mathscr{K}^*$ denote the P-operator defined by $\mathscr{K}^*g(s) = [-f(s)]^{-1}\mathscr{K}g(s)$, $a \leqq s \leqq b$, $g \in \bar{Q}_L$. Then $\mathscr{K}^*$ satisfies the hypotheses of Theorem 2.4, since $\mathscr{I} - \mathscr{A}^*$ is a contraction mapping. Hence, $\mathscr{K}^{*-1}$ exists and is a P-operator. It then follows immediately from Theorems 1.1 and 2.1 that $\mathscr{K}^{-1}$ exists, and $K^{(-1)}(t, s) = [-f(s)]^{-1}K^{*(-1)}(t, s)$ so that $\mathscr{K}^{-1}$ is also a P-operator.

If $\inf_{s\in(a,b]} |K(s-, s)| = 0$ then either there exists a number q in $[a, b)$ such that $\lim_{s\to q^+} K(s-, s) = 0$ or a number p in $(a, b]$ such that $K(p-, p) = 0$. In the first case it can be shown from Definition 3.2 that if c is a positive number there exists a positive number d such that $V_{\xi=q}^{s} K(\xi, s) < c$, $q < s < q+d$. Hence $K(q, q+) = 0$, $\mathscr{K}$ does not have an inverse which is a P-operator, and if $q = a$, $\mathscr{K}$ has no inverse.

In the second case, it can be shown in a similar manner that if c is a positive number, there exists a positive number d such that if s and t are in $(p-d, p]$, $t < s$, then $V_{\xi=t}^{s} K(\xi, s) < c$. Hence $|\mathscr{K}\tau_t(s)| < c$ for t in $(p-d, p]$ and $t < s \leqq p$ by Lemma 1.1. From this it follows that there exists a strictly increasing sequence of numbers, $\{t_n\}_{n=1}^{\infty}$, $t_n < p$, $n = 1, 2, 3, \cdots$, such that if for each positive integer n, g_n is defined by

$$g_n(s) = \begin{cases} \mathscr{K}\tau_{t_n}(s), & a \leqq s \leqq p \\ 0, & p < s \leqq b\,, \end{cases}$$

then the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly to zero on $[a, b]$. Suppose now that $\mathscr{K}$ has an inverse and $\mathscr{K}^{-1}$ is a P-operator. Since $\mathscr{K}^{-1}$ is bounded, $\{\mathscr{K}^{-1}g_n\}_{n=1}^{\infty}$ is also uniformly convergent on $[a, b]$. But $\mathscr{K}^{-1}(g_n - \mathscr{K}\tau_{t_n})(s) = 0$, $a \leqq s \leqq p$, if $\mathscr{K}^{-1}$ is a P-operator. Or, $\mathscr{K}^{-1}g_n(s) = \tau_{t_n}(s)$, $a \leqq s \leqq p$, and $\{\tau_{t_n}\}_{n=1}^{\infty}$ is not uniformly convergent on $[a, p]$. This completes the proof.

If $\mathscr{K}$ is a stationary operator then Theorem 3.1 yields a stronger result, because in this case $K(s-, s) = \mathscr{K}\tau_a(a+)$, $a < s \leqq b$. Therefore either $K(s-, s) \equiv 0$ or $\inf |K(s-, s)| > 0$ on $(a, b]$. Consequently a necessary and sufficient condition that a stationary operator, $\mathscr{K}$, have an inverse is that $\mathscr{K}\tau_a(a+) \neq 0$. Furthermore the inverse of a stationary operator must be a P-operator, and it can be shown by applying the construction used in the proof of Theorem 2.4, and the remarks following Theorem 2.4, to a stationary generating function that the inverse must be stationary also. This is analogous to Lane's result for the operators in T_{OL} [3].

From Theorem 3.1 and its special form for a stationary operator, it may be concluded that if $\mathscr{K}$ is a P_1-operator and there exists a stationary operator, $\mathscr{S}$, which has no inverse and dominates $\mathscr{K}$, then $\mathscr{K}$ has no inverse, since $|\mathscr{K}\tau_a(s)| \leqq |\mathscr{S}\tau_a(s)|$, $a \leqq s \leqq b$, from Definition 3.2.

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