

FRACTIONAL POWERS OF OPERATORS

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A definition of fractional (or complex) powers A^α , $\alpha \in \mathbb{C}$, is given for closed linear operators A in a Banach space X with the resolvent set containing the negative real ray $(-\infty, 0)$ and such that $\{\lambda(\lambda + A)^{-1}; 0 < \lambda < \infty\}$ is bounded; fundamental properties such as additivity ($A^\alpha A^\beta = A^{\alpha+\beta}$), coincidence with the iterations $A^\alpha = A^n$ for integers $\alpha = n$, and analytic dependence on α are discussed. Since the fractional powers A^α are generally unbounded in both of the cases $\operatorname{Re} \alpha > 0$ and $\operatorname{Re} \alpha < 0$, attention is paid to the domains $D(A^\alpha)$, which are related to the spaces D^σ and R^τ of $x \in X$ defined by the regularity of $(\lambda + A)^{-1}x$ at ∞ and 0 . When $-A$ generates a bounded continuous semi-group or a bounded analytic semi-group, more detailed results are obtained.

The study of fractional powers of operators has a long history, which may go back to Abel's work on the tautochrone, the Riemann-Liouville integral, and its generalizations by M. Riesz. However, it is only recently that the general theory was developed. When A is the negative of the infinitesimal generator of a bounded semi-group of operators, Hille [6] and Phillips [16] showed that fractional powers could be treated in the framework of an operational calculus which they originated. This program was carried out thoroughly by Balakrishnan [1]. Later Balakrishnan [2] gave a new definition and extended his theory to a wider class of operators. About the same time two different definitions were introduced by Krasnosel'skii-Sobolevskii [13] and Kato [10]; further results were obtained by them, Yosida [22], Kato [11] and Watanabe [20]. These theories, with the exception of [11], as well as some classical results on the Riemann-Liouville integral (Hardy-Littlewood [5], Love-Young [14]), will be reconstructed from a unified point of view.

Our definition of fractional powers is essentially the same as Balakrishnan's second definition and if, in particular, $\operatorname{Re} \alpha > 0$, they are identical. In order to see that this definition is a natural one, let us consider the case in which A is bounded and the resolvent set $\rho(A)$ contains the negative real axis $(-\infty, 0]$. The most natural definition of A^α for such an operator A is given by the Dunford integral

$$(1.1) \quad A^\alpha = \frac{1}{2\pi i} \int_\Gamma \zeta^\alpha (\zeta - A)^{-1} d\zeta,$$

where the path Γ encircles the spectrum $\sigma(A)$ counterclockwise avoiding

the negative real axis and ζ^α takes the principal branch. It is then quite easy to see that A^α is a bounded operator which is an entire function of α , satisfying $A^\alpha = A^n$ if α is an integer n , and $A^\alpha A^\beta = A^{\alpha+\beta}$, for all $\alpha, \beta \in \mathbf{C}$.

Now we notice that the integration path Γ may be considered to encircle the negative real axis clockwise. Thus we can write the integral as

$$(1.2) \quad \begin{aligned} A^\alpha &= \frac{1}{2\pi i} \int_{-\infty}^0 [(\zeta + i0)^\alpha - (\zeta - i0)^\alpha](\zeta - A)^{-1} d\zeta \\ &= -\frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^\alpha (\lambda + A)^{-1} d\lambda \end{aligned}$$

in the sense of Fantappiè and Sato [17]. The last integral loses its ordinary meaning at ∞ if $\operatorname{Re} \alpha \geq 0$ and at 0 if $\operatorname{Re} \alpha \leq -1$. However, we have a convergent expression valid for $-(m+1) < \operatorname{Re} \alpha < n+1$:

$$(1.3) \quad \begin{aligned} A^\alpha &= -\frac{\sin \pi\alpha}{\pi} \left[\int_0^N (-1)^m \lambda^{\alpha+m} (\lambda + A)^{-1} A^{-m} d\lambda \right. \\ &\quad \left. + \sum_{k=-m}^n (-1)^{k+1} \frac{N^{\alpha-k}}{\alpha-k} A^k + \int_N^\infty (-1)^{n+1} \lambda^{\alpha-n-1} (\lambda + A)^{-1} A^{n+1} d\lambda \right], \end{aligned}$$

where N is an arbitrary fixed positive real number. This may be proved directly from (1.1). Another proof is obtained from (1.2) by expanding $(\lambda + A)^{-1}$ into a Taylor series around ∞ and 0 and applying the formula

$$(1.4) \quad \int_0^N \lambda^\beta d\lambda = -\int_N^\infty \lambda^\beta d\lambda = \frac{N^{\beta+1}}{\beta+1},$$

which holds for every $\beta \neq -1$ in the sense of Fantappiè-Sato.

Throughout this paper we will assume that A is a closed linear operator such that the resolvent set contains $(-\infty, 0)$ and the resolvent satisfies

$$(1.5) \quad \|\lambda(\lambda + A)^{-1}\| \leq M, \quad \lambda > 0,$$

with a constant M independent of λ . The bounded operator A considered above and the negatives of infinitesimal generators of bounded semi-groups satisfy these conditions. We remark that the operator $A(\lambda + A)^{-1} = 1 - \lambda(\lambda + A)^{-1}$ is also uniformly bounded. The bound is denoted L :

$$(1.6) \quad \|A(\lambda + A)^{-1}\| \leq L, \quad \lambda > 0.$$

The constants M and L will have the same meanings throughout this paper.

Our definition is based on the formula (1.2) or (1.3). But we

cannot expect that $(\lambda + A)^{-1}$ has a Taylor expansion at ∞ or 0, nor that $A^\alpha x$ is defined for all x . In place of a Taylor expansion we use the asymptotic expansion of $(\lambda + A)^{-1}x$ which depends on x . Thus, our study begins with the regularities of $(\lambda + A)^{-1}x$ at ∞ and 0. If x satisfies

$$(\lambda + A)^{-1}x = \lambda^{-1}x_0 - \lambda^{-2}x_1 + \dots + (-1)^n \lambda^{-n-1}x_n + O(\lambda^{-\sigma-1})$$

as $\lambda \rightarrow \infty$, where $n \leq \sigma < n + 1$ (if $n = \sigma$, $O(\lambda^{-\sigma-1})$ must be replaced by $o(\lambda^{-\sigma-1})$), then x is said to belong to the space D^σ . Similarly the space R^σ is defined by the regularity at 0. The spaces D^σ and R^σ are shown to be closely related to the domain $D(A^n)$ and the range $R(A^n)$, and give a kind of interpolation spaces (Sections 2, 3).

If $\text{Re } \alpha > 0$ and if $\sigma > \text{Re } \alpha$, then $A^\alpha x$ for $x \in D^\sigma$ is defined by (1.2) or its justification as (1.3). It will be proved that the operator A_σ^α thus defined has the smallest closed extension A_\pm^α independent of σ , which we define to be the fractional power. In the same way A_-^α , $\text{Re } \alpha < 0$, and A_0^α , $\alpha \in \mathbb{C}$, are defined by their restrictions to R^σ and $D^\sigma \cap R^\tau$, respectively (Section 4).

A_+^α , A_-^α and A_0^α are closed linear operators with dense domains in $\overline{D(A)}$, $\overline{R(A)}$ and $\overline{D(A)} \cap \overline{R(A)}$, respectively, and dense ranges in $\overline{D(A)} \cap \overline{R(A)}$. When P is an operator (or a relation) in a space X , and Y is a subspace of X , let us call the operator Q in Y defined by $D(Q) = \{y \in D(P) \cap Y; Py \in Y\}$ and $Qy = Py, y \in D(Q)$, the maximal restriction of P in Y . Then A_0^α for $\text{Re } \alpha > 0$ ($\text{Re } \alpha < 0$) is the maximal restriction of A_+^α (A_-^α) in $\overline{D(A)} \cap \overline{R(A)}$. When α is an integer n , A_+^α , A_-^α and A_0^α coincide with the maximal restrictions of A^n in $\overline{D(A)}$, $\overline{R(A)}$ and $\overline{D(A)} \cap \overline{R(A)}$ respectively (Section 4).

More precisely the domain $D((\mu + A)_\pm^\alpha)$ does not depend on $\mu \geq 0$ and coincides with the range $R((\mu + A)_0^{-\alpha})$, $\mu > 0$, while $R((\mu + A)^{-\alpha})$ is contained in $D^{\text{Re } \alpha}$ unless $\text{Re } \alpha$ is an integer and in $D^{\text{Re } \alpha - \varepsilon}$ for any $\varepsilon > 0$ if $\text{Re } \alpha$ is an integer. In particular, it follows that $D(A_+^\alpha) \subset D(A_+^\beta)$ if $\text{Re } \alpha > \text{Re } \beta$. Since $D^{\text{Re } \alpha + \varepsilon} \subset D(A_+^\alpha)$, $\varepsilon > 0$, $D^{\text{Re } \alpha}$ is almost equal to $D(A_+^\alpha)$. Similarly, if $\mu > 0$, we have $D(A_-^\alpha) = D((A(\mu + A)^{-1})_-^\alpha) = R((A(\mu + A)^{-1})_0^{-\alpha}) \subset R^{-\text{Re } \alpha - \varepsilon}$ or $R^{-\text{Re } \alpha}$ according as $\text{Re } \alpha$ is an integer or not (Section 6).

A_\pm^α maps $D^\sigma \cap D(A_\pm^\alpha)$ into $D^{\sigma - \text{Re } \alpha}$ if $\alpha - \text{Re } \alpha$ is positive and not an integer.

If $\text{Re } \alpha \cdot \text{Re } \beta > 0$, then $A_*^\alpha A_*^\beta = A_*^{\alpha + \beta}$ holds in the sense of the product of two operators, where the subscripts are assumed to be the same. For every α and β the smallest closed extension of $A_0^\alpha A_0^\beta$ is the same as $A_0^{\alpha + \beta}$. If $x \in D(A_0^\beta) \cap D(A_0^{\alpha + \beta})$, then $A_0^\beta x \in D(A_0^\alpha)$ and $A_0^\alpha A_0^\beta x = A_0^{\alpha + \beta} x$. Thus the additivity holds fairly completely (Section 7).

If $x \in D(A_*^\alpha) \cap D(A_*^\beta)$, $\text{Re } \alpha > \text{Re } \beta$, then x belongs to $D(A_*^\gamma)$ for all

$\operatorname{Re} \alpha > \operatorname{Re} \gamma > \operatorname{Re} \beta$. Moreover, $A_*^\gamma x$ is analytic in γ and converges strongly to $A_*^\alpha x (A_*^\beta x)$ as γ tends to $\alpha (\beta)$ in such a way that $|\arg(\alpha - \gamma)| \leq \theta < \pi/2$ ($|\arg(\gamma - \beta)| \leq \theta < \pi/2$). $A_*^\gamma x$ satisfies the convexity inequality

$$(1.7) \quad \|A_*^\gamma x\| \leq C \|A_*^\alpha x\|^\theta \|A_*^\beta x\|^{1-\theta}$$

where $\theta = \operatorname{Re}(\gamma - \beta)/\operatorname{Re}(\alpha - \beta)$ and the constant C depends on γ only in terms of $|\arg(\alpha - \gamma)|$ and $|\arg(\gamma - \beta)|$ (Section 8).

Therefore if B is a closed linear operator such that $D(B) \supset D(A_+^\gamma)$, we have

$$(1.8) \quad \|Bx\| \leq C \|A_+^\alpha x\|^\theta \|x\|^{1-\theta}, \quad x \in D(A_+^\alpha),$$

where $\operatorname{Re} \alpha > \operatorname{Re} \gamma > 0$ and $\theta = \operatorname{Re} \gamma / \operatorname{Re} \alpha$. Conversely if (1.8) holds, then $D(B) \supset D(A_+^\gamma)$ for every γ with $\operatorname{Re} \gamma > \theta \operatorname{Re} \alpha$ (Section 9).

Now we have to specify the operator A more strictly. An operator A is said to be of type $(\omega, M(\theta))$, $0 \leq \omega < \pi$, if the domain $D(A)$ is dense, the resolvent set of $-A$ contains the sector $|\arg \lambda| < \pi - \omega$ and (1.5) holds on each ray $\lambda = re^{i\theta}$, $0 < r < \infty$, $|\theta| < \pi - \omega$, with $M = M(\theta)$. Every operator A with a dense domain which satisfies (1.5) is of type $(\omega, M(\theta))$ for some $\omega < \pi$. If $-A$ is the generator of a bounded semi-group, then A is of type $(\pi/2, M \sec \theta)$. Further it is known that A is of type $(\omega, M(\theta))$ with $\omega < \pi/2$ if and only if $-A$ generates a semi-group T_t which has an analytic extension to the sector $|\arg t| < \pi/2 - \omega$ such that the extension is uniformly bounded on each smaller sector $|\arg t| \leq \pi/2 - \omega - \varepsilon$, $\varepsilon > 0$. Let us call such a semi-group an analytic semi-group.

If A is of type $(\omega, M(\theta))$ and if $0 < \alpha < \pi/\omega$, then A_+^α is of type $(\alpha\omega, M_\alpha(\theta))$, where $M_\alpha(\theta)$ is bounded by $3M((\theta + (\alpha - 1)\pi)^+/\alpha)$. In particular $-A_+^\alpha$ generates an analytic semi-group $\exp(-tA_+^\alpha)$ if $0 < \alpha < \pi/(2\omega)$. The semi-group $\exp(-tA_+^\alpha)$ is continuous in the uniform operator topology in t and α in the domain $|\arg t| < \pi/2 - \alpha\omega$, $0 < \alpha < \pi/(2\omega)$. For every $x \in \overline{R(A)}$, $\exp(-tA_+^\alpha)x$ converges to $\exp(-t)x$ as α tends to 0. If A is of type $(\omega, M(\theta))$ and if $0 < \alpha < \pi/\omega$ and $\operatorname{Re} \beta > 0$, then the multiplicativity $(A_+^\alpha)^\beta = A_+^{\alpha\beta}$ holds (Section 10).

In case A is the generator of a bounded semi-group T_t , the value of the fractional power $A_*^\alpha x$ may be related to the Riemann-Liouville integral of $T_t x$. Let C^σ , $\sigma \geq 0$, be the set of elements x such that $T_t x$ is $n = [\sigma]$ times continuously differentiable and the n -th derivative is uniformly Hölder continuous with exponent $\rho = \sigma - n$. Then we have $D^\sigma \supset C^\sigma \supset D(A_+^\sigma)$ for $\operatorname{Re} \alpha = \sigma$. If $\sigma > \operatorname{Re} \alpha > 0$, $A_*^\alpha x$, $x \in D^\sigma$, is expressed as the fractional derivative of $T_t x$ at 0. If X is sequentially weakly complete, it follows that $x \in D(A_+^\alpha)$ if and only if $\langle T_s x, y' \rangle \in D((-d/ds)_+^\alpha)$ for all $y' \in X'$. As in the case of D^σ , A_+^α maps $C^\sigma \cap D(A_+^\alpha)$

into $C^{\sigma - \text{Re } \alpha}$ if $\sigma - \text{Re } \alpha$ is positive and not an integer. The semi-group $\exp(-tA_+^\alpha)$ generated by $-A_+^\alpha$, $0 < \alpha < 1$, is uniformly bounded in $t \geq 0$ and converges strongly to T_t as α tends to 1 (Section 11).

Let $-A$ be the generator of a bounded semi-group T_t . Then A is of type $(\omega, M(\theta))$ with an $\omega < \pi/2$, or T_t is analytic, if and only if there exists a constant C such that

$$\|A_+^\alpha T_t\| \leq Ct^{-\text{Re } \alpha}.$$

If this is the case, and if σ is not an integer, it follows that $C^\sigma = D^\sigma$, and that $x \in C^\sigma$ if and only if $\|A_+^\alpha T_t x\| = O(t^{\sigma - \text{Re } \alpha})$ as $t \rightarrow 0$ for $\text{Re } \alpha > \sigma$ (Section 12).

Section 13 deals with the cases where A is a normal operator in a Hilbert space, a restriction, an adjoint operator or a continuous image of another operator which satisfies (1.5). As an application we will prove that the convolution of a C_0^α function and a C_0^β function belongs to $C_0^{\alpha + \beta - \varepsilon}$ for any $\varepsilon > 0$, where C_0^α denotes the class of uniformly Hölder continuous functions with exponent α and with compact supports in $(-\infty, \infty)$. Another application to a nonlinear partial differential equation will also be given. The rest of the section is concerned with the fractional differentiation of functions on the unit circle.

We collect several counterexamples in Section 14, among which there is a simple example of an operator of type $(\pi/2, 2 \sec \theta)$ which does not generate any semi-group of type (A).

The author failed to give an example of operator A such that $-A$ generates a bounded semi-group but for which $D^\sigma \neq C^\sigma$.

2. Regularity of $(\lambda + A)^{-1}x$ at infinity. If $x \in D(A^n)$ for a non-negative integer n , $(\lambda + A)^{-1}x$ has the expression

$$(2.1) \quad (\lambda + A)^{-1}x = \lambda^{-1}x - \lambda^{-2}Ax + \dots + (-1)^{n-1}\lambda^{-n}A^{n-1}x + R_{n-1}(\lambda, x)$$

for $\lambda \in \rho(-A)$, $\lambda \neq 0$, where

$$(2.2) \quad R_k(\lambda, x) = (-1)^{k+1}\lambda^{-k-1}A(\lambda + A)^{-1}A^kx = (-1)^{k+1}\lambda^{-k}(\lambda^{-1} - (\lambda + A)^{-1})A^kx.$$

Because of (1.5) we have the order estimate $R_{n-1}(\lambda, x) = O(\lambda^{-n-1})$ as λ tends to infinity along the real axis. If X is reflexive, the converse is also true.

THEOREM 2.1. *In order that there exist $y_k \in X$, $k = 0, 1, \dots, n$, such that*

$$(2.3) \quad (\lambda + A)^{-1}x = \lambda^{-1}y_0 - \lambda^{-2}y_1 + \dots + (-1)^n\lambda^{-n-1}y_n + o(\lambda^{-n-1})$$

as $\lambda \rightarrow \infty$, it is necessary and sufficient that $x \in D(A^n)$, $A^n x \in \overline{D(A)}$ and $y_k = A^k x$, $k = 0, 1, \dots, n$. Here the convergence can be either in the weak or strong sense and λ may increase, taking only values of a sequence $\lambda_j \rightarrow \infty$.

Moreover, let X be reflexive. Then it follows that if $(\lambda + A)^{-1}x$ satisfies

$$(2.4) \quad (\lambda + A)^{-1}x = \lambda^{-1}y_0 - \lambda^{-2}y_1 + \dots + (-1)^{n-1}\lambda^{-n}y_{n-1} + O(\lambda^{-n-1})$$

as $\lambda = \lambda_j \rightarrow \infty$, there exists a $y_n \in X$ such that (2.3) holds. Thus (2.4) is a necessary and sufficient condition in order that $x \in D(A^n)$. In particular we have $\overline{D(A)} = X$, since (2.4) for $n = 0$ holds for all $x \in X$.

Proof. In case $n = 0$ this is precisely an ergodic theorem due to Hille [6], Kato [9] and Yosida [23]. We shall, however, reproduce the proof for the sake of convenience. If $x \in D(A)$, then we have by (2.1) $(\lambda + A)^{-1}x - \lambda^{-1}x = O(\lambda^{-2}) = o(\lambda^{-1})$. Since $\lambda((\lambda + A)^{-1} - \lambda^{-1})$ is uniformly bounded by assumption (1.5), it follows from the Banach-Steinhaus theorem that $\lambda((\lambda + A)^{-1} - \lambda^{-1})x \rightarrow 0$ strongly for every $x \in \overline{D(A)}$.

Conversely, let the set $\{\lambda_j(\lambda_j + A)^{-1}x\}$ have an accumulation point y_0 relative to the weak topology, where $\lambda_j \rightarrow \infty$. We note that this is the case if $\lambda_j(\lambda_j + A)^{-1}x$ converges weakly to y_0 , or if X is reflexive. Clearly $y_0 \in \overline{D(A)}$, so that we have $\lambda_j(\lambda_j + A)^{-1}y_0 \rightarrow y_0$. Thus 0 is a weak accumulation point of $\{\lambda_j(\lambda_j + A)^{-1}(x - y_0)\}$. On the other hand we have for any $\mu \in \rho(-A)$

$$\begin{aligned} & (\mu + A)^{-1}\lambda_j(\lambda_j + A)^{-1}(x - y_0) \\ &= \lambda_j(\lambda_j + A)^{-1}(\mu + A)^{-1}(x - y_0) \rightarrow (\mu + A)^{-1}(x - y_0). \end{aligned}$$

Hence $(\mu + A)^{-1}(x - y_0)$ is the only accumulation point of $\{(\mu + A)^{-1}\lambda_j(\lambda_j + A)^{-1}(x - y_0)\}$. Since $(\mu + A)^{-1}$ is weakly continuous, this implies $(\mu + A)^{-1}(x - y_0) = 0$. Thus we have $x = y_0$.

Next let $n > 0$, and assume that the theorem has been proved for smaller n .

If $x \in D(A^n)$ and $A^n x \in \overline{D(A)}$, then in (2.1) $\lambda^{n+1}R_{n-1}(\lambda, x) = (-1)^n \lambda(\lambda + A)^{-1}A^n x$ converges strongly to $(-1)^n A^n x$ as $\lambda \rightarrow \infty$. This proves (2.3).

Conversely let (2.4) hold. Then by the induction hypotheses we have $x \in D(A^{n-1})$, $A^{n-1}x \in \overline{D(A)}$ and $y_k = A^k x$ for $k \leq n-1$. Thus the remainder term in (2.4) is equal to $(-1)^n \lambda^{-n}(1 - \lambda(\lambda + A)^{-1})A^{n-1}x = R_{n-1}(\lambda, x)$. If X is reflexive or if (2.3) holds when $\lambda_j \rightarrow \infty$, $\{\lambda_j^{n+1}R_{n-1}(\lambda_j, x)\}$ has a weak accumulation point $(-1)^n y_n$. Since $A^{n-1}x \in \overline{D(A)}$ and $(\lambda + A)^{-1}$

maps $\overline{D(A)}$ into $\overline{D(A)}$, y_n belongs to $\overline{D(A)}$. We have $(-1)^n \lambda_j^{n+1} R_{n-1}(\lambda_j, x) = A \lambda_j (\lambda_j + A)^{-1} A^{n-1} x$, and $\lambda_j (\lambda_j + A)^{-1} A^{n-1} x \rightarrow A^{n-1} x$. Therefore the pair $(A^{n-1} x, y_n)$ belongs to the graph of A . This proves $A^{n-1} x \in D(A)$ and $y_n = A^n x$.

DEFINITION OF THE SPACE D^σ . Let $\sigma \geq 0$ and $\sigma = n + \rho$ where $n = 0, 1, 2, \dots$ and $0 \leq \rho < 1$. σ, n and ρ will have the same meanings in the following unless the contrary is explicitly stated. We define D^σ as the set of all $x \in X$ such that $x \in D(A^n)$ and the remainder $R_n(\lambda, x)$ in

$$(2.5) \quad (\lambda + A)^{-1} x = \lambda^{-1} x - \lambda^{-2} A x + \dots + (-1)^n \lambda^{-n-1} A^n x + R_n(\lambda, x)$$

satisfies the order estimate

$$(2.6) \quad R_n(\lambda, x) = \begin{cases} o(\lambda^{-\sigma-1}) & \text{in case } \rho = 0 \\ O(\lambda^{-\sigma-1}) & \text{in case } \rho > 0. \end{cases}$$

Theorem 2.1 shows that $x \in D^\sigma$ if and only if $x \in D(A^n)$ and $A^n x \in \overline{D(A)}$.

Clearly D^σ is a vector space. We introduce in it the following topology: By definition a net (directed sequence) $x_\nu \in D^\sigma$ converges to $x \in D^\sigma$ if (i) $A^k x_\nu \rightarrow A^k x$ strongly for $k = 0, \dots, n$, and in case $\rho > 0$ if (ii)

$$(2.7) \quad \int_1^\infty \langle \lambda^{\sigma+1} R_n(\lambda, x_\nu), y' \rangle dm(\lambda) \rightarrow \int_1^\infty \langle \lambda^{\sigma+1} R_n(\lambda, x), y' \rangle dm(\lambda)$$

for every $y' \in X'$ and every measure m on $[1, \infty)$ of finite variation.

When we consider only a (countable) sequence x_j , the requirement (ii) is simplified as follows.

PROPOSITION 2.2. A sequence $x_j \in D^\sigma$ converges to $x \in D^\sigma$ if and only if (i) $A^k x_j \rightarrow A^k x$ for $k = 0, \dots, n$ and (ii) $\lambda^{\sigma+1} R_n(\lambda, x_j)$ is uniformly bounded in $\lambda \geq 1$ and j .

Proof. If (i) holds, then $R_n(\lambda, x_j)$ converges strongly to $R_n(\lambda, x)$ for every λ . Thus the sufficiency easily follows from Lebesgue's theorem. To prove the necessity let $x_j \rightarrow x_0$ in D^σ and $\{\lambda^{\sigma+1} R_n(\lambda, x_j)\}$ be unbounded. Then there is a sequence $\lambda_k \geq 1$ and a $y' \in X'$ such that $\{\langle \lambda_k^{\sigma+1} R_n(\lambda_k, x_j), y' \rangle\}$ is unbounded. On the other hand it follows from the hypothesis that the sequence $\xi_j = (\langle \lambda_k^{\sigma+1} R_n(\lambda_k, x_j), y' \rangle)$ in the space m of bounded sequences in k converges to ξ_0 in the weak* topology on m induced by l^1 . Hence $\{\xi_j\}$ is bounded in m contrary to the above. This completes the proof.

Since A is closed we see easily that D^σ is a Banach space with

the norm $\|x\| + \|Ax\| + \dots + \|A^n x\|$. When $\rho > 0$, D^σ is a locally convex space but it is not generally metrizable. There is, however, a semi-norm in D^σ which will be used as a substitute for a norm.

Set for every $x \in D^\sigma$

$$(2.8) \quad q_\sigma(x) = \sup_{0 < \lambda < \infty} \lambda^{\sigma+1} \|R_n(\lambda, x)\|.$$

This defines a semi-norm in D^σ , because by the definition $\sup_{\lambda \geq 1} \lambda^{\sigma+1} \|R_n(\lambda, x)\|$ is finite, and if $0 < \lambda < 1$, we have

$$\lambda^{\sigma+1} \|R_n(\lambda, x)\| = \lambda^{\sigma-n} \|A(\lambda + A)^{-1} A^n x\| \leq L \|A^n x\|,$$

proving that $q_\sigma(x)$ is finite for every $x \in D^\sigma$. The semi-norm property is clear.

THEOREM 2.3. *There exists a constant $C(\sigma)$ depending only on σ , M and L such that*

$$(2.9) \quad \|A^k x\| \leq C(\sigma) q_\sigma(x)^{k/\sigma} \|x\|^{(\sigma-k)/\sigma}$$

for $x \in D^\sigma$ and $k = 0, 1, \dots, n$. A set B in D^σ is bounded if and only if $\sup_{x \in B} \|x\|$ and $\sup_{x \in B} q_\sigma(x)$ are finite.

Proof. First we will prove, by induction on n , that

$$(2.10) \quad \|A^n x\| \leq (2(ML)^{1/2})^n \|A^{n+1} x\|^{n/(n+1)} \|x\|^{1/(n+1)}.$$

If $x \in D(A^2)$, we have

$$Ax = (\lambda + A)^{-1} A^2 x + \lambda A(\lambda + A)^{-1} x,$$

so that

$$\|Ax\| \leq \lambda^{-1} M \|A^2 x\| + \lambda L \|x\|$$

for any $\lambda > 0$. Taking the minimum of the right-hand side, we obtain (2.10) for $n = 1$. Let (2.10) have been proved for n . Then the proof for $n + 1$ is obtained by combining (2.10) for n with

$$\|A^{n+1} x\| \leq 2(ML) \|A^{n+2} x\| \|A^n x\|^{1/2}.$$

Similarly from

$$A^n x = A(\lambda + A)^{-1} A^n x + \lambda A(\lambda + A)^{-1} A^{n-1} x$$

and (2.8) we get

$$\|A^n x\| \leq (1 + \rho) \rho^{-\rho/(1+\rho)} q_\sigma(x)^{1/(1+\rho)} (L \|A^{n-1} x\|)^{\rho/(1+\rho)}.$$

Thus the combination with (2.10) for $n - 1$ gives

$$\| A^n x \| \leq C(\rho) q_\sigma(x)^{n/\sigma} \| x \|^{n/\sigma} ,$$

where $C(\rho) = [(1 + \rho)^{1+\rho} \rho^{-\rho} (2(ML)^{1/2})^{(n-1)\rho} L^\rho]^{n/\sigma}$. Now we can prove (2.9) for smaller k step by step using the inequality (2.10).

To prove the second part let B be a bounded set in D^σ . Then in particular $\sup \| x \|$ and $\sup \| A^n x \|$ are finite. By the same method as in the proof of Proposition 2.2 we can easily show that

$$\sup_{\lambda \geq 1} \lambda^{\sigma+1} \| R_n(\lambda, x) \|$$

is finite. Since $\sup_{0 < \lambda < 1} \lambda^{\sigma+1} \| R_n(\lambda, x) \| \leq L \| A^n x \|$, we conclude that $\sup q_\sigma(x)$ is finite.

Conversely let $\sup_{x \in B} \| x \|$ and $\sup_{x \in B} q_\sigma(x)$ be finite. Then by (2.9) $\sup \| A^k x \|$ is finite for $k = 0, 1, \dots, n$. Furthermore,

$$\sup_{x \in B} \left| \int_1^\infty \langle \lambda^{\sigma+1} R_n(\lambda, x), y' \rangle dm \right| \leq \sup_{x \in B} q_\sigma(x) \| y' \| |m|$$

is finite. Thus B is bounded in D^σ . The proof is completed.

The space D^σ is quasi-complete. Although we do not use this fact, we briefly sketch its proof. Let x_ν be a bounded Cauchy net in D^σ . Then x_ν converges to an x in D^n relative to the topology of D^n . Since X is metrizable and since the mapping which maps $A^n x_\nu \in X$ into $\langle \lambda^{\sigma+1} R_n(\lambda, x_\nu), y' \rangle \in C^{[1, \infty)}$ is continuous, Lebesgue's theorem proves that

$$\int_1^\infty \langle \lambda^{\sigma+1} R_n(\lambda, x_\nu), y' \rangle dm(\lambda) \rightarrow \int_1^\infty \langle \lambda^{\sigma+1} R_n(\lambda, x), y' \rangle dm(\lambda) .$$

We have introduced a sophisticated topology in D^σ . The reason will be explained in the following lemma and theorem.

LEMMA 2.4. *If $\mu > 0$, then $\mu(\mu + A)^{-1}$ is a one-to-one continuous mapping from D^σ onto $D^{\sigma+1}$. Moreover, for every $x \in D^\sigma$*

$$(2.11) \quad \mu(\mu + A)^{-1} x \rightarrow x (D^\sigma) \quad \text{as} \quad \mu \rightarrow \infty .$$

Proof. Let $x^\mu = \mu(\mu + A)^{-1} x$ for $x \in D^\sigma$. Then we have

$$(\lambda + A)^{-1} x^\mu = \lambda^{-1} y_0^\mu - \lambda^{-2} y_1^\mu + \dots + (-1)^n \lambda^{-n-1} y_n^\mu + R_n(\lambda, x^\mu) ,$$

where

$$y_k^\mu = A^k x^\mu = \mu(\mu + A)^{-1} A^k x$$

and

$$(2.12) \quad R_n(\lambda, x^\mu) = \mu(\mu + A)^{-1} R_n(\lambda, x) \\ = (-1)^{n+1} \lambda^{-n-2} A^{n+1} x^\mu - \mu A(\mu + A)^{-1} \lambda^{-1} R_n(\lambda, x) .$$

Here we have used the fact that $\mu(\mu + A)^{-1}$ maps $D(A^n)$ into $D(A^{n+1})$. The last equality shows that $x^\mu \in D^{\sigma+1}$, and the first equality in (2.12) with (1.6) proves that $\lambda^{\sigma+1}R_n(\lambda, x^\mu)$ is uniformly bounded. By definition $A^k x \in \overline{D(A)}$ for $k = 0, 1, \dots, n$, so that $A^k x^\mu$ converges to $A^k x$, $\mu \rightarrow \infty$. Hence it follows from Proposition 2.2 that x^μ converges to x in the topology of D^σ . We note, however, that x^μ does not necessarily converge to x in the norm $\|x\| + q_\sigma(x)$.

The operator $\mu(\mu + A)^{-1}$ is, of course, one-to-one, and, as is clearly seen from (2.12), maps D^σ continuously into $D^{\sigma+1}$. If $z \in D^{\sigma+1}$, it is easily shown that $\mu^{-1}(\mu + A)z \in D^\sigma$. Thus $\mu(\mu + A)^{-1}$ is a mapping onto $D^{\sigma+1}$.

THEOREM 2.5. *If $\sigma' > \sigma$, then $D^{\sigma'} \subset D^\sigma$, $D^{\sigma'}$ is dense in D^σ , and the embedding mapping $\varphi_{\sigma'}^{\sigma'} : D^{\sigma'} \rightarrow D^\sigma$ is continuous.*

Proof. The fact that $D^{\sigma'} \subset D^\sigma$ and the continuity of $\varphi_{\sigma'}^{\sigma'}$ are obvious from the definition of D^σ and its topology. The above lemma proves that $D^{\sigma+1}$ is dense in D^σ , so that $D^{\sigma+m}$ is dense in D^σ for every nonnegative integer m . For a given $\sigma' > \sigma$, take an m such that $\sigma + m > \sigma'$. Since $D^{\sigma'} \supset D^{\sigma+m}$, $D^{\sigma'}$ is dense in D^σ . This completes the proof. $D^{\sigma'}$ is not necessarily dense in D^σ with the norm topology defined by $\|x\| + q_\sigma(x)$.

We can introduce a topology in $D(A^n)$ by the norm

$$\|x\|_{D(A^n)} = \|x\| + \dots + \|A^n x\|.$$

(Note that this norm is equivalent to $\|x\| + \|A^n x\|$ because of Theorem 2.3.) From the closedness of A it easily follows that $D(A^n)$ forms a Banach space with this norm. $D(A^n)$ contains D^n as a closed subspace. If X is reflexive, they are identical. But this is not necessarily the case otherwise. Moreover, it is easily shown that $D(A^n) \subset D^\sigma$, for all $\sigma < n$, that the embedding is continuous, and that it has a dense range. Thus we have the following inclusion relation:

$$X \supset \overline{D(A)} = D^0 \supset D^0 \supset D(A) \supset D^1 \supset D^{1+\rho} \supset \dots$$

In case $D(A)$ is not dense in X , the spaces D^σ for A are virtually determined by a restriction of A .

THEOREM 2.6. *Let A_D be the restriction of A to the domain $D(A_D) = \{x \in D(A); Ax \in \overline{D(A)}\}$. Then A_D , considered as a closed operator in $\overline{D(A)}$, satisfies (1.5) and has a dense domain. Moreover, the spaces $D^\sigma(A_D)$ corresponding to A_D are the same as $D^\sigma(A)$ corresponding to A . In particular we have $D^n(A) = D(A_D^n)$ for every integer $n \geq 0$.*

Proof. It is easily shown that A_D has a resolvent $(\lambda + A_D)^{-1}$ which

is equal to the restriction $(\lambda + A)^{-1}|_{\overline{D(A)}}$ of $(\lambda + A)^{-1}$ to $\overline{D(A)}$ for $\lambda \in \rho(-A)$. Thus $(\lambda + A_D)^{-1}$ satisfies (1.5). From Theorem 2.1 it follows that for every $x \in \overline{D(A)}$, $\lambda(\lambda + A_D)^{-1}x \rightarrow x$ as $\lambda \rightarrow \infty$. This shows that $D(A_D)$ is dense in $\overline{D(A)}$. The identity of the spaces $D^\sigma(A)$ and $D^\sigma(A_D)$ follows easily from Theorem 2.1.

3. Regularity of $(\lambda + A)^{-1}x$ at zero. A discussion parallel to Section 2 is possible concerning the regularity of $(\lambda + A)^{-1}x$ at zero. Suppose that x belongs to the range $R(A^n)$ and $x = A^n y$. Then we have

$$(3.1) \quad (\lambda + A)^{-1}x = A^{n-1}y - \lambda A^{n-2}y + \dots + (-1)^{n-1} \lambda^{n-1} y + (-1)^n \lambda^n (\lambda + A)^{-1}y$$

for any $\lambda \in \rho(-A)$. Hence we have the asymptotic relation

$$(3.2) \quad (\lambda + A)^{-1}x = y_{-1} - \lambda y_{-2} + \dots + (-1)^{n-2} \lambda^{n-2} y_{-n+1} + O(\lambda^{n-1})$$

as $\lambda \rightarrow 0$, where $y_{-k} = A^{n-k}y$. The counterpart of Theorem 2.1 is the following theorem.

THEOREM 3.1. *In order that $(\lambda + A)^{-1}x$ satisfy*

$$(3.3) \quad (\lambda + A)^{-1}x = y_{-1} - \lambda y_{-2} + \dots + (-1)^{n-1} \lambda^{n-1} y_{-n} + o(\lambda^{n-1})$$

as $\lambda \rightarrow 0$ with $y_{-k} \in X$, it is necessary and sufficient that there exists a $y_{-n} \in D(A^n) \cap \overline{R(A)}$ such that $x = A^n y_{-n}$ and $y_{-k} = A^{n-k} y_{-n}$. The remainder term is then given by

$$(3.4) \quad R_{-n}(\lambda, x) = (-1)^n \lambda^n (\lambda + A)^{-1} y_{-n}.$$

For the necessity it is sufficient that (3.3) holds for a sequence $\lambda_j \rightarrow 0$ relative to the weak topology.

Furthermore, let X be reflexive. Then for every $x \in X$ there is an $x_h \in N(A) = \{y; Ay = 0\}$ such that

$$(3.5) \quad (\lambda + A)^{-1}x = \lambda^{-1}x_h + o(\lambda^{-1}), \quad \text{as } \lambda \rightarrow 0,$$

and $x - x_h \in \overline{R(A)}$. The expression $x = x_h + (x - x_h)$ with $x_h \in N(A)$ and $x - x_h \in \overline{R(A)}$ is unique, so that X is decomposed directly:

$$(3.6) \quad X = N(A) + \overline{R(A)} \quad (\text{direct}).$$

In case X is reflexive and $n > 0$ it follows that if (3.2) is satisfied as λ tends to 0 taking values of a sequence $\lambda_j \rightarrow 0$, then there exists y_{-n} such that (3.3) holds. Thus (3.2) holds if and only if $x \in R(A^n)$.

Proof. We prove the theorem by induction on n . Let $x \in R(A)$

and $x = Ay$. Then

$$\lambda(\lambda + A)^{-1}x = \lambda A(\lambda + A)^{-1}y \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Thus by the Banach-Steinhaus theorem we have $\lambda(\lambda + A)^{-1}x \rightarrow 0$ for every $x \in \overline{R(A)}$. Conversely let $\lambda_j(\lambda_j + A)^{-1}x$ converge weakly to 0 as $\lambda_j \rightarrow 0$. Then using $x = \lambda_j(\lambda_j + A)^{-1}x + A(\lambda_j + A)^{-1}x$ we see that x is the limit point of $A(\lambda_j + A)^{-1}x \in R(A)$.

In case X is reflexive, choose a sequence $\lambda_j \rightarrow 0$ and let x_k be a weak accumulation point of $\{\lambda_j(\lambda_j + A)^{-1}x\}$. Since $A\lambda_j(\lambda_j + A)^{-1}x \rightarrow 0$ as $\lambda_j \rightarrow 0$ and A is closed, it follows that $x_k \in D(A)$ and $Ax_k = 0$. 0 is then a weak accumulation point of $\{\lambda_j(\lambda_j + A)^{-1}(x - x_k)\} = \{\lambda_j(\lambda_j + A)^{-1}x - x_k\}$. Hence we obtain $x - x_k \in \overline{R(A)}$ as above. Clearly the decomposition is unique.

Let $n > 0$. If $y_{-n} \in D(A^n) \cap \overline{R(A)}$ and $x = A^n y_{-n}$, then (3.3) follows from (3.1) and the theorem for $n = 0$. Conversely let (3.2) be true. Then by the induction hypothesis it follows that $y_{-n+1} \in D(A^{n-1}) \cap \overline{R(A)}$ and $y_{-k} = A^{n-k-1}y_{-n+1}$. The remainder in (3.2) is equal to $R_{-n+1}(\lambda, x) = (-1)^{n-1}\lambda^{n-1}(\lambda + A)^{-1}y_{-n+1}$. Now let $(-1)^{n-1}\lambda_j^{n-1}R_{-n+1}(\lambda_j, x)$ converge weakly to y_{-n} as $\lambda_j \rightarrow 0$. Since A is closed and

$$A(\lambda_j + A)^{-1}y_{-n+1} = y_{-n+1} - \lambda_j(\lambda_j + A)^{-1}y_{-n+1} \rightarrow y_{-n+1},$$

it follows that $y_{-n} \in D(A)$ and $y_{-n+1} = Ay_{-n}$. Noticing that $(\lambda + A)^{-1}$ maps $\overline{R(A)}$ into $\overline{R(A)}$ we have also $y_{-n} \in \overline{R(A)}$.

DEFINITION OF THE SPACE R^σ . Let σ, n and ρ be as in the previous section. The space R^σ is by definition the set of all $x \in R(A^n) \cap \overline{R(A)}$ such that the remainder $R_{-n}(\lambda, x)$ in (3.3) satisfies

$$(3.7) \quad R_{-n}(\lambda, x) = \begin{cases} o(\lambda^{\sigma-1}) & \text{in case } \rho = 0 \\ O(\lambda^{\sigma-1}) & \text{in case } \rho > 0. \end{cases}$$

Note that $x \in R^n$ if and only if there is a $y_{-n} \in D(A^n) \cap \overline{R(A)}$ such that $x = A^n y_{-n}$.

We introduce a locally convex topology in R^σ in the same way as in D^σ by the following family of semi-norms:

- (i) $\|y_{-k}\|$ for $k = 1, \dots, n$,
- (ii) $\left| \int_0^1 \langle \lambda^{-\sigma+1} R_{-n}(\lambda, x), y' \rangle dm(\lambda) \right|$

for all $y' \in X'$ and all measure m of finite variation on $(0, 1]$.

The following theorem which is analogous to Theorem 2.5 permits us to reduce most of the discussion about R^σ to the case of D^σ .

THEOREM 3.2. *Let A_R be the restriction of A to $D(A_R) = D(A) \cap \overline{R(A)}$.*

Then A_R satisfies the assumption (1.5) in $R(A)$, is one-to-one and has a dense range there. The spaces $R^\sigma(A)$ corresponding to A are the same as $R^\sigma(A_R)$ corresponding to A_R . y_{-k} in (3.3) is precisely $A_R^{-k}x$ and the remainder is given by

$$R_{-n}(\lambda, x) = (-1)^n \lambda^n (\lambda + A_R)^{-1} A_R^{-n} x .$$

Furthermore, the inverse A_R^{-1} has the resolvent

$$(3.8) \quad (\lambda + A_R^{-1})^{-1} = \lambda^{-1} A (\lambda^{-1} + A)^{-1} |_{\overline{R(A)}}$$

which satisfies (1.5); and $R^\sigma(A)$ is the same as and is isomorphic to $D^\sigma(A_R^{-1})$.

Proof. First we prove that A_R is one-to-one. This is a consequence of Theorem 3.1. In fact, if $x \in D(A_R)$ and $A_R x = 0$, then

$$x = \lambda(\lambda + A)^{-1}x + (\lambda + A)^{-1}A_R x \rightarrow 0 \quad \text{as } \lambda \rightarrow 0 ,$$

so that x must be 0.

The range $R(A_R)$ is dense in $\overline{R(A)}$ because if $x \in \overline{R(A)}$, then, by Theorem 3.1, $A(\lambda + A)^{-1}x \rightarrow x$ as $\lambda \rightarrow 0$ and $(\lambda + A)^{-1}$ maps $\overline{R(A)}$ into $\overline{R(A)}$. As is easily shown, $(\lambda + A_R)^{-1}$ coincides with the restriction of $(\lambda + A)^{-1}$ to $\overline{R(A)}$:

$$(3.9) \quad (\lambda + A_R)^{-1} = (\lambda + A)^{-1} |_{\overline{R(A)}} .$$

Thus A_R satisfies (1.5).

Now it is easy to see that $R^\sigma(A)$ coincides with $R^\sigma(A_R)$. In particular we have $R^n = R(A_R^n)$ for every integer $n \geq 0$.

Since $\lambda + A_R^{-1} = (\lambda A_R + I)A_R^{-1}$, and since A_R^{-1} and $\lambda A_R + I$, $\lambda^{-1} \in \rho(-A)$, are one-to-one, (3.8) follows from (3.9).

It remains to prove that $D^\sigma(A_R^{-1}) = R^\sigma(A)$. If σ is an integer n , then we have $D^n(A_R^{-1}) = D(A_R^{-n}) = R(A_R^n) = R^n(A)$. Further, it follows from (3.8) that

$$(3.10) \quad \lambda R_n(A_R^{-1}, \lambda, x) = -\lambda^{-1} R_{-n}(A_R, \lambda^{-1}, x)$$

for every $x \in D^n(A_R^{-1}) = R^n(A_R)$. Now it readily follows that $D^\sigma(A_R^{-1}) = R^\sigma(A_R) = R^\sigma(A)$ and that the topologies of the two spaces are identical. This completes the proof.

The results in § 2 may be translated in terms of R^σ . We just state two of them.

LEMMA 3.3. *If $\mu > 0$, then $A(\mu + A)^{-1}$ is a one-to-one continuous mapping from R^σ onto $R^{\sigma+1}$. We have also*

$$A(\mu + A)^{-1}x \rightarrow x \quad (R^\sigma) \quad \text{as } \mu \rightarrow 0$$

for every $x \in R^\sigma$.

THEOREM 3.4. *If $\sigma' > \sigma$, then $R^{\sigma'} \subset R^\sigma$, $R^{\sigma'}$ is dense in R^σ and the embedding map $\psi_{\sigma'}^{\sigma'} : R^{\sigma'} \rightarrow R^\sigma$ is continuous.*

Before leaving R^σ , let us investigate the relation between R^σ and $R(A^n)$. As we have shown above, $R^n = R(A^n)$. Therefore, $R^n \subset R(A^n)$. If X is reflexive, they coincide with each other because of (3.6). We define a norm in $R(A^n)$ by

$$\|x\|_{R(A^n)} = \inf_{A^n y = x} \|y\|_{D(A^n)}.$$

Since the set $\{y; A^n y = 0\}$ is closed in $D(A^n)$, $R(A^n)$ becomes a Banach space with this norm. The topology of R^n is defined by the same norm, so that R^n is a closed subspace of $R(A^n)$. If $\sigma < n$, we have $R^\sigma \supset R(A^n)$. Therefore, there is the following inclusion relation:

$$\overline{R(A)} = R^0 \supset R^\sigma \supset R(A) \supset R^1 \supset R^{1+\sigma} \supset \dots.$$

THE SPACE $D^\sigma \cap R^\tau$. Let σ and $\tau \geq 0$. We introduce in the set $D^\sigma \cap R^\tau$ the weakest topology that makes both the embedding mappings into D^σ and into R^τ continuous. Since D^σ and R^τ are quasi-complete, it follows that $D^\sigma \cap R^\tau$ is quasi-complete.

LEMMA 3.5. *Let $\mu > 0$. Then $\mu(\mu + A)^{-1}(A(\mu + A)^{-1})$ is a one-to-one continuous mapping from $D^\sigma \cap R^\tau$ onto $D^{\sigma+1} \cap R^\tau$ ($D^\sigma \cap R^{\tau+1}$, respectively). We have also*

$$\begin{aligned} \mu(\mu + A)^{-1}x &\rightarrow x \quad (D^\sigma \cap R^\tau) \quad \text{as } \mu \rightarrow \infty \\ (A(\mu + A)^{-1})x &\rightarrow x \quad (D^\sigma \cap R^\tau) \quad \text{as } \mu \rightarrow 0 \end{aligned}$$

for every $x \in D^\sigma \cap R^\tau$.

Proof. It is enough to prove that $\mu(\mu + A)^{-1}x \in R^\tau$, that

$$\mu(\mu + A)^{-1}x \rightarrow x \quad (R^\tau) \quad \text{as } \mu \rightarrow \infty,$$

and the corresponding facts for $A(\mu + A)^{-1}$ and D^σ . If $(\lambda + A)^{-1}x$ satisfies (3.3) with the remainder $R_{-n}(\lambda, x) = (-1)^n \lambda^n (\lambda + A)^{-1} y_{-n} = O(\lambda^{\sigma-1})$, then $x^\mu = \mu(\mu + A)^{-1}x$ satisfies

$$(\lambda + A)^{-1}x^\mu = y_{-1}^\mu - \lambda \mu y_{-2}^\mu + \dots + (-1)^{n-1} \lambda^{n-1} \mu^{n-1} y_{-n}^\mu + R_{-n}(\lambda, x^\mu),$$

where $y_{-k}^\mu = \mu(\mu + A)^{-1}y_{-k}$ and $R_{-n}(\lambda, x^\mu) = \mu(\mu + A)^{-1}R_{-n}(\lambda, x) = (-1)^n \lambda^n (\lambda + A)^{-1} y_{-n}^\mu$. Thus x^μ belongs to R^τ . Since $y_{-k} \in D(A^k) \subset \overline{D(A)}$, it follows from Lemma 2.4 that $y_{-k}^\mu \rightarrow y_{-k}$ as $\mu \rightarrow \infty$, which proves

$x^\mu \rightarrow x (R^\tau)$. The statements about $A(\mu + A)^{-1}$ are proved similarly. Hence it follows that:

THEOREM 3.6. *If $\sigma' \geq \sigma$ and $\tau' \geq \tau$, then $D^{\sigma'} \cap R^{\tau'}$ is densely contained in $D^\sigma \cap R^\tau$, and the embedding map is continuous.*

THEOREM 3.7. *Let $A_B = A_D \cap A_R$, or the restriction of A to $D(A_B) = \{x \in D(A) \cap \overline{R(A)}, Ax \in \overline{D(A)}\}$. Then A_B is one-to-one, the domain $D(A_B)$ and the range $R(A_B)$ are both dense in $\overline{D(A)} \cap \overline{R(A)}$, and A_B and A_B^{-1} both satisfy assumption (1.5) in $\overline{D(A)} \cap \overline{R(A)}$. Moreover, the spaces $D^\sigma \cap R^\tau$ corresponding to A remain unchanged if A is replaced by A_B .*

4. Definition of fractional powers of operators. In this section we give the definition of the fractional power A^α of an operator A which satisfies the assumption (1.5), separating α into three cases: (i) $\text{Re } \alpha > 0$, (ii) $\text{Re } \alpha < 0$, and (iii) $\text{Re } \alpha = 0$. Let us begin with

THE CASE $\text{Re } \alpha > 0$. First, we take a number $\sigma > \text{Re } \alpha$, and define $A_\sigma^\alpha x, x \in D^\sigma$, by

$$(4.1) \quad \begin{aligned} A_\sigma^\alpha x &= A^\alpha x \text{ if } \alpha \text{ is an integer and} \\ A_\sigma^\alpha x &= -\frac{\sin \pi \alpha}{\pi} \left[\int_0^N \lambda^\alpha (\lambda + A)^{-1} x d\lambda \right. \\ &\quad \left. + \int_N^\infty \lambda^\alpha R_n(\lambda, x) d\lambda + \sum_{k=0}^n (-1)^{k+1} \frac{N^{\alpha-k}}{\alpha - k} \right], \text{ otherwise.} \end{aligned}$$

Here N is an arbitrary fixed positive number and $\lambda^\alpha = \exp(\alpha \log \lambda)$. Clearly the integrals converge and the right-hand side represents a continuous linear operator from D^σ into $X = X_w$ equipped with the weak topology. By a simple computation it is seen that $A_\sigma^\alpha x$ does not depend on N and that if $\sigma < \sigma'$, then $A_\sigma^\alpha x = A_{\sigma'}^\alpha x$ for every $x \in D^{\sigma'}$.

If $n < \text{Re } \alpha < \sigma$, we may let $N \rightarrow 0$ and obtain

$$(4.2) \quad A_\sigma^\alpha x = (-1)^n \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^{\alpha-n-1} A(\lambda + A)^{-1} A^n x d\lambda .$$

This expression was used by Balakrishnan [2] to define A^α .

PROPOSITION 4.1. (Balakrishnan [2]). A_σ^α has a closed extension as an operator in X , and its smallest closed extension does not depend on $\sigma > \text{Re } \alpha$.

Proof. Since A and $(\lambda + A)^{-1}$ are commutative with $(\mu(\mu + A)^{-1})^l$,

$l = 1, 2, \dots, \mu > 0$, it is easily shown that

$$(4.3) \quad A_\sigma^\alpha (\mu(\mu + A)^{-1})^l x = (\mu(\mu + A)^{-1})^l A_\sigma^\alpha x, \quad x \in D^\sigma.$$

$(\mu(\mu + A)^{-1})^l$ maps X continuously onto $D(A^l)$ and, hence, into $D^{l-\varepsilon}$, $\varepsilon > 0$. Thus, the left-hand side of (4.3) represents a continuous linear operator in X , if $l > \sigma$.

In order to prove that A_σ^α has a closed extension, let $x_j \in D^\sigma$ be a sequence such that $x_j \rightarrow 0$ and $A_\sigma^\alpha x_j \rightarrow y$ in X . Replacing x in (4.3) by x_j , and taking the limits of both sides, we have

$$0 = (\mu(\mu + A)^{-1})^l y.$$

This proves $y = 0$.

Let $\sigma < \sigma'$. It follows from Theorem 2.5 and the continuity of A_σ^α that the closure of the graph $G(A_\sigma^\alpha) = \{(x, A_\sigma^\alpha x); x \in D^\sigma\}$ in $D^\sigma \times X_w$ is identical with $G(A_{\sigma'}^\alpha)$. Since the closure of $G(A_\sigma^\alpha)$ in $X_w \times X_w$ is equal to the closure of its closure in $D^\sigma \times X_w$, we have the statement. This completes the proof.

The smallest closed extension of A_σ^α , whose existence and uniqueness have been proved above, is defined to be the fractional power of A and denoted by A_+^α , or A^α if there will be no confusion.

PROPOSITION 4.2. The fractional power A_+^α coincides with the fractional power $A_{D^+}^\alpha$ of the operator A_D . Furthermore, if α is an integer n , then A_+^α is equal to A_D^n , the n times iteration of A_D . In particular we have $D(A_+^n) = D(A_D^n) = D^n$.

Proof. In view of Theorem 2.6 we see that A_σ^α is the same as $A_{D^\sigma}^\alpha$. Thus the former part follows.

Now let $x \in D^n$. For every $\mu > 0$ we have $\mu(\mu + A)^{-1}x \in D^{n+1} \subset D(A_+^n)$ and

$$\begin{aligned} A_+^n (\mu(\mu + A)^{-1})x &= A^n (\mu(\mu + A)^{-1})x \\ &= \mu(\mu + A)^{-1} A_D^n x. \end{aligned}$$

Here let μ tend to infinity. Then we obtain $\mu(\mu + A)^{-1}x \rightarrow x$ and $\mu(\mu + A)^{-1} A_D^n x \rightarrow A_D^n x$ by Lemma 2.4. Hence $x \in D(A_+^n)$ and $A_+^n x$ coincides with $A_D^n x$. Conversely if $x \in D^\sigma$ with $\sigma > n$, then by definition we have $A_+^n x = A^n x = A_D^n x$.

Since A_D has a nonvoid resolvent set in $\overline{D(A)}$, A_D^n is closed in $\overline{D(A)}$ (cf. Theorem 6.1 of Taylor [18]), and hence in X . Thus A_D^n is an extension of A_+^n .

PROPOSITION 4.3. The domain $D(A_+^\alpha)$ is contained and dense in $\overline{D(A)}$, while the range $R(A_+^\alpha)$ is contained in $\overline{D(A)} \cap \overline{R(A)}$.

Proof. Since D^σ , $\alpha > \text{Re } \alpha$, is contained and dense in $\overline{D(A)}$, the former part is clear. To prove the latter part it suffices to show $R(A_\sigma^\alpha) \subset \overline{D(A)} \cap \overline{R(A)}$ for a σ . As we have seen above A_σ^α is an operator in $\overline{D(A)}$. Thus the range is contained in $\overline{D(A)}$. We remark that $R_n(\lambda, x)$ and $A^k x$ for $k = 1, \dots, n$ are elements in $R(A)$. Now let N in (4.1) tend to zero. Then the first term and the term $N^\alpha x / \alpha$ tend to zero. Therefore, $A_\sigma^\alpha x$ is a limit point of elements in $\overline{R(A)}$.

Later we will prove that the range $R(A_+^\alpha)$ is dense in $\overline{D(A)} \cap \overline{R(A)}$.

PROPOSITION 4.4. A_+^α commutes with $(\mu(\mu + A)^{-1})^l$ and $(A(\mu + A)^{-1})^l$, whenever $\mu > 0$ and $l = 1, 2, \dots$. Namely, if $x \in D(A_+^\alpha)$, then $(\mu(\mu + A)^{-1})^l x$ and $(A(\mu + A)^{-1})^l x$ belong to $D(A_+^\alpha)$ and

$$(4.4) \quad A_+^\alpha (\mu(\mu + A)^{-1})^l x = (\mu(\mu + A)^{-1})^l A_+^\alpha x,$$

$$(4.5) \quad A_+^\alpha (A(\mu + A)^{-1})^l x = (A(\mu + A)^{-1})^l A_+^\alpha x.$$

Proof. (4.4) follows easily from (4.3). (4.5) is proved similarly.

PROPOSITION 4.5. If $x \in D(A_+^\alpha)$, then for every $l = 1, 2, \dots$

$$A_+^\alpha (\mu(\mu + A)^{-1})^l x \rightarrow A_+^\alpha x \quad \text{as } \mu \rightarrow \infty.$$

Conversely if $y_j = A_+^\alpha (\mu_j(\mu_j + A)^{-1})^l x$ exist for an $x \in \overline{D(A)}$, and if y_j converges weakly to an element $y \in X$ as $\mu_j \rightarrow \infty$, then $x \in D(A_+^\alpha)$ and $A_+^\alpha x = y$.

Especially let X be reflexive. Then x belongs to $D(A_+^\alpha)$ if and only if $\{A_+^\alpha (\mu_j(\mu_j + A)^{-1})^l x\}$ is bounded for an l and a sequence $\mu_j \rightarrow \infty$.

Proof. The first statement is an immediate consequence of Proposition 4.4. The second and third statements follow easily from the closedness of A_+^α .

Lemma A5 in Kato [11] holds without the reflexivity of X .

PROPOSITION 4.6. Let $n < \text{Re } \alpha < n + 1$ and $x \in D^n$. If

$$(4.6) \quad (-1)^n \frac{\sin \pi \alpha}{\pi} \int_0^{N_j} \lambda^{\alpha-n-1} A(\lambda + A)^{-1} A^n x d\lambda$$

converges weakly to a $y \in X$ as $N_j \rightarrow \infty$, then $x \in D(A_+^\alpha)$ and $A_+^\alpha x = y$.

In case X is reflexive, $x \in D(A_+^\alpha)$ with $n < \text{Re } \alpha < n + 1$ if (4.6) is uniformly bounded for a sequence $N_j \rightarrow \infty$.

Proof. Since the integrand belongs to $\overline{D(A)}$, we have $y \in \overline{D(A)}$. If $\mu > 0$, then $\mu(\mu + A)^{-1} x \in D^{n+1}$, so that $A_+^\alpha (\mu(\mu + A)^{-1}) x$ is defined

by (4.2). Therefore,

$$\begin{aligned} & A_+^\alpha(\mu(\mu + A)^{-1})x \\ &= (-1)^n \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-n-1} A(\lambda + A)^{-1} A^n \mu(\mu + A)^{-1} x d\lambda \\ &= w\text{-}\lim_{j \rightarrow \infty} (-1)^n \frac{\sin \pi\alpha}{\pi} \int_0^{N^j} \lambda^{\alpha-n-1} A(\lambda + A)^{-1} A^n \mu(\mu + A)^{-1} x d\lambda \\ &= \mu(\mu + A)^{-1} w\text{-}\lim_{j \rightarrow \infty} (-1)^n \frac{\sin \pi\alpha}{\pi} \int_0^{N^j} \lambda^{\alpha-n-1} A(\lambda + A)^{-1} A^n x d\lambda \\ &= \mu(\mu + A)^{-1} y . \end{aligned}$$

This shows, by Proposition 4.5, that $x \in D(A_+^\alpha)$ and $y = A_+^\alpha x$.

PROPOSITION 4.7. If $x \in D^\sigma$, then $A_+^\alpha x$ is an analytic function of α on the strip $0 < \text{Re } \alpha < \sigma$. Moreover,

$$\| A_+^\alpha x \| \leq C(\sigma, \alpha) q_\sigma(x)^{\text{Re } \alpha / \sigma} \| x \|^{(\sigma - \text{Re } \alpha) / \sigma}$$

holds for every $x \in D^\sigma$ with a constant $C(\sigma, \alpha)$ depending only on σ and α .

Proof. It is clear by (4.1) that $A_+^\alpha x$ is analytic in α unless α is an integer n . But $\alpha = n$ is not a singularity because $A_+^\alpha x \rightarrow A^n x$ as α tends to n . Estimating each term in (4.1) we obtain

$$\begin{aligned} \| A_+^\alpha x \| &\leq \left| \frac{\sin \pi\alpha}{\pi} \right| \left[\frac{N^{\text{Re } \alpha}}{\text{Re } \alpha} M \| x \| + \sum_{k=0}^n \frac{N^{\text{Re } \alpha - k}}{|\alpha - k|} \| A^k x \| \right. \\ &\quad \left. + \frac{N^{\text{Re } \alpha - \sigma}}{\sigma - \text{Re } \alpha} q_\sigma(x) \right] . \end{aligned}$$

$|\sin \pi\alpha|/|\alpha - k|$ is bounded by $\text{const } e^{\pi|\text{Im } \alpha|}$, and we have by Theorem 2.3 the estimates

$$\| A^k x \| \leq C(\sigma)(N^{k-\sigma} q_\sigma(x) + N^k \| x \|), \quad k = 0, \dots, n, \quad 0 < N .$$

Hence it follows that

$$(4.7) \quad \| A_+^\alpha x \| \leq C_1(\sigma) e^{\pi|\text{Im } \alpha|} \left[\frac{|\alpha|}{\text{Re } \alpha} N^{\text{Re } \alpha} \| x \| + \frac{|\alpha - n|}{\sigma - \text{Re } \alpha} N^{\text{Re } \alpha - \sigma} q_\sigma(x) \right] ,$$

with a constant $C_1(\sigma)$ depending only on σ . Taking the minimum of the right-hand side we obtain the desired inequality.

PROPOSITION 4.8. Let $\text{Re } \alpha > 0$ and $m = 1, 2, 3, \dots$. If $\sigma > m + \text{Re } \alpha$, then

$$(4.8) \quad A_B^m A_+^\alpha x = A_+^\alpha A_B^m x = A_+^{m+\alpha} x , \quad x \in D^\sigma .$$

Proof. By definition $x \in D^\sigma$ if and only if $x \in D(A_D^m)$ and $A_D^m x \in D^{\sigma-m}$. Thus A_D^m belongs to $D(A_+^\alpha)$. Making use of expansion (3.1), we have

$$\int_0^N \lambda^\alpha (\lambda + A)^{-1} A_D^m x d\lambda = (-1)^m \int_0^N \lambda^{m+\alpha} (\lambda + A)^{-1} x d\lambda + \sum_{k=0}^{m-1} (-1)^{m+k+1} \frac{N^{\alpha+m-k}}{\alpha + m - k} A^k x .$$

In view of definition (4.1), this proves that

$$A_{\sigma-m}^\alpha A_D^m x = A_\sigma^{m+\alpha} x , \quad x \in D^\sigma .$$

Notice that the integrals defining $A^\alpha x$ converge in the sense of Riemann and that the integrands are commutative with A_D^m . Then the former part of (4.8) follows easily from the closedness of A_D^m .

PROPOSITION 4.9. Let $\text{Re } \alpha, \text{Re } \beta > 0$, and σ be sufficiently large. Then we have

$$(4.9) \quad A_+^\alpha A_+^\beta x = A_+^{\alpha+\beta} x , \quad x \in D^\sigma .$$

Proof. Combining Proposition 4.7 with Proposition 4.8 we see that if m is an integer such that $0 \leq m < \sigma$, then for every $x \in D^\sigma$, $A_+^\beta x$ is a D^m valued analytic function of α on the strip $0 < \text{Re } \beta < \sigma - m$. If $\text{Re } \alpha < m$, A_+^α is a continuous operator from D^m into X . Thus $A_+^\alpha A_+^\beta x$ is analytic in β in the domain defined by $0 < \text{Re } \alpha < m$ and $0 < \text{Re } \beta < \sigma - m$.

From Proposition 4.7 it follows that $A_+^\alpha A_+^\beta x$ is analytic in α in the same domain. Therefore, it is analytic in α and β . Similarly $A^{\alpha+\beta} x$ is analytic in α and β .

Thus in order to prove (4.9) in general it is enough to prove it for $0 < \text{Re } \alpha + \text{Re } \beta < 1$ and say for all $x \in D(A^2)$. We have by (4.2)

$$\begin{aligned} A_+^\alpha A_+^\beta x &= \frac{\sin \pi \alpha}{\pi} \frac{\sin \pi \beta}{\pi} \int_0^\infty \int_0^\infty \lambda^{\alpha-1} \tau^{\beta-1} (\lambda + A)^{-1} (\tau + A)^{-1} A^2 x d\lambda d\tau \\ &= \frac{\sin \pi \alpha}{\pi} \frac{\sin \pi \beta}{\pi} \int_0^\infty \int_0^\infty \lambda^{\alpha-1} \tau^{\beta-1} \frac{\lambda(\lambda + A)^{-1} - \tau(\tau + A)^{-1}}{\lambda - \tau} A x d\lambda d\tau . \end{aligned}$$

Since the integrand is analytic, we may take for the path of integration of λ a line parallel to and slightly above the real axis and for τ a line slightly under the real axis. Splitting the integrand into two terms, we obtain

$$A_+^\alpha A_+^\beta x = - \frac{\sin \pi \alpha}{\pi} \frac{\sin \pi \beta}{\pi} \left[\int_0^\infty \lambda^\alpha (\lambda + A)^{-1} A x d\lambda \int_0^\infty \tau^{\beta-1} (\tau - \lambda)^{-1} d\tau \right]$$

$$\begin{aligned}
 & + \int_0^\infty \tau^\beta (\tau + A)^{-1} A x d\tau \int_0^\infty \lambda^{\alpha-1} (\lambda - \tau)^{-1} d\lambda \Big] \\
 & = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^\alpha (\lambda + A)^{-1} A x (-\lambda - i0)^{\beta-1} d\lambda \\
 & \quad + \frac{\sin \pi \beta}{\pi} \int_0^\infty \tau^\beta (\tau + A)^{-1} A x (-\tau + i0)^{\alpha-1} d\tau \\
 & = \frac{\sin \pi (\alpha + \beta)}{\pi} \int_0^\infty \lambda^{\alpha+\beta-1} (\lambda + A)^{-1} A x d\lambda = A_+^{\alpha+\beta} x .
 \end{aligned}$$

THE CASE $\operatorname{Re} \alpha < 0$. Similarly as in the case $\operatorname{Re} \alpha > 0$, we define $A_{-\sigma}^\alpha x$ for $x \in R^\sigma$ and for $-\sigma < \operatorname{Re} \alpha < 0$ by

$$\begin{aligned}
 (4.10) \quad & A_{-\sigma}^\alpha x = y_\alpha \text{ if } \alpha \text{ is an integer or} \\
 & = -\frac{\sin \pi \alpha}{\pi} \left[\int_0^N \lambda^\alpha R_{-n}(\lambda, x) d\lambda \right. \\
 & \quad \left. + \sum_{k=1}^n (-1)^{k+1} \frac{N^{\alpha+k}}{\alpha+k} y_{-k} + \int_N^\infty \lambda^\alpha (\lambda + A)^{-1} x d\lambda \right] \text{ otherwise,}
 \end{aligned}$$

where $R_{-n}(\lambda, x)$ and y_{-k} are as in (3.3) and (3.4) and N is an arbitrary fixed positive number. $A_{-\sigma}^\alpha x$ does not depend on N and σ as far as it has a meaning.

Now we might pursue the analogy of the case $\operatorname{Re} \alpha > 0$ and could prove that $A_{-\sigma}^\alpha$ has the smallest closed extension independent of σ , etc. We remark, however, that $A_{-\sigma}^\alpha$ defined above is the same as $(A_{-R}^{-1})_{\sigma}^{-\alpha}$. For, the domains $R^\sigma(A)$ and $D^\sigma(A_{-R}^{-1})$ are one and the same as was shown in Theorem 3.2, and it is easily proved by (3.10) that the expression (4.10) is identical with the expression (4.1) for A_{-R}^{-1} .

Consequently we have the following proposition as a corollary of Propositions 4.1-4.9.

PROPOSITION 4.10. Let $\operatorname{Re} \alpha < 0$ and $A_{-\sigma}^\alpha$ be defined by (4.10). Then $A_{-\sigma}^\alpha$ has the smallest closed extension A_-^α in X which is independent of σ . A_-^α is identical with $A_{-R}^\alpha = (A_{-R}^{-1})_{-}^{-\alpha}$ defined by A_{-R} and if especially α is a negative integer $-n$, A_-^α coincides with A_{-R}^{-n} in the usual sense. The domain $D(A^\alpha)$ is contained and dense in $\overline{R(A)}$, and the range $R(A_-^\alpha)$ is contained $\overline{D(A)} \cap \overline{R(A)}$. A_-^α is commutative with bounded operators $(\mu(\mu + A)^{-1})^l$ and $(A(\mu + A)^{-1})^l$, $\mu > 0$, $l = 1, 2, \dots$.

For every $x \in R^\sigma$, $A^\alpha x$ is analytic in α on the strip $-\sigma < \operatorname{Re} \alpha < 0$. If $\operatorname{Re} \alpha, \operatorname{Re} \beta < 0$, then there is a $\sigma \geq 0$ such that

$$A_-^\alpha A_-^\beta x = A_-^{\alpha+\beta} x$$

holds for every $x \in R^\sigma$.

THE CASE $\text{Re } \alpha = 0$. If $\text{Re } \alpha = 0$, the integral (1.2) becomes singular both at infinity and at zero. Therefore, first we restrict the domain to $D^\sigma \cap R^\tau$, where $\sigma = n + \rho$ and $\tau = m + \pi$ are positive. $A^\alpha x$ for $x \in D^\sigma \cap R^\tau$ is defined by

$$(4.11) \quad \begin{aligned} A_{\sigma\tau}^\alpha x &= x \text{ if } \alpha = 0 \text{ or} \\ &= -\frac{\sin \pi\alpha}{\pi} \left[\int_0^N \lambda^\alpha R_{-m}(\lambda, x) d\lambda \right. \\ &\quad \left. + \sum_{k=-m}^n (-1)^{k+1} \frac{N^{\alpha-k}}{\alpha-k} A_B^k x + \int_N^\infty \lambda^\alpha R_n(\lambda, x) d\lambda \right], \text{ otherwise.} \end{aligned}$$

As before N is an arbitrary fixed positive number, and it has nothing to do with the value of $A_{\sigma\tau}^\alpha x$.

The right-hand side of (4.12) expresses an analytic function of α on the strip $-\tau < \text{Re } \alpha < \sigma$ and it coincides with $A_{-\tau}^\alpha x$ and $A_\sigma^\alpha x$ in the subdomain $-\tau < \text{Re } \alpha < 0$ and $0 < \text{Re } \alpha < \sigma$, respectively, Thus it is possible to give another definition of fractional powers by means of the operator $A_{\sigma\tau}^\alpha$ even in the case $\text{Re } \alpha \neq 0$.

PROPOSITION 4.11. For every complex number α , $A_{\sigma\tau}^\alpha$ has the smallest closed extension A_0^α independent of σ and τ such that $-\tau < \text{Re } \alpha < \sigma$.

Proof. This may be proved in the same way as Proposition 4.1 by making use of operators $(\mu(\mu + A)^{-1})^l (A(\nu + A)^{-1})^k$ in place of $(\mu(\mu + A)^{-1})^l$.

PROPOSITION 4.12. A_0^α coincides with the fractional power $(A_B)_0^\alpha$ of the operator A_B . If α is an integer n , A_0^α is equal to A_B^n .

In case $\text{Re } \alpha > 0$ ($\text{Re } \alpha < 0$), A_0^α is the restriction of A_+^α (A_-^α) to the domain $D(A_0^\alpha) = D(A_+^\alpha) \cap \overline{R(A)}$ ($D(A_-^\alpha) \cap \overline{D(A)}$), respectively).

Proof. In view of Theorem 3.7, the first statement is clear. The second statement follows from the third statement and Proposition 4.2 and 4.10.

To prove the third statement, let $\text{Re } \alpha > 0$. Obviously A_0^α is a restriction of A_+^α and we have $D(A_0^\alpha) \subset D(A_+^\alpha) \cap \overline{R(A)}$. Conversely let $x \in D(A_+^\alpha) \cap \overline{R(A)}$. If l is sufficiently large, $x^{\mu\nu} = (\mu(\mu + A)^{-1})^l A(\nu + A)^{-1} x$ belongs to $D^{l-\varepsilon} \cap R^{1-\varepsilon} \subset D(A_0^\alpha)$, and it follows from Proposition 4.4 and Theorem 3.5 that $x^{\mu\nu} \rightarrow x$ and $A_0^\alpha x^{\mu\nu} \rightarrow A_+^\alpha x$ as $\mu \rightarrow \infty$ and $\nu \rightarrow 0$. Thus we have $x \in D(A_0^\alpha)$. The proof in the case $\text{Re } \alpha < 0$ is similar.

Analogously to Proposition 4.9 we have

PROPOSITION 4.13. Let α and β be complex numbers. If σ and τ are sufficiently large, we have

$$A_0^\alpha A_0^\beta x = A_0^{\alpha+\beta} x, \quad x \in D^\sigma \cap R^\tau.$$

Proof. We omit the details of the proof which is essentially the same as that of Proposition 4.9. We note only that if an integer m satisfies $-\tau < m < \sigma$, then A_B^m maps $D^\sigma \cap R^\tau$ onto $D^{\sigma-m} \cap R^{\tau+m}$ homeomorphically and that if, in addition, the real parts of a complex number α and $\alpha + m$ satisfy the same condition, then we have

$$A_0^\alpha A_B^m x = A_B^m A_0^\alpha x = A_0^{\alpha+m} x, \quad x \in D^\sigma \cap R^\tau.$$

5. The case where A^{-1} is bounded. In this section we confine our attention to the class of operators A which has a bounded inverse A^{-1} and satisfies (1.5). $\mu + A$ satisfies these conditions if $\mu > 0$ and A satisfies (1.5).

The distinctive feature of this case is that $(\lambda + A)^{-1}$ is regular at zero. In fact, (3.3) is satisfied for every n by setting $y_{-k} = A^{-k}x$, $k = 1, \dots, n$. Thus it follows that $R^\sigma = X$ and, as is easily seen, these spaces are isomorphic. (Conversely, we can prove that if $R^\sigma = X$ for a $\sigma > 0$, then A has a bounded inverse.)

In particular, $A^\alpha x$ is defined everywhere and analytic in α if $\operatorname{Re} \alpha < 0$. Furthermore, if $x \in D^\sigma$, then $A^\alpha x$ is analytic in the half-plane $\operatorname{Re} \alpha < \sigma$.

The following proposition gives a useful representation of A^α . This was used by Krasnosel'skii-Sobolevskii [13] to define A^α and $A^{-\alpha}$.

PROPOSITION 5.1. Let A have a continuous inverse and n be a positive integer. Then for every α with $-(n+1) < \operatorname{Re} \alpha < 0$ we have

$$(5.1) \quad A^\alpha = -\frac{\sin \pi \alpha}{\pi} \frac{n!}{(\alpha+1) \cdots (\alpha+n)} \int_0^\infty \lambda^{\alpha+n} (\lambda + A)^{-n-1} d\lambda,$$

where the integral converges in the uniform operator topology.

Proof. It is evident that the integral (5.1) converges in the uniform operator topology and represents an analytic function of α for $-(n+1) < \operatorname{Re} \alpha < 0$. Integrating (5.1) by parts, we see that the integral does not depend on n . When $n = 0$, (5.1) coincides with (4.10). Thus it follows from the unique continuation property of analytic functions that (5.1) holds for $-(n+1) < \operatorname{Re} \alpha < 0$.

We remark that $A_0^\alpha = A_+^\alpha$ for $\operatorname{Re} \alpha > 0$, while $A_0^\alpha = A_-^\alpha |_{\overline{D(A)}}$ for $\operatorname{Re} \alpha < 0$ and that the latter operator is bounded in $\overline{D(A)}$.

PROPOSITION 5.2. If $\operatorname{Re} \alpha < 0$ ($\operatorname{Re} \alpha > 0$) and $x \in D(A_0^\beta)$ ($x \in D(A_0^{\alpha+\beta})$), then $x \in D(A_0^{\alpha+\beta})$ ($x \in D(A_0^\beta)$, $A_0^\beta x \in D(A_0^\alpha)$, respectively) and

$$(5.2) \quad A_0^\alpha A_0^\beta x = A_0^{\alpha+\beta} x .$$

In particular, $D(A_0^\alpha) \subset D(A_0^\beta)$ if $\operatorname{Re} \alpha > \operatorname{Re} \beta$.

Proof. Let $\operatorname{Re} \alpha < 0$. By Proposition 4.13, (5.2) holds for all $x \in D^\sigma$ if σ is sufficiently large. If $x \in D(A_0^\beta)$, it can be approximated by a sequence $x_j \in D^\sigma$ in such a way that $x_j \rightarrow x$ and $A_0^\beta x_j \rightarrow A_0^\beta x$. Since A_0^α is bounded, $A_0^\alpha A_0^\beta x_j = A_0^{\alpha+\beta} x_j$ converges to $A_0^\alpha A_0^\beta x$. This proves by the closedness of $A_0^{\alpha+\beta}$ that $x \in D(A_0^{\alpha+\beta})$ and $A_0^{\alpha+\beta} x = A_0^\alpha A_0^\beta x$. The proof in the case $\operatorname{Re} \alpha > 0$ is similar.

COROLLARY 5.3. If $\operatorname{Re} \alpha > 0$, then $A_+^\alpha = A_0^\alpha$ is the inverse of $A_0^{-\alpha} = A_-^{-\alpha} |_{D(A)}$. In particular the domain $D(A_+^\alpha)$ is contained in the range $R(A_-^{-\alpha})$.

PROPOSITION 5.4. Let $\operatorname{Re} \alpha > 0$. If $\operatorname{Re} \alpha$ is not an integer, then the range $R(A_-^{-\alpha})$ is contained in $D^{\operatorname{Re} \alpha}$ and there exists a constant $K(\alpha)$ depending only on M, L and α such that

$$(5.3) \quad q_{\operatorname{Re} \alpha}(A_-^{-\alpha} x) \leq K(\alpha) \|x\| , \quad x \in X .$$

If α is an integer n , then $A_-^{-\alpha} = A^{-n}$, so that $R(A_-^{-\alpha}) = D(A^n)$. Finally, if $\operatorname{Re} \alpha$ is an integer n and $\alpha \neq n$, then $R(A_-^{-\alpha}) \subset D(A^{n-1})$ and there exists a constant $K_1(\alpha)$ depending only on $M, L, \|A^{-1}\|$ and α such that

$$(5.4) \quad \|A^n(\lambda + A)^{-1} A_-^{-\alpha} x\| \leq K_1(\alpha) \lambda^{-1} \log(\lambda + 2) \|x\| ,$$

$$x \in X, 0 < \lambda < \infty .$$

Consequently $R(A_-^{-\alpha})$ is contained continuously in $D^{\operatorname{Re} \alpha - \varepsilon}$ for every $\varepsilon > 0$.

Proof. Let $\alpha = n + \beta$, where $n = 0, 1, 2, \dots$ and $0 < \operatorname{Re} \beta \leq 1$. In view of Proposition 4.10, we can show that

$$A_-^{-\alpha} = A^{-n} A_-^{-\beta}$$

similarly to Proposition 5.2. Hence we have $R(A_-^{-\alpha}) \subset D(A^n)$ and

$$A^n A_-^{-\alpha} x = A_-^{-\beta} x , \quad x \in X .$$

Thus the problem may be reduced to the case $n = 0$. The statement is obvious in the case $\beta = 1$. Let $\beta \neq 1$. Then we have by Proposition 5.1

$$\frac{\pi(1 - \beta)}{\sin \pi\beta} A(\lambda + A)^{-1} A^{-\beta} x = \int_0^\infty \tau^{1-\beta} A(\lambda + A)^{-1} (\tau + A)^{-2} x d\tau .$$

We split the integral into two parts and estimate each of them as follows:

$$\begin{aligned} & \| A(\lambda + A)^{-1} \int_\lambda^\infty \tau^{1-\beta} (\tau + A)^{-2} x d\tau \| \\ & \leq LM^2 \int_\lambda^\infty \tau^{1-\operatorname{Re}\beta} \tau^{-2} d\tau \| x \| \\ & \leq LM^2 (\operatorname{Re} \beta)^{-1} \lambda^{-\operatorname{Re}\beta} \| x \| , \\ & \| (\lambda + A)^{-1} \int_0^\lambda (A(\tau + A)^{-1} \tau^{1-\beta} (\tau + A)^{-1} x d\tau \| \\ & \leq M\lambda^{-1} LM \int_0^\lambda \tau^{1-\operatorname{Re}\beta} \tau^{-1} d\tau \| x \| \\ & \leq LM^2 (1 - \operatorname{Re} \beta)^{-1} \lambda^{-\operatorname{Re}\beta} \| x \| \quad \text{if } \operatorname{Re} \beta < 1 . \end{aligned}$$

In case $\operatorname{Re} \beta = 1$, the last integral becomes infinity. We note, however, that there is a constant K such that $\| (\tau + A)^{-1} \| \leq K(1 + \tau)^{-1}$, $0 < \tau < \infty$. Thus the last integral may be replaced by

$$\begin{aligned} & M\lambda^{-1} LK \int_0^\lambda (1 + \tau)^{-1} d\tau \| x \| \\ & = MLK\lambda^{-1} \log (1 + \lambda) \| x \| . \end{aligned}$$

PROPOSITION 5.5 (Krasnosel'skii-Sobolevskii [13]). Let α, β and γ be complex numbers such that $\operatorname{Re} \alpha > \operatorname{Re} \gamma > \operatorname{Re} \beta$. Then we have

$$(5.5) \quad \| A_\delta^\lambda x \| \leq C(\alpha, \beta, \gamma) \| A_\delta^\alpha x \|^\theta \| A_\delta^\beta x \|^{1-\theta} , \quad x \in D(A_\delta^\alpha) ,$$

where

$$(5.6) \quad \theta = \operatorname{Re} (\gamma - \beta) / \operatorname{Re} (\alpha - \beta)$$

in case $\operatorname{Re} (\alpha - \beta)$ is not an integer or $\alpha - \beta$ is an integer, and an arbitrary number greater than the right-hand side of (5.6) otherwise. The constant $C(\alpha, \beta, \gamma)$ depends only on α, β, γ and θ .

Proof. In view of Proposition 5.2 we may assume that $\beta = 0$. Then the statement is an immediate consequence of Propositions 5.4 and 4.7.

6. Domain $D(A^\alpha)$. We examine the domain $D(A_+^\alpha)$ ($D(A_-^\alpha)$) in connection with $D^\sigma(A)$ and $D((\mu + A)_+^\alpha)$, $\mu > 0$ ($R^\sigma(A)$ and $D((A(\mu + A)^{-1})_-^\alpha)$, $\mu > 0$, respectively). First we note

PROPOSITION 6.1. $D^\sigma(\mu + A)$ is identical with and isomorphic to

$D^\sigma(A)$ for every $\sigma \geq 0$ and every $\mu \geq 0$.

We omit the proof which is straightforward.

PROPOSITION 6.2. Let $\mu > 0$ and $\nu \geq 0$. Then the bounded operator $(\nu + A)(\mu + A)^{-1}$ satisfies (1.5). If $\text{Re } \alpha > 0$, then its fractional power $((\nu + A)(\mu + A)^{-1})_+^\alpha$ is bounded and expressed as

$$(6.1) \quad ((\nu + A)(\mu + A)^{-1})_+^\alpha = 1 - \frac{\sin \pi \alpha}{\pi} \int_\nu^\mu (\lambda - \nu)^\alpha (\mu - \lambda)^{-\alpha} (\lambda + A)^{-1} d\lambda ,$$

where the integral is taken in the sense of Sato. In terms of the convergent integral this is equal to

$$(6.2) \quad 1 + \alpha(\nu - \mu)(\mu + A)^{-1} + \dots \\ + \frac{\alpha(\alpha - 1) \dots (\alpha - n + 1)}{n!} (\nu - \mu)^n (\mu + A)^{-n} \\ - \frac{\sin \pi \alpha}{\pi} \int_\nu^\mu (\lambda - \nu)^\alpha (\mu - \lambda)^{-\alpha+n} (\lambda + A)^{-1} (\mu + A)^{-n} d\lambda ,$$

where n is an integer such that $n + 1 > \text{Re } \alpha$.

Proof. It is easy to see that if $\lambda > 0$, then $(\lambda + (\nu + A)(\mu + A)^{-1})$ is one-to-one and the inverse is given by

$$(6.3) \quad (\lambda + 1)^{-1} (1 + (\mu - \nu)(\lambda + 1)^{-1} (\lambda(\mu - \nu)(\lambda + 1)^{-1} + \nu + A)^{-1}) .$$

Thus $(\nu + A)(\mu + A)^{-1}$ satisfies condition (1.5). Since $(\nu + A)(\mu + A)^{-1}$ is bounded, it follows that $D^\sigma((\nu + A)(\mu + A)^{-1}) = X$ for all $\sigma \geq 0$ and that they are isomorphic to each other. Therefore, $((\nu + A)(\mu + A)^{-1})^\alpha$ is bounded for $\text{Re } \alpha > 0$ and analytic in α . On the other hand, an easy calculation shows that expression (6.2) does not depend on n as far as $0 < \text{Re } \alpha < n + 1$ and represents a bounded operator valued analytic function of α for $\text{Re } \alpha > 0$. Thus it is enough to prove that (6.1) is true for $0 < \alpha < 1$. This can be done by expressing $((\nu + A)(\mu + A)^{-1})_+^\alpha$ as integral (4.2) in terms of (6.3) and then by changing the variable as $\tau = (\lambda\mu + 1)/(\lambda + 1)$.

PROPOSITION 6.3. Let $\text{Re } \alpha > 0$, $\mu > 0$ and $\nu \geq 0$. If $x \in \overline{D(A)}$, then $(\mu + A)_0^{-\alpha} x \in D((\nu + A)_+^\alpha)$ and

$$(6.4) \quad (\nu + A)_+^\alpha (\mu + A)_0^{-\alpha} x = ((\nu + A)(\mu + A)^{-1})_+^\alpha x .$$

If $x \in D((\nu + A)_+^\alpha)$, then

$$(6.5) \quad (\mu + A)_0^{-\alpha} (\nu + A)_+^\alpha x = (\nu + A)_+^\alpha (\mu + A)_0^{-\alpha} x \\ = ((\nu + A)(\mu + A)^{-1})_+^\alpha x .$$

Proof. First we prove that (6.5) is valid for every $x \in D^\sigma$ if σ is sufficiently large. In view of Proposition 5.1, we see easily that $(\mu + A)^{-\alpha}$ maps D^σ into D^σ for every σ and that it commutes with A^k and $(\lambda + A)^{-k}$. Therefore, we have

$$(\mu + A)^{-\alpha}(\nu + A)^{\alpha}_{\pm}x = (\nu + A)^{\alpha}_{\pm}(\mu + A)^{-\alpha}x$$

for every $x \in D^\sigma$ if $\sigma > \operatorname{Re} \alpha$.

Let $x \in D^\sigma$, $\sigma > 0$. Then it follows that $(\mu + A)^{-\alpha}(\nu + A)^{\alpha}_{\pm}x$ is analytic for $0 < \operatorname{Re} \alpha < \sigma$ since $(\mu + A)^{-\alpha}$ is analytic for $\operatorname{Re} \alpha > 0$ and $(\nu + A)^{\alpha}_{\pm}x$ is analytic for $0 < \operatorname{Re} \alpha < \sigma$. $((\nu + A)(\mu + A)^{-1})^{\alpha}_{\pm}x$ is also analytic for $\operatorname{Re} \alpha > 0$, and expressed by integral (6.1). Therefore, to prove (6.4) for $x \in D^\sigma$, it is sufficient to show that $(\nu + A)^{\alpha}(\mu + A)^{-\alpha}x$ is equal to the integral (6.1) for $0 < \alpha < 1$ and all $x \in D(A)$. By (4.2) and (5.1) for $n = 0$ we have

$$\begin{aligned} & (\nu + A)^{\alpha}_{\pm}(\mu + A)^{-\alpha}x \\ &= \left(\frac{\sin \pi \alpha}{\pi}\right)^2 \int_{\nu}^{\infty} (\lambda - \nu)^{\alpha-1} d\lambda \int_{\mu}^{\infty} (\tau - \mu)^{-\alpha} (\lambda + A)^{-1} (\tau + A)^{-1} (\nu + A) x d\tau \\ &= \left(\frac{\sin \pi \alpha}{\pi}\right)^2 \int_{\nu}^{\infty} (\lambda - \nu)^{\alpha-1} d\lambda \int_{\mu}^{\infty} (\tau - \mu)^{-\alpha} \frac{(\tau + A)^{-1} - (\lambda + A)^{-1}}{\lambda - \tau} (\nu + A) x d\tau. \end{aligned}$$

Now by a similar computation as in the proof of Proposition 4.9 we obtain

$$\begin{aligned} & (\nu + A)^{\alpha}_{\pm}(\mu + A)^{-\alpha}x \\ &= \frac{\sin \pi \alpha}{\pi} \left[\int_{\nu}^{\infty} (\lambda - \nu)^{\alpha-1} (\mu - \lambda - i0)^{-\alpha} (\lambda + A)^{-1} (\nu + A) x d\lambda \right. \\ & \quad \left. + \int_{\mu}^{\infty} (\tau - \mu)^{-\alpha} (\nu - \tau + i0)^{\alpha-1} (\tau + A)^{-1} (\nu + A) x d\lambda \right] \\ &= \frac{\sin \pi \alpha}{\pi} \int_{\nu}^{\mu} (\lambda - \nu)^{\alpha-1} (\mu - \lambda)^{-\alpha} (\lambda + A)^{-1} (\nu + A) x d\lambda \\ &= x - \frac{\sin \pi \alpha}{\pi} \int_{\nu}^{\mu} (\lambda - \nu)^{\alpha} (\mu - \lambda)^{-\alpha} (\lambda + A)^{-1} x d\lambda. \end{aligned}$$

Thus we have proved that (6.5) holds for every $x \in D^\sigma$ when σ is large enough.

Hence the statements of the theorem follow by the same arguments as in the proof of Proposition 5.2.

THEOREM 6.4. *If $\operatorname{Re} \alpha > 0$, the domain $D((\nu + A)^{\alpha}_{\pm})$ does not depend on $\nu \geq 0$ and coincides with the range $R((\mu + A)^{-\alpha}_0) = R((\mu + A)^{-\alpha} |_{D(A)})$ for $\mu > 0$.*

Proof. It follows from Proposition 6.3 and Corollary 5.3 that if

$\mu > 0$ and $\nu \geq 0$,

$$D((\mu + A)_+^\alpha) = R((\mu + A)_0^{-\alpha}) \subset D((\nu + A)_+^\alpha).$$

Thus if μ and ν are positive, then we have $D((\mu + A)_+^\alpha) = D((\nu + A)_+^\alpha)$ and $D((\mu + A)_+^\alpha)$ is contained in $D(A_+^\alpha)$.

To prove that $D(A_+^\alpha) \subset D((\mu + A)_+^\alpha)$, we show that the restriction of A_+^α to $D((\mu + A)_+^\alpha)$ is closed in X . Proposition 6.3 gives

$$\begin{aligned} (A_+^\alpha - (\mu + A)_+^\alpha)(\mu + A)_0^{-\alpha}x &= \alpha(-\mu)(\mu + A)^{-1}x + \dots \\ &+ \frac{\alpha(\alpha - 1) \dots (\alpha - n + 1)}{n!} (-\mu)^n(\mu + A)^{-n}x \\ &- \frac{\sin \pi\alpha}{\pi} \int_0^\mu \lambda^\alpha(\mu - \lambda)^{-\alpha+n}(\lambda + A)^{-1}(\mu + A)^{-n}x d\lambda \end{aligned}$$

for every $x \in \overline{D(A)}$. We notice that the right-hand side remains bounded if we operate $\mu + A$ on both sides. Therefore, there is a constant C such that

$$\|(\mu + A)(A_+^\alpha - (\mu + A)_+^\alpha)x\| \leq C \|(\mu + A)_+^\alpha x\|, \quad x \in D((\mu + A)_+^\alpha).$$

Let $x = (\mu + A)^{-1}y$ with $y \in D((\mu + A)_+^\alpha)$ and use Propositions 4.4, 5.2 and 5.5. Then we obtain the estimate

$$\begin{aligned} \|(A_+^\alpha - (\mu + A)_+^\alpha)y\| &\leq C \|(\mu + A)_0^{\alpha-1}y\| \\ &\leq \frac{1}{2} \|(\mu + A)_+^\alpha y\| + C_1 \|y\| \end{aligned}$$

with a constant C_1 independent of $y \in D((\mu + A)_+^\alpha)$. Hence it follows that

$$\|(\mu + A)_+^\alpha x\| \leq 2 \|A_+^\alpha x\| + 2C_1 \|x\|, \quad x \in D((\mu + A)_+^\alpha).$$

Now it is easy to see that A_+^α restricted to $D((\mu + A)_+^\alpha)$ is closed.

THEOREM 6.5. *Let $\operatorname{Re} \alpha > 0$. If α is real or $\operatorname{Re} \alpha$ is not an integer, then $D(A_+^\alpha)$ is contained in the space $D^{\operatorname{Re} \alpha}$. There is a constant $K(\alpha)$ depending only on M, L and α such that*

$$(6.6) \quad q_{\operatorname{Re} \alpha}(x) \leq K(\alpha) \|A_+^\alpha x\|, \quad x \in D(A_+^\alpha).$$

Thus the embedding map is continuous. Even when $\operatorname{Re} \alpha$ is an integer, $D(A_+^\alpha)$ is contained in every $D^{\operatorname{Re} \alpha - \varepsilon}$ for $\varepsilon > 0$ continuously. In particular, we have $D(A_+^\alpha) \subset D(A_+^\beta)$ if $0 < \operatorname{Re} \beta < \operatorname{Re} \alpha$.

Proof. The statements except (6.6) are immediate consequences of Theorem 6.4 and Proposition 5.4. Let $\operatorname{Re} \alpha = n + \rho$, $0 < \rho < 1$. Then (6.6) is equivalent to

$$(6.7) \quad \|A^{n+1}(\lambda + A)^{-1}x\| \leq K(\alpha)\lambda^{-\rho} \|A_+^\alpha x\|, \quad \lambda > 0, \quad x \in D(A_+^\alpha).$$

It follows from Proposition 5.4 that if $\nu > 0$,

$$\|(\nu + A)^{n+1}(\lambda + \nu + A)^{-1}x\| \leq K(\alpha)\lambda^{-\rho} \|(\nu + A)_+^\alpha x\|$$

for every $x \in D(A_+^\alpha)$, where $K(\alpha)$ does not depend on ν . Let ν tend to 0 with λ fixed. Then clearly the left-hand side converges to that of (6.7). Further, it follows from Propositions 6.2 and 6.3 that $(\nu + A)_+^\alpha(\mu + A)^{-\alpha}$ converges to $A_+^\alpha(\mu + A)^{-\alpha}$ in the uniform operator topology of $\overline{D(A)}$. Thus we have $(\nu + A)_+^\alpha x \rightarrow A_+^\alpha x$ for every $x \in D(A_+^\alpha)$. This proves (6.7) and the proof is completed.

In view of Theorem 3.2 and in particular identity (3.8) the discussions concerning the domain $D(A_-^\alpha)$ may be reduced to the case considered above. We summarize the results in the following propositions and theorems.

PROPOSITION 6.6. $R^\sigma(A)$ is identical with and isomorphic to $R^\sigma(A(\mu + A)^{-1})$ for every $\sigma \geq 0$ and every $\mu > 0$.

PROPOSITION 6.7. Let $\operatorname{Re} \alpha < 0$ and $\mu > 0$. If $x \in \overline{R(A)}$, then $(A(\mu + A)^{-1})_0^{-\alpha} x \in D(A_-^\alpha)$ and we have

$$A_-^\alpha(A(\mu + A)^{-1})_0^{-\alpha} x = (\mu + A)_0^\alpha x.$$

If $x \in D(A_-^\alpha)$, then it follows that

$$A_-^\alpha(A(\mu + A)^{-1})_0^{-\alpha} x = (A(\mu + A)^{-1})_0^{-\alpha} A_-^\alpha x = (\mu + A)_0^\alpha x.$$

THEOREM 6.8. If $\operatorname{Re} \alpha < 0$ and $\mu > 0$, we have

$$D(A_-^\alpha) = D((A(\mu + A)^{-1})_-^\alpha) = R((A(\mu + A)^{-1})_-^\alpha |_{\overline{R(A)}}).$$

THEOREM 6.9. If $\operatorname{Re} \alpha < 0$ and if α is real or $\operatorname{Re} \alpha$ is not an integer, then we have $D(A_-^\alpha) \subset R^{-\operatorname{Re} \alpha}$, and the embedding map is continuous. When $\operatorname{Re} \alpha$ is an integer, $D(A_-^\alpha)$ is not necessarily contained in $R^{-\operatorname{Re} \alpha}$, but it is contained in $R^{-\operatorname{Re} \alpha - \varepsilon}$ for any $\varepsilon > 0$ and the embedding map is continuous. Thus we have $D(A_-^\alpha) \subset D(A_-^\beta)$ if $\operatorname{Re} \alpha < \operatorname{Re} \beta$.

7. Additivity and range.

THEOREM 7.1. If $\operatorname{Re} \alpha \cdot \operatorname{Re} \beta > 0$, then we have

$$(7.1) \quad A_\pm^\alpha A_\pm^\beta = A_0^\alpha A_\pm^\beta = A_\pm^{\alpha+\beta}$$

in the sense of the product of operators.

Proof. It is sufficient to consider the case $\operatorname{Re} \alpha > 0$ and $\operatorname{Re} \beta > 0$. The case $\operatorname{Re} \alpha < 0$ may be reduced to this case. We know that if σ is sufficiently large,

$$(7.2) \quad A_+^\alpha A_+^\beta x = A_+^{\alpha+\beta} x$$

holds for every $x \in D^\sigma$. Theorem 6.5 shows $D(A_+^{\alpha+\beta}) \subset D(A_+^\beta)$. Thus there is a constant C such that

$$(7.2) \quad \|A_+^\beta x\| \leq C(\|A_+^{\alpha+\beta} x\| + \|x\|), \quad x \in D(A_+^{\alpha+\beta}).$$

Now let $x \in D(A_+^{\alpha+\beta})$. Then there exists a sequence $x_j \in D^\sigma$ such that $x_j \rightarrow x$ and $A_+^{\alpha+\beta} x_j \rightarrow A_+^{\alpha+\beta} x$. Hence we have, by (7.2) and (7.3), $A_+^\beta x_j \rightarrow A_+^\beta x$ and $A_+^\alpha A_+^\beta x_j \rightarrow A_+^{\alpha+\beta} x$. This proves $A_+^\beta x \in D(A_+^\alpha)$ and $A_+^\alpha A_+^\beta x = A_+^{\alpha+\beta} x$. Since $A^\beta x \in \overline{R(A)}$, it follows that $A_+^\alpha A_+^\beta x = A_0^\alpha A_0^\beta x$.

Conversely let $x \in D(A_+^\alpha A_+^\beta)$. If l is a sufficiently large integer, we have $x^\mu = (\mu(\mu + A)^{-1})^l \in D^\sigma$ and $x^\mu \rightarrow x$ as $\mu \rightarrow \infty$. From (7.2) for $x \in D^\sigma$ and Proposition 4.4 it follows that

$$\begin{aligned} A_+^{\alpha+\beta} x^\mu &= A_+^\alpha A_+^\beta (\mu(\mu + A)^{-1})^l x \\ &= (\mu(\mu + A)^{-1})^l A_+^\alpha A_+^\beta x \\ &\rightarrow A_+^\alpha A_+^\beta x \text{ as } \mu \rightarrow \infty. \end{aligned}$$

Hence we have $x \in D(A_+^{\alpha+\beta})$ and $A_+^{\alpha+\beta} x = A_+^\alpha A_+^\beta x$.

If we use $(\mu(\mu + A)^{-1})^l (A(\nu + A)^{-1})^h$ instead of $(\mu(\mu + A)^{-1})^l$, we can prove that

$$A_0^\alpha A_0^\beta \subset A_0^{\alpha+\beta}$$

for all α and β . On the other hand, we know that if σ and τ are large enough, $D^\sigma \cap R^\tau$ is contained both in $D(A_0^\alpha A_0^\beta)$ and in $D(A_0^{\alpha+\beta})$, and $A_0^{\alpha+\beta}$ is the smallest closed extension of its restriction to $D^\sigma \cap R^\tau$. Thus we obtain the following theorem.

THEOREM 7.2. *For arbitrary α and β we have*

$$(7.4) \quad [A_0^\alpha A_0^\beta]_\sigma = A_0^{\alpha+\beta},$$

where $[A_0^\alpha A_0^\beta]_\sigma$ denotes the smallest closed extension of $A_0^\alpha A_0^\beta$.

More precisely the following theorem holds.

THEOREM 7.3. *If $x \in D(A_0^\beta) \cap D(A_0^{\alpha+\beta})$, then $A_0^\beta x \in D(A_0^\alpha)$ and satisfies*

$$(7.5) \quad A_0^\alpha A_0^\beta x = A_0^{\alpha+\beta} x.$$

In particular, A_0^α is one-to-one for every α and the inverse is the same as $A_0^{-\alpha}$.

Proof. Let $x^{\mu\nu} = (\mu(\mu + A)^{-1})^l (A(\nu + A)^{-1})^h$ with l and h sufficiently large. Then it follows that $x^{\mu\nu} \rightarrow x$, $A_0^\beta x^{\mu\nu} \rightarrow A_0^\beta x$ and $A_0^{\alpha+\beta} x^{\mu\nu} = A_0^\alpha \cdot A_0^\beta x^{\mu\nu} \rightarrow A_0^{\alpha+\beta} x$ as $\mu \rightarrow \infty$ and $\nu \rightarrow 0$. Since A_0^α is closed, we have thus the statement of the theorem.

COROLLARY 7.4. *If A has a bounded inverse and if $\operatorname{Re} \alpha > 0$, then*

$$(7.6) \quad A_0^\alpha A_0^\beta = A_0^{\alpha+\beta}.$$

Proof. By Proposition 5.2, $D(A_0^\beta) \cap D(A_0^{\alpha+\beta}) = D(A_0^{\alpha+\beta})$.

The results obtained above give us information about the ranges of operators A_\pm^α and A_0^α restriction to certain subspaces. We give here only one proposition as an illustration.

PROPOSITION 7.5. *Let $\operatorname{Re} \alpha > \operatorname{Re} \beta > 0$. Then*

$$(7.7) \quad A_+^\beta (D(A_+^\alpha)) \subset D(A_+^\gamma)$$

if $\gamma = \alpha - \beta$ or $-\operatorname{Re} \beta < \operatorname{Re} \gamma < \operatorname{Re}(\alpha - \beta)$.

Proof. It follows from Theorem 7.1 that

$$A_+^\alpha = A_0^{\alpha-\beta} A_+^\beta.$$

This proves the proposition for $\gamma = \alpha - \beta$. Similarly we have

$$A_+^\beta = A_0^{-\gamma} A_+^{\beta+\gamma}$$

for $0 > \operatorname{Re} \gamma > -\operatorname{Re} \beta$. Hence Theorem 7.3 shows that $R(A_+^\beta) \subset D(A_+^\gamma)$. Now the proposition for a general γ follows from Proposition 4.12 and Theorem 6.5.

In this connection the following theorem will be interesting.

THEOREM 7.6. (i) *Let $\sigma > \operatorname{Re} \alpha > 0$ and let $\sigma - \operatorname{Re} \alpha$ be not an integer. Then A_+^α maps D^σ into $D^{\sigma-\operatorname{Re} \alpha}$.* (ii) *Similarly let $\sigma \geq 0$, $\operatorname{Re} \alpha < 0$ and let $\sigma - \operatorname{Re} \alpha$ be not an integer. Then A_-^α maps $D^\sigma \cap D(A_-^\alpha)$ into $D^{\sigma-\operatorname{Re} \alpha}$.*

Proof. (i) We may assume that $0 < \operatorname{Re} \alpha < 1$ without loss of generality. First we consider the case $\operatorname{Re} \alpha < \rho$. It is clear from the foregoing discussion that $A_+^\alpha x \in D(A^*)$. Using (4.2) and the commutativity of operators, A , A_+^α and $(\lambda + A)^{-1}$, we obtain

$$\begin{aligned} A^{n+1}(\lambda + A)^{-1} A_+^\alpha x &= \frac{\sin \pi \alpha}{\pi} \int_0^\infty \tau^{\alpha-1} A(\lambda + A)^{-1} A(\tau + A)^{-1} A^n x d\tau \\ &= \sin \pi \alpha / \pi (I_1 + I_2), \end{aligned}$$

where I_1 and I_2 denote the integrals over $(0, \lambda)$ and (λ, ∞) , respectively. Since

$$(7.8) \quad \|A(\lambda + A)^{-1}A(\tau + A)^{-1}A^n x\| \leq \|A(\tau + A)^{-1}\| \|A^{n+1}(\lambda + A)^{-1}x\| \\ \leq Lq_\sigma(x)\lambda^{-\rho},$$

we have

$$\|I_1\| \leq Lq_\sigma(x)(\operatorname{Re} \alpha)^{-1}\lambda^{\operatorname{Re} \alpha - \rho}.$$

In the same way we have

$$(7.9) \quad \|I_2\| \leq L \int_\lambda^\infty |\tau^{\alpha-1}| q_\sigma(x)\tau^{-\rho} d\tau \\ = Lq_\sigma(x)(\operatorname{Re} \alpha - \rho)^{-1}\lambda^{\operatorname{Re} \alpha - \rho}.$$

Thus $A_+^\alpha x$ belongs to $D^{\sigma - \operatorname{Re} \alpha}$.

In the case $\operatorname{Re} \alpha > \rho$, it is enough to make use of the estimates

$$(7.10) \quad \|A(\lambda + A)^{-1}A(\tau + A)^{-1}A^{n-1}x\| \leq Mq_\sigma(x)\lambda^{-1}\tau^{-\rho} \\ \|A(\lambda + A)^{-1}A(\tau + A)^{-1}A^{n-1}x\| \leq Mq_\sigma(x)\lambda^{-\rho}\tau^{-1}$$

for I_1 and I_2 , respectively.

(ii) We may restrict ourselves to the case $0 > \operatorname{Re} \alpha > -1$ as above. First, let $\rho - \operatorname{Re} \alpha < 1$. Since $D^\sigma \cap D(A_-^\alpha)$ is contained in $\overline{D(A)} \cap \overline{R(A)}$, $A_-^\alpha x = A_0^\alpha x$ and $A^n x = A_0^n x$ hold for $x \in D^\sigma \cap D(A_-^\alpha)$. Hence we have by Theorem 7.3 $A_-^\alpha x \in D^n$ and

$$(7.11) \quad A^{n+1}(\lambda + A)^{-1}A_-^\alpha x = A_0^{1+\alpha}(\lambda + A)^{-1}A^n x \\ = -\frac{\sin \pi \alpha}{\pi} \int_0^\infty \tau^\alpha A(\lambda + A)^{-1}(\tau + A)^{-1}A^n x d\tau.$$

In a similar way to (i), employing inequalities (7.10), we obtain

$$A^{n+1}(\lambda + A)^{-1}A_-^\alpha x = O(\lambda^{\operatorname{Re} \alpha - \rho}), \quad \lambda \rightarrow \infty.$$

If $\rho - \operatorname{Re} \alpha > 1$, we can apply A to (7.11) under the integral sign, obtaining

$$A^{n+2}(\lambda + A)^{-1}A_-^\alpha x = -\frac{\sin \pi \alpha}{\pi} \int_0^\infty \tau^\alpha A(\lambda + A)^{-1}A(\tau + A)^{-1}A^n x d\tau.$$

From (7.8) and a similar inequality it follows that

$$A^{n+2}(\lambda + A)^{-1}A_-^\alpha x = O(\lambda^{\operatorname{Re} \alpha - \rho + 1}), \quad \lambda \rightarrow \infty.$$

8. Convexity and continuity in α .

THEOREM 8.1. *Let $\operatorname{Re} \alpha > \operatorname{Re} \gamma > \operatorname{Re} \beta$. Then there is a constant*

$C(\alpha, \beta, \gamma)$ depending only on $M, L, \alpha, \beta, \arg(\alpha - \gamma)$, and $\arg(\gamma - \beta)$, such that

$$(8.1) \quad \|A_*^\gamma x\| \leq C(\alpha, \beta, \gamma) \|A_*^\alpha x\|^\theta \|A_*^\beta x\|^{1-\theta}, \quad x \in D(A_*^\alpha) \cap D(A_*^\beta),$$

where $\theta = \operatorname{Re}(\gamma - \beta)/\operatorname{Re}(\alpha - \beta)$. Subscripts for A_*^α, A_*^β and A_*^γ are assumed to be the same.

Proof. If $\operatorname{Re} \alpha \cdot \operatorname{Re} \beta \leq 0$, A may be replaced by A_B , so that we have $A_*^\alpha x = A_0^{\alpha-\beta} A_0^\beta x$, and $A_*^\gamma x = A_0^{\gamma-\beta} A_0^\beta x$. If $\operatorname{Re} \beta > 0$, then by Theorem 7.1 we have also $A_*^\alpha x = A_0^{\alpha-\beta} A_+^\beta x$ and $A_*^\gamma x = A_0^{\gamma-\beta} A_+^\beta x$. Lastly the case $\operatorname{Re} \alpha < 0$ is reduced to the above case by considering $A_{\bar{R}}^{-1}$ instead of A . Thus we may assume that $\beta = 0$ and $A = A_B$ without loss of generality.

If $\operatorname{Re} \alpha$ is not an integer, then it follows from Theorem 6.5 and (4.7) that

$$\|A_+^\gamma x\| \leq C(\alpha) \left[\frac{|\gamma|}{\operatorname{Re} \gamma} N^{\operatorname{Re} \gamma} \|x\| + \frac{C_1(\alpha)}{\operatorname{Re}(\alpha - \gamma)} N^{\operatorname{Re}(\gamma - \alpha)} \|A_+^\alpha x\| \right]$$

for all $N > 0$. Thus taking the minimum of the right-hand side, we obtain (8.1) with $C(\alpha, \beta, \gamma)$ depending also on $\operatorname{Re}(\alpha - \gamma)$. We may, however, change the roles of α and β by Theorem 7.3. Therefore, taking $\min\{C(\alpha, \beta, \gamma), C(\beta, \alpha, \gamma)\}$ for $C(\alpha, \beta, \gamma)$, we have the assertion.

If $\operatorname{Re} \alpha$ is an integer, choose δ_1 and δ_2 such that $0 < \delta_1 < \delta_2 < \operatorname{Re} \alpha$ and that any one of $\delta_1, \delta_2, \operatorname{Re} \alpha - \delta_1$ and $\operatorname{Re} \alpha - \delta_2$ is not an integer. Applying the same method as in the proof of Theorem 2.3, we obtain (8.1) for $\gamma = \delta_1$ and δ_2 . If we use these inequalities and (8.1) for $\alpha = \delta_2$ or for $\beta = \delta_1$, we can similarly prove the statement of the theorem.

THEOREM 8.2. *Let $x \in D(A_*^\alpha) \cap D(A_*^\beta)$, $\operatorname{Re} \alpha > \operatorname{Re} \beta$. Then $A_*^\gamma x$ is analytic in the strip $\operatorname{Re} \beta < \operatorname{Re} \gamma < \operatorname{Re} \alpha$, and it converges to $A_*^\alpha x$ ($A_*^\beta x$) as γ tends to α (β) in a Stolz region $\arg(\alpha - \gamma) \leq \theta < \pi/2$ ($\arg(\gamma - \beta) \leq \theta < \pi/2$, respectively).*

Proof. The analyticity is already known. Theorem 8.1 implies that the operator A_*^γ from the space $D(A_*^\alpha) \cap D(A_*^\beta)$ normed with $\|x\| + \|A_*^\alpha x\| + \|A_*^\beta x\|$ into X is uniformly bounded if γ is in a fixed Stolz region. If $x \in D^\sigma$ with σ sufficiently large (or $D^\sigma \cap R^\sigma$ or R^σ according as $\operatorname{Re} \beta > 0$ or $\operatorname{Re} \alpha \cdot \operatorname{Re} \beta \leq 0$ or $\operatorname{Re} \alpha < 0$), then $A_*^\gamma x$ is analytic at α and β . Since D^σ (or $D^\sigma \cap R^\sigma$ or R^σ) is dense in $D(A_*^\alpha) \cap D(A_*^\beta)$, the statement follows from the Banach-Steinhaus theorem.

PROPOSITION 8.3. *Let $x \in D(A_+^\alpha)$ for an α . Then $A_+^\gamma x$ converges to x as $\gamma \rightarrow 0$ in a Stolz region if and only if $x \in \overline{R(A)}$.*

Proof. If $x \in \overline{R(A)}$, then it follows that $x \in D(A_0^0) \cap D(A_0^\alpha)$. Thus $A_+^\gamma x$ converges to x as $\gamma \rightarrow 0$. Conversely if $A_+^\gamma x \rightarrow x$, then x belongs to $\overline{R(A)}$ as the limit of elements in $\overline{R(A)}$.

PROPOSITION 8.4. Let $\operatorname{Re} \alpha > 0$ and γ_j be a sequence such that $\gamma_j \rightarrow \alpha$ in a Stolz region $\arg(\alpha - \gamma_j) \leq \theta < \pi/2$. Then $x \in D(A_+^\alpha)$ if and only if $x \in D(A_+^{\gamma_j})$ for all j and $A_+^{\gamma_j} x$ converges (weakly).

Proof. Necessity is clear from Theorem 8.2. Let $A_+^{\gamma_j} x$ converge weakly to an element y . Since $x \in D^{\operatorname{Re} \alpha - \varepsilon}$ for $\varepsilon > 0$, we have $(\mu(\mu + A)^{-1})x \in D(A_+^\alpha)$, $\mu > 0$. Thus it follows from the necessity part and Proposition 4.4 that

$$\begin{aligned} A_+^\alpha(\mu(\mu + A)^{-1})x &= \lim A_+^{\gamma_j}(\mu(\mu + A)^{-1})x \\ &= w\text{-}\lim (\mu(\mu + A)^{-1})A_+^{\gamma_j}x \\ &= (\mu(\mu + A)^{-1})y . \end{aligned}$$

Let $\mu \rightarrow \infty$. Then the right-hand side tends to y because y is in $\overline{D(A)}$. Therefore, we have by Proposition 4.5 $x \in D(A_+^\alpha)$ and $A_+^\alpha x = y$.

9. Domination. The results of this section have been announced by Krasnosel'skii-Sobolevskii [13].

PROPOSITION 9.1. If B is a closed linear operator from X to another Banach space Y such that the domain $D(B)$ contains $D(A_+^\alpha)$, then for any β with $\operatorname{Re} \beta > \operatorname{Re} \alpha$, there is a constant C such that

$$(9.1) \quad \|Bx\| \leq C(\|A_+^\beta x\| + \|x\|)^\theta \|x\|^{1-\theta}, \quad x \in D(A_+^\beta),$$

where $\theta = \operatorname{Re} \alpha / \operatorname{Re} \beta$.

Proof. It follows from the closed graph theorem that there is a constant C_1 such that

$$(9.2) \quad \|Bx\| \leq C_1(\|A_+^\alpha x\| + \|x\|), \quad x \in D(A_+^\alpha).$$

(9.1) is an easy consequence of this inequality and (8.1).

THEOREM 9.2. If a closed linear operator B satisfies (9.1) for a β , $\operatorname{Re} \beta > 0$, and a θ , $0 < \theta < 1$, then $D(B)$ contains $D(A_+^\alpha)$ for every α with $\operatorname{Re} \alpha > \theta \operatorname{Re} \beta$.

Proof. We are concerned only with the domain $D(A_+^\alpha)$, so that we may replace A by $A + 1$ and assume that A has a bounded inverse without loss of generality. Then $\|A_+^\alpha x\|$ gives an equivalent norm to

$\|A_+^\alpha x\| + \|x\|$ in $D(A_+^\alpha)$. Further we may assume that β in (9.1) is a positive integer m , because if (9.1) is valid for a β , then it follows from Theorem 8.1 that (9.1) holds for every β with a greater real part without changing the value $\theta \operatorname{Re} \beta$.

It is plain from (1.5) and (1.6) that

$$\|B(\lambda + A)^{-m}x\| \leq CL^{m\theta}M^{m(1-\theta)}(\lambda + 1)^{-m(1-\theta)}\|x\|, \quad x \in \overline{D(A)}.$$

This implies by Proposition 5.1 that for any $\operatorname{Re} \alpha > m\theta$ there is a constant C_1 such that

$$\|BA^{-\alpha}x\| \leq C_1\|x\|, \quad x \in \overline{D(A)}.$$

Since A_+^α is equal to $(A_-^\alpha)^{-1}|_{D(A)}$, we have (9.2), proving that $D(B) \supset D(A_+^\alpha)$.

PROPOSITION 9.3. Let A and B be operators satisfying (1.5) and let $D(B_+^\beta) \supset D(A_+^\alpha)$ for an α and a β . Then we have $D(B_+^\delta) \supset D(A_+^\gamma)$ and

$$\|B_+^\delta x\| \leq C(\|A_+^\gamma x\| + \|x\|), \quad x \in D(A_+^\gamma)$$

for any γ and δ such that $0 < \operatorname{Re} \delta < \operatorname{Re} \beta$ and $\operatorname{Re} \delta / \operatorname{Re} \beta < \operatorname{Re} \gamma / \operatorname{Re} \alpha$.

Proof. It follows from Theorem 8.1 that

$$\begin{aligned} \|B_+^\delta x\| &\leq C \|B_+^\beta x\|^{\operatorname{Re} \delta / \operatorname{Re} \beta} \|x\|^{1 - \operatorname{Re} \delta / \operatorname{Re} \beta} \\ &\leq C_1 (\|A_+^\alpha x\| + \|x\|)^{\operatorname{Re} \delta / \operatorname{Re} \beta} \|x\|^{1 - \operatorname{Re} \delta / \operatorname{Re} \beta}. \end{aligned}$$

Now the statement follows from Theorem 9.2.

10. Resolvent and multiplicativity. This section is devoted to the resolvent of A_+^α . We note that A_+^α is an operator in $\overline{D(A)}$, and hence the resolvent is considered only in $\overline{D(A)}$. It has been shown, however, that A_+^α is determined by its restriction A_D which has a dense domain in $\overline{D(A)}$. Therefore, we may assume that $D(A)$ is dense in X . Further we shall restrict ourselves to the case $\alpha > 0$, because this is the only case important in application. For the general spectral mapping theorem the reader is referred to Balakrishnan [2].

First we need a specification of the operator A . From the assumption (1.5) and the resolvent equation it follows easily that $(\lambda + A)^{-1}$ exists at least for λ in the sector $|\arg \lambda| < \arcsin(M^{-1})$ and that $\lambda(\lambda + A)^{-1}$ is uniformly bounded on each ray in the sector. Write

$$M(\theta) = \sup_{|\arg \gamma| = \theta} \|\lambda(\lambda + A)^{-1}\|, \quad \theta \geq 0.$$

Then by the Phragmén-Lindelöf theorem $M(\theta)$ is an increasing function of θ and satisfies some convexity property.

Modifying the definition of Kato [10], we call an operator A of type $(\omega, (M(\theta)))$, $0 \leq \omega < \pi$, if A is a densely defined closed linear operator such that the resolvent set of $-A$ contains the sector $|\arg \lambda| < \mu - \omega$, and

$$(10.1) \quad \sup_{|\arg \lambda| = \theta} \|\lambda(\lambda + A)^{-1}\| \leq M(\theta) < \infty$$

holds for $0 \leq \theta < \pi - \omega$. An operator A is of type $(\omega, M(\theta))$ for an $\omega < \pi/2$ if and only if $-A$ generates a semi-group T_t which has an analytic extension to the sector $|\arg t| < \pi/2 - \omega$ such that the extended T_t is uniformly bounded on each smaller sector $|\arg t| \leq \pi/2 - \omega - \varepsilon$, (Kato [10], Komatsu [12]).

THEOREM 10.1. *If A is of type $(\omega, M(\theta))$ and if $|\arg c| < \pi - \omega$, then cA satisfies (1.5) and we have*

$$(10.2) \quad (cA)_*^\alpha = c^\alpha A_*^\alpha \quad \text{for all } \alpha.$$

Proof. Since $(\lambda + cA)^{-1} = c^{-1}(c^{-1}\lambda + A)^{-1}$, the former statement is evident. To prove (10.2) it is enough to show that $(cA)_*^\alpha x = c^\alpha A_*^\alpha x$ holds for all $x \in D^\sigma$ with σ sufficiently large (or $D^\sigma \cap R^\sigma$ or R^σ). This is proved by rotating the integration path in (4.1) (or (4.11) or (4.10)) since the remainders $R_{+\varepsilon}(\lambda, x)$ are estimated uniformly on each smaller sector.

PROPOSITION 10.2. *If A is of type $(\omega, M(\theta))$ and if $0 < \alpha < \pi/\omega$, then every $\lambda > 0$ belongs to the resolvent set of $-A_+^\alpha$ and the resolvent is expressed by*

$$(10.3) \quad \begin{aligned} (\lambda + A_+^\alpha)^{-1} &= \sum_{i=1}^h (\alpha \zeta_i^{\alpha-1})^{-1} (-\zeta_i + A)^{-1} \\ &+ \frac{\sin \pi \alpha}{\pi} \int_0^\infty \frac{\tau^\alpha}{\lambda^2 + 2\lambda \tau^\alpha \cos \pi \alpha + \tau^{2\alpha}} (\tau + A)^{-1} d\tau, \end{aligned}$$

where ζ_i 's are all roots of $\lambda + \zeta^\alpha = 0$ satisfying $-\pi < \arg \zeta \leq \pi$, and the integral term disappears when α is an integer.

REMARK. The number h of ζ_i is 0 when $0 < \alpha < 1$, 1 when $\alpha = 1$, 2 when $1 < \alpha < 3$, 3 when $\alpha = 3$, etc. This formula was used by Kato [10] to define A^α for $0 < \alpha < 1$. A different proof is given by Balakrishnan [2] for sufficiently small complex α .

Proof. Let us denote the right-hand side of (10.3) by $I(\lambda, \alpha)$. It is clear that the integral converges in the uniform operator topology, and hence $I(\lambda, \alpha)$ is a bounded linear operator. If we prove that

$$(10.4) \quad (\lambda + A_+^\alpha)I(\lambda, \alpha)x = I(\lambda, \alpha)(\lambda + A_+^\alpha)x = x$$

holds for every $x \in D^\sigma$ with sufficiently large σ , then the proposition follows easily as the limits of both sides.

Since the integrand in (10.3) commutes with A and $(\lambda + A)^{-1}$, the commutativity is easy to see. Now we note that

$$(10.5) \quad I(\lambda, \alpha) = \frac{\sin \pi \alpha}{\pi} \int_0^\varepsilon \frac{\tau^\alpha}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} (\tau + A)^{-1} d\tau \\ - \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (\lambda + \zeta^\alpha)^{-1} (-\zeta + A)^{-1} d\zeta,$$

where ε is a small positive number and the integration path Γ_ε runs from $\infty e^{i(\omega+\varepsilon)}$ to $-\varepsilon + i0$ and from $-\varepsilon - i0$ to $\infty e^{-i(\omega+\varepsilon)}$. This is proved by the standard technique of shift of integration path. Hence it follows that $I(\lambda, \alpha)$ is analytic in $\alpha \in (0, \pi/\omega)$. If $x \in D^\sigma$, $(\lambda + A_+^\alpha)x$ is analytic in $\alpha \in (0, \sigma)$, so that $I(\lambda, \alpha)(\lambda + A_+^\alpha)x$ is an analytic function of α in the interval $0 < \alpha < \min(\pi/\omega, \sigma)$. Thus it is sufficient to prove (10.4) only for $0 < \alpha < 1$. Let $x \in D(A)$. Then in a similar way to Proposition 4.9 we obtain

$$I(\lambda, \alpha)A_+^\alpha x = \frac{\sin \pi \alpha}{\pi} \left[\int_0^\infty \frac{\tau^{\alpha-1}}{\lambda + (-\tau + i0)^\alpha} (\tau + A)^{-1} A x d\tau \right. \\ \left. + \int_0^\infty \frac{\tau^\alpha (-\tau - i0)^{\alpha-1}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} (\tau + A)^{-1} A x d\tau \right] \\ = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \frac{\lambda\tau^{\alpha-1}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} (1 - \tau(\tau + A)^{-1}) x d\tau.$$

Now it is easy to see that this is equal to $x - \lambda I(\lambda, \alpha)x$.

THEOREM 10.3 (Kato [10]). *If A is of type $(\omega, M(\theta))$ and if $0 < \alpha < \pi/\omega$, then A^α is of type $(\alpha\omega, M_\alpha(\theta))$, where*

$$(10.6) \quad M_\alpha(\theta) \leq \left(\frac{h}{\alpha} + \left| 1 - \frac{h}{\alpha} \right| \right) M \left(\left(\frac{1}{\alpha} \theta + \frac{\alpha-1}{\alpha} \pi \right)^+ \right),$$

with h as in Proposition 10.2, and the resolvent $(\lambda + A^\alpha)^{-1}$ is analytic in α and λ in the domain $0 < \alpha < \pi/\omega$, $|\arg \lambda| < \pi - \alpha\omega$.

Proof. When $\arg \theta = 0$, this is an easy consequence of the representation (10.3), since we have

$$\frac{\sin \pi \alpha}{\pi} \int_0^\infty \frac{\tau^{\alpha-1}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} d\tau = \left(1 - \frac{h}{\alpha} \right) \frac{1}{\lambda}.$$

The case $\arg \theta \neq 0$ can be reduced to this case because of Theorem 10.1. The analyticity follows from (10.5).

PROPOSITION 10.4. If $x \in \overline{R(A)}$, then $(\lambda + A_+^\alpha)^{-1}x$ converges strongly to $(\lambda + 1)^{-1}x$ as $\alpha > 0$ tends to 0.

Proof. We may consider only the case $\lambda > 0$. Since $(\lambda + A_+^\alpha)^{-1}$ is bounded uniformly in α and $D(A) \cap R(A)$ is dense in $\overline{R(A)}$, it is enough to show that

$$(\lambda + A^\alpha)^{-1}x \rightarrow (\lambda + 1)^{-1}x, \quad x \in D(A) \cap R(A).$$

We have

$$\begin{aligned} & (\lambda + A^\alpha)^{-1}x - (\lambda + 1)^{-1}x \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\tau^\alpha}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} ((\tau + A)^{-1} - (\tau + 1)^{-1})x d\tau. \end{aligned}$$

If α is sufficiently small, $\tau^\alpha(\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha})^{-1}$ is uniformly bounded. Further, from the assumption that $x \in D(A) \cap R(A)$ it follows that

$$\|((\tau + A)^{-1} - (\tau + 1)^{-1})x\| = \|(\tau + 1)^{-1}(\tau + A)^{-1}(1 - A)x\| \leq C(\tau + 1)^{-2}.$$

Thus we have

$$\|(\lambda + A^\alpha)^{-1}x - (\lambda + 1)^{-1}x\| \leq \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{C_1}{(\tau + 1)^2} d\tau \rightarrow 0.$$

THEOREM 10.5 (Yosida [22], Kato [10], Watanabe [20]). *If A is an operator of type $(\omega, M(\theta))$ and $0 < \alpha\omega < \pi/2$, then $-A_+^\alpha$ is the generator of strongly continuous semi-group $\exp(-tA_+^\alpha)$ which is analytic in the sector $|\arg t| < \pi/2 - \alpha\omega$ and uniformly bounded on each smaller sector $|\arg t| \leq \pi/2 - \alpha\omega - \varepsilon, \varepsilon > 0$. $\exp(-tA_+^\alpha)$ is continuous in t and α in the uniform operator topology in the domain $|\arg t| < \pi/2 - \alpha\omega, 0 < \alpha < \pi/(2\omega)$. Moreover, if $x \in \overline{R(A)}$, then $\exp(-tA_+^\alpha)x$ converges strongly to $\exp(-t)x$ uniformly on every compact set of the half plane $\operatorname{Re} t > 0$ as $\alpha > 0$ tends to 0.*

Proof. The first and the second statements are direct consequences of Theorem 10.3. The third statement follows from Proposition 10.4 and Theorem 3.2 by Trotter's theorem (Trotter [19]).

THEOREM 10.6 (Watanabe [20]). *Let A be of type $(\omega, M(\theta))$. Then*

$$(10.7) \quad (A_+^\alpha)_+^\beta = A_+^{\alpha\beta}$$

if $0 < \alpha < \pi/\omega$ and $\operatorname{Re} \beta > 0$.

Proof. $(A_+^\alpha)_+^\beta$ and $A_+^{\alpha\beta}$ are defined to be the smallest closed exten-

sions of their restrictions to $D^\sigma(A_+^\alpha)$ and $D^\tau(A)$, respectively, where σ and τ are sufficiently large. On the other hand, since $(A_+^\alpha)^n = A_+^{\alpha n}$ for positive integer n by Theorem 7.1, it follows from Theorem 6.5 that $D^\sigma(A_+^\alpha) = D((A_+^\alpha)^\sigma) = D(A_+^{\alpha\sigma}) \subset D^{\alpha\sigma-\varepsilon}(A) \subset D(A_+^{\alpha(\sigma-1)}) = D^{\sigma-1}(A_+^\alpha)$ for positive integer σ . Thus we have only to prove that

$$(10.8) \quad (A_+^\alpha)^\beta x = A_+^{\alpha\beta} x, \quad x \in D^\tau(A)$$

holds if τ is sufficiently large. We may assume that $\tau = \alpha(\sigma - 1/2)$ with a positive integer σ . Then, both sides of (10.8) are analytic in β in the domain $0 < \operatorname{Re} \beta < \sigma - 1$. Thus it is enough to show (10.8) for $0 < \beta < \min\{1, 1/\alpha\}$, and, say, for $x \in D(A)$.

By an easy computation it follows from (10.5) that

$$A_+^\alpha(\lambda + A_+^\alpha)^{-1} = -\frac{1}{2\pi i} \int_{r_\theta} \zeta^\alpha(\lambda + \zeta^\alpha)^{-1} \zeta^{-1}(-\zeta + A)^{-1} A x d\zeta$$

for $\lambda > 0$ and $x \in D(A)$, where Γ_θ is composed of two rays $re^{i\theta}$, $\infty > r \geq 0$, and $re^{-i\theta}$, $0 \leq r < \infty$, and θ must satisfy $\pi > \alpha\theta > \alpha\omega$.

Therefore we have

$$\begin{aligned} (A_+^\alpha)^\beta x &= \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{\beta-1} A_+^\alpha(\lambda + A_+^\alpha)^{-1} x d\lambda \\ &= -\frac{1}{2\pi i} \int_{r_\theta} \zeta^{-1}(-\zeta + A)^{-1} A x d\zeta \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{\beta-1} \zeta^\alpha(\lambda + \zeta^\alpha)^{-1} d\lambda \\ &= -\frac{1}{2\pi i} \int_{r_\theta} \zeta^{\alpha\beta-1}(-\zeta + A)^{-1} A x d\zeta, \quad x \in D(A). \end{aligned}$$

We may let $\theta = \pi$ in the last integral without changing the value and obtain

$$(A_+^\alpha)^\beta x = \frac{\sin \pi\alpha\beta}{\pi} \int_0^\infty \tau^{\alpha\beta-1}(\tau + A)^{-1} A x d\tau = A_+^{\alpha\beta} x.$$

11. The case where $-A$ generates a bounded semi-group. Throughout this section we assume that $-A$ is the infinitesimal generator of a strongly continuous semi-group T_t which is uniformly bounded in t :

$$(11.1) \quad \|T_t\| \leq M, \quad 0 \leq t < \infty.$$

A closed linear operator A satisfies this condition if and only if the domain $D(A)$ is dense, the negative real axis is contained in the resolvent set $\rho(A)$ and

$$(11.2) \quad \|\lambda^m(\lambda + A)^{-m}\| \leq M, \quad 0 < \lambda < \infty, \quad m = 1, 2, \dots.$$

A typical example is the differential operator $p = -d/ds$ in the

space $B[0, \infty)$ of all bounded uniformly continuous functions $x(s)$ on $[0, \infty)$. The corresponding semi-group T_t is given by

$$(11.3) \quad T_t x(s) = x(s + t), \quad x(s) \in B[0, \infty).$$

It is well-known that the half-plane $\text{Re } \lambda > 0$ is contained in the resolvent set $\rho(-A)$ and the resolvent is expressible as the Laplace transform of the semi-group T_t :

$$(11.4) \quad (\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda t} T_t x dt, \quad \text{Re } \lambda > 0.$$

Thus T_t is commutative with $(\lambda + A)^{-1}$. We note that hence it follows that $T_t A^n \subset A^n T_t, n = 1, 2, \dots, T_t A_+^\alpha \subset A_+^\alpha T_t, \text{Re } \alpha > 0$, and $T_t A_0^\alpha \subset A_0^\alpha T_t, \alpha$ arbitrary. $A_+^\alpha, \text{Re } \alpha < 0$, coincides with A_0^α since $D(A)$ is dense. (11.4) implies also that A is of type $(\pi/2, M \text{ sec } \theta)$.

PROPOSITION 11.1. Let $\text{Re } \alpha < 0$ and $\mu > 0$. Then

$$(11.5) \quad (\mu + A)_0^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\mu t} T_t x dt, \quad x \in X.$$

Proof. If $0 < -\alpha < 1$, then we have by Proposition 5.1 and (11.4)

$$\begin{aligned} (\mu + A)_0^\alpha x &= \frac{1}{\Gamma(-\alpha)\Gamma(1 + \alpha)} \int_\mu^\infty (\lambda - \mu)^\alpha d\lambda \int_0^\infty e^{-\lambda t} T_t x dt \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\mu t} T_t x dt. \end{aligned}$$

Since both sides of (11.5) are analytic in α , the equality holds for all $\text{Re } \alpha < 0$.

DEFINITION. Let $\sigma = n + \rho \geq 0$ as before. $C^\sigma = C^\sigma(A)$ is defined to be the set of $x \in X$ such that

(i) $x \in D(A^n)$,

and, in case $\rho > 0$,

(ii) $T_t A^n x - A^n x = O(t^\rho)$ as $t \rightarrow 0$.

Let

$$(11.6) \quad p_\sigma(x) = \begin{cases} \|A^n x\| & \rho = 0 \\ \sup_t t^{-\rho} \|T_t A^n x - A^n x\| & \rho > 0. \end{cases}$$

Then C^σ becomes a normed space by the norm

$$\|x\| + p_\sigma(x).$$

When $A = -d/ds$ in $B[0, \infty)$, C^σ is the set of all n -times con-

tinuously differentiable functions $x(s)$ on $[0, \infty)$ such that the n -th derivative $x^{(n)}(s)$ is uniformly Hölder continuous with exponent ρ or simply uniformly continuous when $\rho = 0$.

PROPOSITION 11.2. C^σ is a Banach space and contained in D^σ continuously.

Proof. If $\sigma = n$, then $C^\sigma = D^\sigma$ and these spaces are isomorphic. As we have seen in the proof of Proposition 4.2, the operator A^n is closed. Thus $C^n = D(A^n)$ is complete.

Next let $\rho > 0$. We have

$$(11.7) \quad q_\sigma(x) \leq K(\sigma)p_\sigma(x), \quad x \in C^\sigma,$$

with a constant $K(\sigma)$ depending only on σ . In fact, it follows from (11.4) that

$$\begin{aligned} \|A^{n+1}(\lambda + A)^{-1}x\| &= \|(1 - \lambda(\lambda + A)^{-1})A^n x\| \\ &\leq \int_0^\infty \lambda e^{-\lambda t} \|T_t A^n x - A^n x\| dt \\ &\leq \Gamma(1 + \rho)\lambda^{-\rho} p_\sigma(x). \end{aligned}$$

Hence we have $C^\sigma \subset D^\sigma$. (11.7) implies also that the embedding is continuous.

To prove the completeness of C^σ , let $x_j \in C^\sigma$ be a Cauchy sequence. By (11.7) and Proposition 4.7, x_j forms a Cauchy sequence in C^n . Thus there is an $x \in C^n$ such that $x_j \rightarrow x$ and $A^n x_j \rightarrow A^n x$ in X . It is easy to see that

$$p_\sigma(x_j - x) = \sup_t \lim_{k \rightarrow \infty} t^{-\rho} \|(T_t - 1)A^n(x_j - x_k)\| \rightarrow 0.$$

THEOREM 11.3. Let $\operatorname{Re} \alpha > 0$. If α is real or $\operatorname{Re} \alpha$ is not an integer, then $D(A_+^\alpha)$ is contained in $C^{\operatorname{Re} \alpha}$.

Proof. Since $D(A)$ is dense, $D(A_+^\alpha)$ coincides with $D(A^n) = C^n$. Suppose that $\operatorname{Re} \alpha$ is not an integer. By Theorem 7.1, $x \in D(A_+^\alpha)$ if and only if $x \in D(A^n)$ and $A^n x \in D(A_+^{\alpha-n})$. On the other hand, $x \in C^\sigma$ if and only if $x \in D(A^n)$ and $A^n x \in C^{\sigma-n}$. Therefore we may assume that $0 < \operatorname{Re} \alpha < 1$ without loss of generality. If $x \in D(A_+^\alpha) = D((\mu + A)_+^\alpha)$, $\mu > 0$, then there is a $y \in X$ such that $x = (\mu + A)_+^{-\alpha} y$. Thus we have

$$\begin{aligned} T_h x - x &= \frac{1}{\Gamma(\alpha)} \left[\int_h^\infty \{(t-h)^{\alpha-1} e^{-\mu(t-h)} - t^{\alpha-1} e^{-\mu t}\} T_t y dt \right. \\ &\quad \left. - \int_0^h t^{\alpha-1} e^{-\mu t} T_t y dt \right]. \end{aligned}$$

Clearly the norm of the latter integral is bounded by $M\sigma^{-1}h^\sigma \|y\|$ with $\sigma = \text{Re } \alpha$. The former integral is estimated as follows:

$$\begin{aligned} M \|y\| &\int_h^\infty |(t-h)^{\alpha-1}e^{-\mu(t-h)} - t^{\alpha-1}e^{-\mu t}| dt \\ &\leq M \|y\| \frac{|1-\alpha|}{1-\sigma} \int_h^\infty dt \int_{t-h}^t \left(\frac{1-\sigma}{s} + \mu\right) s^{\sigma-1} e^{-\mu s} ds \\ &\leq \frac{|1-\sigma|}{\sigma(1-\sigma)} M h^\sigma \|y\|. \end{aligned}$$

PROPOSITION 11.4 (Balakrishnan [1]). Let $\sigma > \text{Re } \alpha > 0$. Then for every $x \in D^\sigma$

$$(11.8) \quad A_+^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} T_t x dt$$

in the sense of Sato or

$$(11.8)' \quad \begin{aligned} A_+^\alpha x &= \frac{1}{\Gamma(-\alpha)} \left[\int_0^N t^{-\alpha-1} (T_t x - x - tAx - \dots - \frac{t^n}{n!} A^n x) dt \right. \\ &\quad \left. + \sum_{k=0}^n \frac{N^{k-\alpha}}{k-\alpha} A^k x + \int_N^\infty t^{-\alpha-1} T_t x dt \right], \end{aligned}$$

where $0 < N < \infty$. In particular, if $0 < \text{Re } \alpha < 1$ and $\sigma > \text{Re } \alpha$, then

$$(11.9) \quad A_+^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} (T_t x - x) dt, \quad x \in D^\sigma.$$

Proof. If $x \in D^\sigma$, it follows from Theorem 11.3 that $x \in C^{\sigma-\varepsilon}$ for $\varepsilon > 0$. Thus the integrals of (11.8)' converge absolutely. Further, it is easy to see that the right-hand side of (11.8)' represents an analytic function of α for $0 < \text{Re } \alpha < \sigma$ independent of N . In case $0 < \text{Re } \alpha < \sigma < 1$, we can make N tend to infinity and obtain (11.9). Since $A_+^\alpha x$ is also analytic in α and (11.8)' does not depend on $\sigma > \text{Re } \alpha$, it is enough to prove equality (11.9). By (4.2) and (11.4) we have

$$\begin{aligned} A_+^\alpha x &= -\frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^\alpha d\lambda \int_0^\infty e^{-\lambda t} (T_t x - x) dt \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} (T_t x - x) dt, \end{aligned}$$

completing the proof.

THEOREM 11.5. Let $p = -d/ds$ in $B[0, \infty)$ and $\text{Re } \alpha > 0$. Then $x \in D(A_+^\alpha)$ if and only if

(11.10) $\langle T_s x, y' \rangle \in D(p_+^\alpha)$ for all $y' \in X'$, and
there is a function $f(s) \in X$ independent of y' such that

$$(11.11) \quad p_+^\alpha \langle T_s x, y' \rangle = \langle f(s), y' \rangle.$$

If X is sequentially weakly complete, the second condition may be omitted.

Proof. First we remark that $\langle T_s y, y' \rangle \in B[0, \infty)$ for all $y \in X$ and $y' \in X'$. In view of Proposition 11.1 we have

$$(11.12) \quad \begin{aligned} \langle T_s(\mu + A)_0^{-\alpha} y, y' \rangle &= \frac{1}{\Gamma(\alpha)} \int_s^\infty (t-s)^{\alpha-1} e^{-\mu(t-s)} \langle T_t y, y' \rangle dt \\ &= (\mu + p)_0^{-\alpha} \langle T_s y, y' \rangle \end{aligned}$$

for $\mu > 0$. Let $x \in D(A_+^\alpha)$. Then there is a $y \in X$ such that $x = (\mu + A)_0^{-\alpha} y$. Thus we have by (11.12)

$$\langle T_s x, y' \rangle = (\mu + p)_0^{-\alpha} \langle T_s y, y' \rangle \in D(p_+^\alpha).$$

Similarly it is shown by Propositions 6.2 and 6.3 that

$$(11.13) \quad p_+^\alpha \langle T_s x, y' \rangle = \langle T_s A_+^\alpha x, y' \rangle, \quad x \in D(A_+^\alpha).$$

This proves (11.11) with $f(s) = T_s A_+^\alpha x$.

To prove the converse, first let α be an integer n . Since assumptions (11.10) and (11.11) show that $T_s x$ is n times weakly differentiable at $s = 0$, it follows that $x \in D(A)$ (see Komatsu [12]), and we have by (11.13)

$$(11.14) \quad p \langle T_s x, y' \rangle = \langle T_s A x, y' \rangle.$$

Assume that the theorem is true for smaller integers. Then it follows from (11.14) that $Ax \in D(A^{n-1})$. Hence we have $x \in D(A^n)$.

Next if α is not an integer, take an integer n such that $n + 1 > \operatorname{Re} \alpha$. (11.12) with Theorem 7.1 gives

$$\langle T_s(\mu + A)_0^{-\beta} x, y' \rangle = (\mu + p)_0^{-\beta} \langle T_s x, y' \rangle \in D((\mu + p)^{n+1}) = D(p^{n+1}),$$

if $\beta = n + 1 - \alpha$ and $\mu > 0$. Moreover, we have by Propositions 6.2 and 6.3

$$\begin{aligned} p^{n+1} \langle T_s(\mu + A)_0^{-\beta} x, y' \rangle &= p_+^\beta (\mu + p)_0^{-\beta} p_+^\alpha \langle T_s x, y' \rangle \\ &= (p(\mu + p)^{-1})_+^\beta \langle f(s), y' \rangle \\ &= \langle f(s), y' \rangle - \frac{\sin \pi \beta}{\pi} \int_0^\mu \lambda^\beta (\mu - \lambda)^{-\beta} d\lambda \int_s^\infty e^{-\lambda(t-s)} \langle f(t), y' \rangle dt. \end{aligned}$$

Since $f(s)$ is uniformly bounded and weakly continuous, it can be shown that the integral $\int_0^\mu \lambda^\beta (\mu - \lambda)^{-\beta} d\lambda \int_s^\infty e^{-\lambda(t-s)} f(t) dt$ converges weakly (see Komatsu [12]). Thus there exists a function $g(s) \in X$ such that

$$p^{n+1} \langle T_s(\mu + A)_0^{-\beta} x, y' \rangle = \langle g(s), y' \rangle, \quad y' \in X'.$$

Applying the theorem in the case of integer $n + 1$, we obtain $(\mu + A)_0^{-\beta} x \in D(A^{n+1})$. Since $D(A)$ is dense, it follows from Theorem 7.1 that $x \in D((\mu + A)_+^\alpha) = D(A_+^\alpha)$.

Lastly let us prove that if X is sequentially weakly complete, the existence of $f(s) \in X$ satisfying (11.11) follows from (11.10). Let x satisfy (11.10). In view of Theorem 11.3 we have $\langle T_s x, y' \rangle \in C^\sigma(p)$ for any $\sigma < \text{Re } \alpha$, i.e., $x(s) = T_s x$ is n times scalarly differentiable and the n -th derivative is scalarly uniformly Hölder continuous with exponent ρ , $\sigma = n + \rho$.

If $\rho > 0$, we can replace the word “scalarly” by “strongly”. It is evident that $x(s)$ is n times weakly* continuously differentiable in X'' and $\{(x^{(n)}(s) - x^{(n)}(t))/|t - s|^\rho\}$ is weakly* bounded in X'' . Since a weakly* bounded set is strongly bounded, $x^{(n)}(s)$ is strongly uniformly Hölder continuous in X'' . Hence it is easily proved that $x^{(k)}(s)$, $k = 1, 2, \dots, n$, is strongly continuous in X'' . In particular the difference quotient

$$\frac{x(s+h) - x(s)}{h} = \frac{1}{h} \int_s^{s+h} x'(t) dt$$

converges strongly to $x'(s)$ in X'' as $h \rightarrow 0$. This proves that $x(s)$ is strongly differentiable. Repeating the same argument, we see that $x(s)$ is n times strongly differentiable and the n -th derivative is strongly uniformly Hölder continuous with exponent ρ .

As we have shown above, it then follows that $x \in D(A^n)$ and $(-d/ds)^n T_s x = T_s A^n x$. Therefore x belongs to $C^\sigma(A)$. In particular we have $x \in D(A_+^\beta)$ for $\text{Re } \beta < \text{Re } \alpha$ and

$$p_+^\beta \langle T_s x, y' \rangle = \langle T_s A_+^\beta x, y' \rangle, \quad y' \in X'.$$

When β tends to α in a Stolz region, the left-hand side converges to $p_+^\alpha \langle T_s x, y' \rangle$. Since X is sequentially weakly complete, there exists the limit

$$f(s) = w\text{-}\lim_{\beta \rightarrow \alpha} T_s A_+^\beta x \in X,$$

which satisfies (11.11).

Incidentally, we have proved that if x satisfies (11.10), then $x \in D(A_+^\beta)$ for $\text{Re } \beta < \text{Re } \alpha$ in any case.

Similarly to Theorem 7.6 the following theorem holds.

THEOREM 11.6 (Hardy-Littlewood [6]). (i) Let $\sigma > \operatorname{Re} \alpha > 0$. If $\sigma - \operatorname{Re} \alpha$ is not an integer, then $A_+^\alpha(C^\sigma)$ is contained in $C^{\sigma - \operatorname{Re} \alpha}$. (ii) If $\operatorname{Re} \alpha < 0$ and if $\sigma - \operatorname{Re} \alpha$ is not an integer, then $A_0^\alpha(C^\sigma \cap D(A_0^\alpha))$ is contained in $C^{\sigma - \operatorname{Re} \alpha}$.

Proof. (i) We may assume that $0 < \operatorname{Re} \alpha < 1$ without loss of generality. First let $\operatorname{Re} \alpha < \rho$. (11.9) gives

$$(T_h - 1)A^n A_+^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} (T_h - 1)(T_t - 1)A^n x dt.$$

Hence, using the estimate

$$\| (T_h - 1)(T_t - 1)A^n x \| \leq (M + 1)p_\sigma(x) \min \{h^\rho, t^\rho\},$$

we obtain

$$\| (T_h - 1)A^n A_+^\alpha x \| \leq \frac{(M + 1)p_\sigma(x)}{|\Gamma(-\alpha)|} \left(\frac{1}{\rho - \operatorname{Re} \alpha} + \frac{1}{\operatorname{Re} \alpha} \right) h^{\rho - \operatorname{Re} \alpha}.$$

In case $\rho < \operatorname{Re} \alpha$, we employ the identity

$$\begin{aligned} (T_h - 1)A^{n-1}A_+^\alpha x &= \frac{1}{\Gamma(-\alpha)} \left[\int_0^h t^{-\alpha-1} dt \int_0^t T_s (T_h - 1)A^n x ds \right. \\ &\quad \left. + \int_h^\infty t^{-\alpha-1} dt \int_0^h T_s (T_t - 1)A^n x ds \right], \end{aligned}$$

which follows from (11.9) and Theorem 11.5. From the estimate

$$\left\| \int_0^t T_s (T_h - 1)A^n x ds \right\| \leq M p_\sigma(x) t h^\rho$$

and the one with t and h interchanged, it is concluded that

$$(T_h - 1)A^{n-1}A_+^\alpha x = O(h^{\rho - \operatorname{Re} \alpha + 1}), \quad \text{as } h \rightarrow 0.$$

(ii) The proof of (ii) in Theorem 7.6 can be modified to this case with no difficulty.

THEOREM 11.7. If $0 < \alpha < 1$, $\lambda > 0$ and $n = 1, 2, \dots$, then there is an analytic function $\varphi^n(\alpha, \lambda, s) > 0$ defined on $(0, \infty)$, independent of A , such that

$$(11.15) \quad (\lambda + A_+^\alpha)^{-n} x = \lambda^{-n} \int_0^\infty \varphi^n(\alpha, \lambda, s) T_s x ds.$$

$\varphi^n(\alpha, \lambda, s)$ satisfies

$$(11.16) \quad \int_0^\infty \varphi^n(\alpha, \lambda, s) ds = 1.$$

In particular we have

$$(11.17) \quad \|(\lambda(\lambda + A_+^\alpha)^{-1})^n\| \leq M, \quad n = 1, 2, \dots .$$

Proof. Combining (11.4) and (10.3), we have

$$(11.18) \quad (\lambda + A_+^\alpha)^{-1}x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\tau^\alpha d\tau}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} \int_0^\infty e^{-\tau s} T_s x ds \\ = \lambda^{-1} \int_0^\infty \varphi^1(\alpha, \lambda, s) T_s x ds ,$$

where

$$\varphi^1(\alpha, \lambda, s) = \frac{\lambda \sin \pi\alpha}{\pi} \int_0^\infty \frac{\tau^\alpha e^{-\tau s}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} d\tau .$$

It is easy to see that $\varphi^1(\alpha, \lambda, s)$ is positive and analytic for $0 < s < \infty$ and satisfies (11.16). Let us define $\varphi^n(\alpha, \lambda, s)$ for $n > 1$ recursively by

$$\varphi^{n+1}(\alpha, \lambda, s) = \varphi^n(\alpha, \lambda, s) * \varphi^1(\alpha, \lambda, s) \\ = \int_0^s \varphi^n(\alpha, \lambda, s-t) \varphi^1(\alpha, \lambda, t) dt .$$

(11.15) for general n is proved by induction.

Other properties of $\varphi^n(\alpha, \lambda, s)$ are clear by the definition.

THEOREM 11.8 (Phillips [16], Yosida [22]). *If $0 < \alpha < 1$, then there exists a positive analytic function $\psi(\alpha, t, s)$ on $(0, \infty)^2$ such that*

$$(11.19) \quad \int_0^\infty \psi(\alpha, t, s) ds = 1 ,$$

and the analytic semi-group $\exp(-tA_+^\alpha)$ generated by $-A_+^\alpha$ has the representation

$$(11.20) \quad \exp(-tA_+^\alpha)x = \int_0^\infty \psi(\alpha, t, s) T_s x ds .$$

$\exp(-tA_+^\alpha)$ is uniformly bounded:

$$(11.21) \quad \|\exp(-tA_+^\alpha)\| \leq M, \quad 0 < t < \infty, \quad 0 < \alpha < 1 ,$$

and converges strongly to $T_t = \exp(-tA)$ as α tends to 1. The convergence is uniform on every compact set in $(0, \infty)$.

Proof. Let

$$\Psi^n(\alpha, t, s) = \int_0^s \varphi^n(\alpha, n/t, r) dr .$$

Then we have by (11.15)

$$(11.22) \quad \left(1 + \frac{tA_+^\alpha}{n}\right)^{-n} x = \int_0^\infty T_s x d\Psi^n(\alpha, t, s),$$

and in particular

$$\left(1 + \frac{t\alpha^\alpha}{n}\right)^{-n} x = \int_0^\infty e^{-as} d\Psi^n(\alpha, t, s), \quad a > 0.$$

Since $\{\Psi^n\}$ is a uniformly bounded sequence of increasing functions, there is a subsequence which converges to an increasing function $\Psi(\alpha, t, s)$ except for a countable set. Now it is easy to see that

$$(11.23) \quad \int_0^\infty e^{-as} d\Psi(\alpha, t, s) = \lim_{n' \rightarrow \infty} \int_0^\infty e^{-as} d\Psi^{n'}(\alpha, t, s) \\ = \exp(-ta^\alpha), \quad a > 0.$$

Letting $a \rightarrow 0$, we have also

$$\int_0^\infty d\Psi(\alpha, t, s) = 1.$$

Thus it follows that

$$\int_0^\infty f(s) d\Psi(\alpha, t, s) = \lim_{n' \rightarrow \infty} \int_0^\infty f(s) d\Psi^{n'}(\alpha, t, s)$$

for every bounded continuous function $f(s)$. In particular $(1 + tA_+^\alpha/n')^{-n'}x$ converges weakly to $\int_0^\infty T_s x d\Psi(\alpha, t, s)$. On the other hand, $(1 + tA_+^\alpha/n)^{-n}x$ converges strongly to $\exp(-tA_+^\alpha)x$ (Hille-Phillips [7]), so that we have

$$\exp(-tA_+^\alpha)x = \int_0^\infty T_s x d\Psi(\alpha, t, s).$$

(11.23) shows that $\exp(-t\lambda^\alpha)$ is the Laplace transform of $d\Psi(\alpha, t, s)$. It is not difficult to derive from this fact that $d\Psi(\alpha, t, s)$ has an analytic density $\psi(\alpha, t, s)$.

According to Trotter [19], a sequence of semi-groups T_t^α converges strongly to a semi-group T_t if T_t^α is uniformly bounded and the resolvent converges at a point $\lambda > 0$. This is surely the case with $T_t^\alpha = \exp(-tA^\alpha)$ by (11.21) and Theorem 10.3.

12. The case where $-A$ generates an analytic semi-group.

THEOREM 12.1. *Let A be an operator of type $(\omega, M(\theta))$ with $\omega < \pi/2$, and let T_t be the analytic semi-group generated by $-A$. If $|\arg t| < \pi/2 - \omega$, $t \neq 0$, then $T_t x \in D(A_+^\alpha)$ for any x and $\operatorname{Re} \alpha > 0$,*

and we have

$$(12.1) \quad A_+^\alpha T_t x = \frac{1}{2\pi i} \int_\Gamma (-\lambda)^\alpha e^{t\lambda} (\lambda + A)^{-1} x d\lambda,$$

where the path Γ consists of two rays from $\infty e^{-i\theta}$ to 0 and from 0 to $\infty e^{i\theta}$ with $\pi/2 < \theta < \pi/2 + |\arg t|$. There is a constant C depending only on $\alpha, \varepsilon > 0$ and A such that

$$(12.2) \quad \|A_+^\alpha T_t\| \leq C |t|^{-\operatorname{Re} \alpha}, \quad |\arg t| \leq \pi/2 - \omega - \varepsilon.$$

Proof. Since $T_t x$ is analytic in the sector $|\arg t| < \pi/2 - \omega$, $T_t x \in D(A_+^\alpha)$ for any $\operatorname{Re} \alpha > 0$ and $A_+^\alpha T_t x$ is analytic in α . The right-hand side of (12.1) expresses also an analytic function of α . Thus it is enough to prove (12.1) for $0 < \alpha < 1$. If $x \in D(A)$, we have the integral representation

$$(12.3) \quad T_t x = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} (\lambda + A)^{-1} x d\lambda$$

(Hille-Phillips [6]), so that we obtain by (11.9)

$$\begin{aligned} A_+^\alpha T_t x &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty s^{-\alpha-1} ds \frac{1}{2\pi i} \int_\Gamma (e^{s\lambda} - 1) e^{t\lambda} (\lambda + A)^{-1} x d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} (\lambda + A)^{-1} x d\lambda \frac{1}{\Gamma(-\alpha)} \int_0^\infty s^{-\alpha-1} (e^{s\lambda} - 1) ds \\ &= \frac{1}{2\pi i} \int_\Gamma (-\lambda)^\alpha e^{t\lambda} (\lambda + A)^{-1} x d\lambda. \end{aligned}$$

Since both sides of (12.1) represent bounded operators, (12.1) holds for every x . (12.2) follows easily from (12.1). Conversely,

THEOREM 12.2. *Let T_t be a bounded semi-group and $-A$ be its generator. If there is a complex number α with $\operatorname{Re} \alpha > 0$ such that*

$$(12.4) \quad \|A_+^\alpha T_t\| \leq C t^{-\operatorname{Re} \alpha}, \quad t > 0,$$

with a constant C , then A is of type $(\omega, M(\theta))$ for an $\omega < \pi/2$.

Proof. Since A_+^α commutes with T_t , $A_+^\alpha T_t x = T_{t/2} A_+^\alpha T_{t/2} x$ is still in $D(A_+^\alpha)$ and we have $(A_+^\alpha)^2 T_t x = A_+^{2\alpha} T_t x = (A_+^\alpha T_{t/2})^2 x$. Similarly we have $T_t x \in D(A_+^{n\alpha})$ and $A_+^{n\alpha} T_t x = (A_+^\alpha T_{t/n})^{n\alpha} x$ for any integer $n > 0$. Thus it follows from (12.4) that

$$\|A_+^{n\alpha} T_t\| \leq (Cn)^{n\operatorname{Re} \alpha} t^{-n\operatorname{Re} \alpha}, \quad t > 0.$$

Taking an n such that $\operatorname{Re} n\alpha > 1$ and applying Theorem 8.1, we obtain

the estimate

$$\|AT_t\| \leq C_1 t^{-1}, \quad t > 0,$$

which is a necessary and sufficient condition that A is of type $(\omega, M(\theta))$ with $\omega < \pi/2$ (Yosida [21], Komatsu [12]).

REMARK. If condition (12.4) is satisfied only for small t , it follows that for every $\varepsilon > 0$, $A + \varepsilon$ is of type $(\omega, M(\theta))$ with $\omega < \pi/2$ because the same calculation leads to

$$\|(A + \varepsilon)e^{-\varepsilon t}T_t\| \leq C_2 t^{-1}, \quad t > 0.$$

THEOREM 12.3. Let A and T_t be as in Theorem 12.1 and let $\sigma > 0$ be not an integer. Then the following conditions for x are equivalent.

(i) For each $\varepsilon > 0$, there is a constant C independent of t in the sector $|\arg t| \leq \pi/2 - \omega - \varepsilon$ such that

$$(12.5) \quad \|T_t A^n x - A^n x\| \leq C |t|^\rho.$$

(ii) $x \in C^\sigma$.

(iii) $x \in D^\sigma$.

(iv) For each $\operatorname{Re} \alpha > \sigma$ and $\varepsilon > 0$ there is a constant C such that

$$(12.6) \quad \|A_+^\alpha T_t x\| \leq C |t|^{\sigma - \operatorname{Re} \alpha},$$

for $|\arg t| \leq \pi/2 - \omega - \varepsilon$.

(v) (12.6) holds for an α , $\operatorname{Re} \alpha > \sigma$, and sufficiently small $t > 0$.

Proof. Implication (i) \Rightarrow (ii) \Rightarrow (iii) and (iv) \Rightarrow (v) are immediate. (iii) \Rightarrow (iv). We may assume that $0 < \sigma < 1$ without loss of generality. If $x \in D^\sigma$, then the estimate

$$(12.7) \quad \|(\lambda + A)^{-1}x - \lambda^{-1}x\| \leq C |\lambda|^{-\sigma-1}$$

holds uniformly on each sector $|\arg \lambda| \leq \pi - \omega - \varepsilon$, $\varepsilon > 0$. In fact, the resolvent equation gives

$$\lambda((\lambda + A)^{-1} - \lambda^{-1})x = (1 + (\mu - \lambda)(\lambda + A)^{-1})\mu((\mu + A)^{-1} - \mu^{-1})x.$$

If λ is in the sector $|\arg \lambda| \leq \pi - \omega - \varepsilon$, we can choose a $\mu > 0$ such that $C_1 |\lambda| \leq \mu \leq C_2 |\lambda|$ with constants C_i independent of λ . Thus we have

$$\|(\lambda + A)^{-1}x - \lambda^{-1}x\| \leq (1 + (C_2 + 1)M(\pi - \omega - \varepsilon))C_1^{-\sigma} |\lambda|^{-\sigma-1} q_\sigma(x).$$

Since $\int_r (-\lambda)^\alpha e^{t\lambda} \lambda^{-1} d\lambda = 0$, it follows from Theorem 12.1 that

$$\begin{aligned} \| A_+^\alpha T_t x \| &= \left\| \frac{1}{2\pi i} \int_r (-\lambda)^\alpha e^{t\lambda} (\lambda + A)^{-1} - \lambda^{-1} x d\lambda \right\| \\ &\leq C_3 \int_0^\infty r^{\operatorname{Re} \alpha - \sigma - 1} e^{-\sigma |t| r} dr \\ &= C_4 |t|^{\sigma - \operatorname{Re} \alpha} . \end{aligned}$$

(v) \Rightarrow (ii). First we note that if (12.6) holds for an α , then it holds for every α with a greater real part. For, if $\operatorname{Re} \beta > 0$, then

$$\| A_+^{\alpha + \beta} T_t x \| \leq \| A_+^\beta T_{t/2} \| \| A_+^\alpha T_{t/2} x \| \leq C t^{\sigma - \operatorname{Re}(\alpha + \beta)} .$$

Therefore we have

$$(12.8) \quad \| (-A)^m T_t x \| \leq C t^{\sigma - m} , \quad 0 < t < t_0 ,$$

for an integer $m > \sigma$. Since $(-A)^m T_t x$ is the m th derivative of $T_t x$, it follows that

$$(12.9) \quad \begin{aligned} \| (-A)^{m-1} T_t x - (-A)^{m-1} T_s x \| &= \left\| \int_s^t (-A)^m T_r x dr \right\| \\ &\leq C |\sigma - m + 1|^{-1} |t^{\sigma - m + 1} - s^{\sigma - m + 1}| , \quad t, s \in (0, t_0) . \end{aligned}$$

If $m - \sigma > 1$, we have (12.8) with m replaced by $m - 1$. If $m - \sigma < 1$, then (12.9) shows that $(-A)^{m-1} T_t x$ converges as $t \rightarrow 0$, so that x belongs to $D(A^{m-1})$. Letting $s \rightarrow 0$ in (12.9), we obtain also

$$\| T_t (-A)^{m-1} x - (-A)^{m-1} x \| \leq C |\sigma - m + 1|^{-1} t^{\sigma - m + 1} .$$

This proves $x \in C^\sigma$.

The implication (iv) \Rightarrow (i) is proved similarly.

Even when σ is an integer > 0 , the equivalence of (iv) and (v) is proved in the same way.

DEFINITION. When $-A$ generates an analytic semi-group, let us denote by B^σ , $\sigma > 0$, the set of all x such that (v) (or (iv)) of Theorem 12.3 is satisfied.

Then Theorems 7.6 and 11.6 are unified in the following way.

THEOREM 12.4. *Let A be as above. If $\sigma > \operatorname{Re} \alpha > 0$, then A_+^α maps B^σ into $B^{\sigma - \operatorname{Re} \alpha}$. If $\sigma > 0$ and $\operatorname{Re} \alpha \leq 0$, then A_0^α maps $B^\sigma \cap D(A_0^\alpha)$ into $B^{\sigma - \operatorname{Re} \alpha}$. Note that $\sigma - \operatorname{Re} \alpha$ can be an integer.*

Proof. Let $x \in B^\sigma \cap D(A^\alpha)$. If $\operatorname{Re} \beta > \sigma - \operatorname{Re} \alpha$, we have by Theorems 7.1 or 7.3

$$\begin{aligned} \| A_+^\beta T_t A_*^\alpha x \| &= \| A_0^\beta A_*^\alpha T_t x \| \\ &= \| A_+^{\alpha + \beta} T_t x \| \\ &\leq C |t|^{\sigma - \operatorname{Re}(\alpha + \beta)} . \end{aligned}$$

13. **Miscellaneous results.** In this section we give a few results and applications related to the fractional powers of operators with and without proofs.

THEOREM 13.1. *Let $X \supset Y$ be two Banach spaces such that the topology of Y is stronger than that of X . (By the closed graph theorem it is enough to assume that if $x_j \rightarrow x$ in X and $x_j \rightarrow y$ in Y , then $x = y$.) If a closed linear operator A in X and its maximal restriction B in Y (see Section 1) satisfy assumption (1.5) in X and Y respectively, and if $D(B)$ is dense in Y , then B_+^α is the maximal restriction of A_+^α in Y . If in particular Y is a closed subspace of X , then the maximal restriction is the simple restriction to Y .*

Proof. It follows from the assumptions that $(\lambda + A)^{-1}$, $\lambda > 0$, maps Y into Y and

$$(13.1) \quad (\lambda + B)^{-1} = (\lambda + A)^{-1}|_Y, \quad \lambda > 0.$$

Hence we have

$$(13.2) \quad B_+^\alpha \lambda^n (\lambda + B)^{-n} = A_+^\alpha \lambda^n (\lambda + A)^{-n}|_Y, \quad n > \operatorname{Re} \alpha.$$

Let $y \in D(B_+^\alpha)$. Then Proposition 4.5 shows

$$B_+^\alpha \lambda^n (\lambda + B)^{-n} y \rightarrow B_+^\alpha y \text{ in } Y \text{ as } \lambda \rightarrow \infty.$$

so that we have

$$A_+^\alpha \lambda^n (\lambda + A)^{-n} y \rightarrow B_+^\alpha y \text{ in } X.$$

Since $y \in \overline{D(B)} \subset \overline{D(A)}$, this proves that $y \in D(A_+^\alpha)$ and $A_+^\alpha y = B_+^\alpha y$.

Conversely let $y \in D(A_+^\alpha) \cap Y$ and $A_+^\alpha y \in Y$. We have

$$\begin{aligned} B_+^\alpha \lambda^n (\lambda + B)^{-n} y &= A_+^\alpha \lambda^n (\lambda + A)^{-n} y \\ &= \lambda^n (\lambda + A)^{-n} A_+^\alpha y \\ &= \lambda^n (\lambda + B)^{-n} A_+^\alpha y. \end{aligned}$$

Since $D(B)$ is dense in Y , it follows that

$$(13.3) \quad B_+^\alpha \lambda^n (\lambda + B)^{-n} y \rightarrow A_+^\alpha y \text{ in } Y \text{ as } \lambda \rightarrow \infty.$$

Therefore we have $y \in D(B_+^\alpha)$ and $B_+^\alpha y = A_+^\alpha y$.

If Y is a closed subspace, (13.3) holds without the assumption that $A_+^\alpha y \in Y$. Thus the simple restriction gives the maximal restriction.

The assumptions of Theorem 13.1 are satisfied if $-A$ is the infinitesimal generator of a bounded semi-group T_t in X , Y is invariant under T_t and the restriction $S_t = T_t|_Y$ forms a bounded semi-group in Y . In this case $-B$ coincides with the infinitesimal generator of S_t .

Let us denote by $c_0(\mathbf{Z})$ the space of all sequences $x = (\xi_k)$, $k = 0, \pm 1, \pm 2, \dots$, such that $\xi_k \rightarrow 0$ as $|k| \rightarrow \infty$ and normed by $\|x\| = \sup |\xi_k|$. Then each of $L^p(T)$, $1 \leq p < \infty$, and $C(T)$ on the unit circle T is embedded in $c_0(\mathbf{Z})$ by the Fourier transformation. The translation semi-group S_t :

$$(13.4) \quad S_t x(e^{is}) = x(e^{i(s-t)})$$

becomes the restriction of the semi-group T_t defined by

$$(13.5) \quad T_t(\xi_k) = (e^{-itk} \xi_k), \quad (\xi_k) \in c_0(\mathbf{Z}).$$

We denote by $-A$ and $-d/ds$ the infinitesimal generators of T_t and S_t respectively. By a simple computation it is shown that

$$A_+^\alpha(\xi_k) = ((ik)^\alpha \xi_k), \\ D(A_+) = \{(\xi_k) \in c_0(\mathbf{Z}); ((ik)^\alpha \xi_k) \in c_0(\mathbf{Z})\}.$$

Thus we have:

PROPOSITION 13.2. Let X be any one of $L^p(T)$, $1 \leq p < \infty$, and $C(T)$. Then a function

$$x(e^{is}) \sim \sum_{k=-\infty}^{\infty} c_k e^{iks} \in X$$

belongs to $D((d/ds)_+^\alpha)$ if and only if the formal sum

$$\left(\frac{d}{ds}\right)^\alpha x(e^{is}) \sim \sum_{k=-\infty}^{\infty} c_k (ik)^\alpha e^{iks}$$

represents a function in X .

The following two theorems may be proved in the same way as Theorem 13.1.

THEOREM 13.3. Let X and Y be Banach spaces, and T be a continuous linear mapping from X to Y . If closed linear operators A and B satisfy

$$(13.6) \quad BTx = TAx, \quad x \in D(A)$$

and assumption (1.5) in X and in Y , respectively, then we have

$$(13.7) \quad B_*^\alpha Tx = TA_*^\alpha x, \quad x \in D(A_*^\alpha),$$

where A_*^α has the same subscript as B_*^α .

THEOREM 13.4 (Love-Young [14]). Let X be a Banach space and let X' be its dual space. If an operator A in X satisfies (1.5) and

has a dense domain, then its transposed operator $B = A'$ also satisfies (1.5) and B_+^α is the maximal restriction of $(A_+^\alpha)'$ in $\overline{D(B)}$. If, moreover, $R(A)$ is dense, then B_-^α and B_0^α are the maximal restrictions of $(A_0^\alpha)'$ in $\overline{R(B)}$ and $\overline{D(B)} \cap \overline{R(B)}$, respectively.

Let us apply Theorem 13.3 to the regularity of the convolution of functions. Let A be the negative of the generator of translation semi-group

$$T_t x(s) = x(s - t)$$

in $B(-\infty, \infty)$, and let B be the corresponding operator in $L^1(-\infty, \infty)$.

PROPOSITION 13.5. If $x(s) \in D(A_+^\alpha)$ and $y(s) \in D(B_+^\beta)$, then the convolution $x * y(s) \in D(A_+^{\alpha+\beta})$.

Proof. The mapping $T: B(-\infty, \infty) \times L^1(-\infty, \infty) \rightarrow B(-\infty, \infty)$ defined by $T(x, y) = x * y$ is a continuous bilinear mapping, and we have

$$\begin{aligned} A(x * y) &= Ax * y, \quad x \in D(A), \quad y \in L^1(-\infty, \infty), \\ A(x * y) &= x * By, \quad x \in B(-\infty, \infty), \quad y \in D(B). \end{aligned}$$

Therefore it follows from Theorem 13.3 that if $x \in D(A_+^\alpha)$ and $y \in L^1(-\infty, \infty)$, then $x * y \in D(A_+^\alpha)$ and $A_+^\alpha(x * y) = (A_+^\alpha x) * y$. If, moreover, $y \in D(B_+^\beta)$, then we have $(A_+^\alpha x) * y \in D(A_+^\beta)$ and $A_+^\beta((A_+^\alpha x) * y) = (A_+^\alpha x) * (B_+^\beta y)$, so that $x * y$ belongs to $D(A_+^{\alpha+\beta})$.

COROLLARY 13.6. Let $x(s) \in B(-\infty, \infty)$ be uniformly Hölder continuous with exponent $0 < \alpha < 1$ and let $y(s) \in C_0(-\infty, \infty)$ be uniformly Hölder continuous with exponent $0 < \beta < 1$. If $\alpha + \beta > 1$, then $x * y(s)$ is differentiable and the derivative is uniformly Hölder continuous with exponent $\alpha + \beta - 1 - \varepsilon$, for an arbitrary $\varepsilon > 0$. If $\alpha + \beta \leq 1$, then $x * y(s)$ is uniformly Hölder continuous with exponent $\alpha + \beta - \varepsilon$, $\varepsilon > 0$.

THEOREM 13.7. Let

$$A = \int_{\mathcal{C}} \zeta dE(\zeta)$$

be a normal operator in a Hilbert space X . Then A is of type $(\omega, M(\theta))$ if and only if the spectrum $\sigma(A)$ is contained in the sector $|\arg \lambda| \leq \omega$. If this is the case, $M(\theta)$ satisfies

$$M(\theta) \leq \begin{cases} 1 & , \quad \theta \leq \pi/2 - \omega \\ \operatorname{cosec}(\omega + \theta) & , \quad \theta > \pi/2 - \omega . \end{cases}$$

In particular, a normal operator is the negative of the generator of a bounded semi-group if and only if A is of type $(\pi/2, M(\theta))$.

In order that an element x belong to D^σ it is necessary and sufficient that

$$q_\sigma(x)^2 = \sup_{0 < \lambda < \infty} \lambda^{2\sigma} \int |\lambda + \zeta|^{-2} |\zeta|^{2n+2} d \|E(\zeta)x\|^2 < \infty .$$

When $-A$ generates a semi-group, D^σ coincides with C^σ .

The fractional power A^α is given by

$$A_+^\alpha = \int_C \zeta^\alpha dE(\zeta) ,$$

$$A_0^\alpha = \int_{C-\{0\}} \zeta^\alpha dE(\zeta) .$$

The domain $D(A_+^\alpha)$ consists of the elements x such that

$$\|A_+^\alpha x\| = \int_C |\zeta^\alpha|^2 d \|E(\zeta)x\|^2 < \infty .$$

If $\operatorname{Re} \alpha = \operatorname{Re} \beta$, then $\|A_+^\alpha x\|$ is equivalent to $\|A_+^\beta x\|$ and $D(A_+^\alpha) = D(A_+^\beta)$. However, $D(A_+^\alpha)$ does not coincide with $D^{\operatorname{Re} \alpha}$ if A is unbounded and $\operatorname{Re} \alpha$ is not an integer.

Proof. We prove only the last statement. We have

$$\lambda^{2\rho} |\lambda + \zeta|^{-2} |\zeta|^2 \leq C |\zeta|^{2\rho}, \quad \lambda > 0, \quad |\arg \zeta| \leq \omega$$

with a constant C depending only on $0 < \rho < 1$ and ω . Thus if $x \in D(A_+^\sigma)$ for a $\sigma = n + \rho > 0$, we have

$$q(\lambda, x) = \int \lambda^{2\rho} |\lambda + \zeta|^{-2} |\zeta|^{2n+2} d \|E(\zeta)x\|^2 \leq C \|A_+^\sigma x\|^2 .$$

Further it follows from Lebesgue's theorem that

$$q(\lambda, x) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0 .$$

If A is unbounded, however, it is not difficult to construct an element x or a measure $d \|E(\zeta)x\|^2$ such that

$$q(\lambda, x) \leq C \quad \text{and} \quad q(\lambda, x) \rightarrow 0 .$$

This completes the proof.

Let us denote by H the space of functions $x(z)$ which are continuous on the unit disk $|z| \leq 1$ and analytic in the open disk $|z| < 1$, and by H^p , $1 \leq p < \infty$, the Hardy space $\subset L^p(T)$ (Hoffman [6]). Then the translation (or rotation) group S_t restricted to H or H^p becomes the boundary group of the analytic semi-group

$$(13.8) \quad S_t x(e^{is}) = x(e^{\text{Im } t} e^{i(s - \text{Re } t)}), \quad \text{Im } t \leq 0.$$

$-id/ds$, which is the negative of the infinitesimal generator of S_t in the direction of the negative imaginary axis, satisfies the conditions of Section 12, so that if $\sigma > 0$ is not an integer we have

$$D^\sigma \left(\frac{1}{i} \frac{d}{ds} \right) = C^\sigma \left(\frac{1}{i} \frac{d}{ds} \right) = B^\sigma \left(\frac{1}{i} \frac{d}{ds} \right).$$

On the other hand, we have shown in the proof of Theorem 12.3 that $D^\sigma(-id/ds) = D^\sigma(d/ds)$. $C^\sigma(-id/ds)$ coincides with $C^\sigma(d/ds)$ because of the identity

$$S_{2h} - 1 = (S_{(1-i)h} - 1) - S_{2h}(S_{(-1-i)h} - 1)$$

and the equivalence of (i) and (ii) in Theorem 12.3. Further it follows from the identity

$$\frac{1}{i} \frac{d}{ds} x(re^{is}) = re^{is} \frac{d}{dz} x(re^{is})$$

that $\| -idx/ds \|$ is equivalent to $\| dx/dz \|$ if r is near 1. Thus we have proved the following theorem.

THEOREM 13.8 (Hardy-Littlewood [5]). *Let X be H or H^p , $1 \leq p < \infty$, and let $0 < \sigma < 1$. Then the following conditions for $x \in X$ are equivalent.*

- (i) $\left\| \int_0^\infty \lambda e^{-\lambda t} (x(e^{i(s-t)}) - x(e^{is})) dt \right\| = O(\lambda^{-\sigma}), \lambda \rightarrow \infty.$
- (ii) $\| x(e^{i(s-h)}) - x(e^{is}) \| = O(h^\sigma), h \rightarrow 0.$
- (iii) $\| x(re^{is}) - x(e^{is}) \| = O((1-r)^\sigma), r \rightarrow 1.$
- (iv) $\left\| \frac{d}{dz} x(re^{is}) \right\| = O((1-r)^{\sigma-1}), r \rightarrow 1.$
- (v) $x \in D^\sigma \left(\frac{1}{i} \frac{d}{ds} \right).$

If $1 < p < \infty$, M. Riesz's theorem (Zygmund [24], Hoffman [6]) states that the natural projection P from $L^p(T)$ to H^p is continuous. Since P is commutative with $(\lambda + d/ds)^{-1}$, we obtain the following theorem.

THEOREM 13.9 (Hardy-Littlewood [5]). *If $1 < p < \infty$ and $0 < \sigma < 1$, then the following are equivalent for $x \in L^p(T)$.*

- (i) $x \in D^\sigma(d/ds).$
- (ii) $x \in C^\sigma(d/ds).$
- (iii) $x = x_1 + \bar{x}_2$, where $x_i \in H^p$ satisfy one of the conditions of Theorem 13.8.

Similarly Privalov's theorem (Zygmund [24]) proves the following theorem.

THEOREM 13.10. *A function $x(e^{is}) \in C(T)$ is uniformly Hölder continuous with exponent $0 < \sigma < 1$ if and only if there are two functions $x^i \in H$ satisfying the conditions of Theorem 13.8 such that*

$$x(e^{is}) = x_1(e^{is}) + \overline{x_2(e^{is})} .$$

Theorem 9.2 may be applied to the theory of partial differential equations.

Let $A(x, \partial/\partial x)$ be a strongly elliptic linear partial differential operator of order 2 in a domain $\Omega \subset R^n$ and let $B(x, \partial/\partial x)$ be a linear partial differential operator on $\partial\Omega$. It is known that if $A(x, \partial/\partial x)$, $B(x, \partial/\partial x)$ and the domain Ω satisfy certain conditions, then the smallest closed extension A in $L^p(\Omega)$, $1 < p < \infty$, of the operator $A(x, \partial/\partial x)$ restricted to the space $\{u \in C^\infty(\Omega); Bu = 0\}$ satisfies:

$$D(A) \subset W_{L^p}^2(\Omega) ,$$

$$(13.9) \quad \|\nabla^2 u\|_{L^p} + \|u\|_{L^p} \leq C_1 \|Au\|_{L^p} , \quad u \in D(A) ,$$

and $-A$ generates an analytic semi-group.

Let us prove that if $n/p < 3$, there are constants $0 < \alpha < 1$ and $C > 0$ such that

$$(13.10) \quad \|\nabla u \cdot u\|_{L^p} \leq C \|A_+^\alpha u\|_{L^p}^2 , \quad u \in D(A_+^\alpha) ,$$

$$(13.11) \quad \|\nabla u \cdot u - \nabla v \cdot v\|_{L^p} \leq C(\|A_+^\alpha u\|_{L^p} + \|A_+^\alpha v\|_{L^p}) \|A_+^\alpha(u - v)\|_{L^p} , \\ u, v \in D(A_+^\alpha) .$$

Obviously we have

$$\|\nabla u \cdot u\|_{L^p} \leq \|\nabla u\|_{L^r} \|u\|_{L^s} ,$$

$$\|\nabla u \cdot u - \nabla v \cdot v\|_{L^p} \leq \|\nabla u\|_{L^r} \|u - v\|_{L^s} + \|\nabla u - \nabla v\|_{L^r} \|v\|_{L^s} ,$$

where $p^{-1} = r^{-1} + s^{-1}$, $r, s \geq 1$. Let

$$l = \max \left\{ \frac{1}{4} \left(\frac{n}{p} + 1 \right), \frac{1}{2} \right\}$$

and choose suitable r and s . Then it follows from the Sobolev-Gagliardo-Nirenberg inequality (Nirenberg [15]) that

$$\|\nabla u\|_{L^r} \leq C(\|\nabla^2 u\|_{L^p} + \|u\|_{L^p})^l \|u\|_{L^p}^{1-l} , \\ \|u\|_{L^s} \leq C(\|\nabla^2 u\|_{L^p} + \|u\|_{L^p})^l \|u\|_{L^p}^{1-l} .$$

Combining these inequalities with (13.9) and applying Theorem 9.2, we

obtain (13.10) and (13.11).

Once inequalities (13.10) and (13.11) are established, the local existence theorem for the Cauchy problem of the nonlinear equation:

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= A\left(x, \frac{\partial}{\partial t}\right)u + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} \cdot u, \quad x \in \Omega, \\ B\left(x, \frac{\partial}{\partial x}\right)u(t, x) &= 0, \quad x \in \partial\Omega, \\ u(0, x) &= u_0(x) \in D(A_+^\alpha), \end{aligned}$$

is proved by the method of Fujita (Fujita-Kato [3], Itô [8]).

14. Counterexamples. This section is devoted to various counterexamples which show that the statements of some theorems cannot be improved in general.

1. If X is not reflexive,

$$(14.1) \quad (\lambda + A)^{-1}x - \lambda^{-1}x = O(\lambda^{-2})$$

does not necessarily imply that $x \in D(A)$. Let $A = -d/ds$ in $B[0, \infty)$. If $x \in B[0, \infty)$ is uniformly Lipschitz continuous, then we have

$$\begin{aligned} |((\lambda + A)^{-1} - \lambda^{-1})x(s)| &= \left| \int_0^\infty e^{-\lambda t} (x(s+t) - x(s)) dt \right| \\ &\leq C \int_0^\infty t e^{-\lambda t} dt = C\Gamma(2)\lambda^{-2}. \end{aligned}$$

Thus x satisfies (14.1), but it is not always differentiable.

On the other hand, by the same computation, we obtain the following theorem as a corollary of Theorem 2.1.

THEOREM 14.1. *Let X be a reflexive Banach space and let $-A$ be the generator of a bounded semi-group T_t in X . Then $x \in D(A)$ if and only if*

$$(14.2) \quad T_t x - x = O(t) \quad \text{as } t \rightarrow 0.$$

In particular a Lipschitz continuous function in $L^p(a, b)$, $1 < p < \infty$, is differentiable in $L^p(a, b)$ (Hardy-Littlewood [5]).

2. The domain $D(A)$ is not necessarily dense even if A satisfies (1.5). In the space m of all bounded sequences, the operator A ,

$$A(\xi_k) = (l\xi_k), \quad D(A) = \{x \in m; Ax \in m\},$$

gives such an operator. $\overline{D(A)}$ coincides with c_0 .

3. (3.6) is not true in general. Let $A = -d/ds$ in $B[0, \infty)$. Evidently the null space $N(A)$ is composed of all constant functions. Whereas for any constant c ,

$$x(s) = \sin \sqrt{s} + c$$

does not satisfy

$$(14.3) \quad \sup_{s \geq 0} \int_s^\infty \lambda e^{\lambda(s-t)} x(t) dt \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

so that $\sin \sqrt{s}$ does not belong to $N(A) + \overline{R(A)}$.

It should be remarked that by a Tauberian theorem (14.3) is equivalent to

$$(14.4) \quad \sup_{s \geq 0} \left| \int_s^{s+N} x(t) dt \right| = o(N) \quad \text{as } N \rightarrow \infty.$$

4. The Weierstrass-Hardy function

$$(14.5) \quad x(e^{is}) = \sum_{k=1}^\infty b^{-\rho k} e^{ib^k s}, \quad b = 2, 3, \dots, \quad 0 < \rho < 1,$$

belongs to $C^\rho(d/ds)$ in H and hence in any one of $C(T)$, H^p and $L^p(T)$, $1 \leq p < \infty$ (Hardy [4]), but its formal fractional derivative

$$\left(\frac{d}{ds}\right)_+^\rho x(s) \sim \sum_{k=1}^\infty i^{\rho k} e^{ib^k s}$$

is not even in $L^1(T)$, so that x does not belong to $D((d/ds)_+^\rho)$ in any of the spaces H , $C(T)$, H^p and $L^p(T)$. Since

$$\begin{aligned} \left\| \frac{1}{i} \frac{d}{ds} \left(\lambda + \frac{1}{i} \frac{d}{ds} \right)^{-1} x \right\|_H &= \left\| \sum_{k=1}^\infty \frac{b^{(1-\rho)k}}{\lambda + b^k} e^{ib^k s} \right\| \\ &\leq \sum_{k=1}^\infty \frac{b^{(1-\rho)k}}{\lambda + b^k} \leq \left(\frac{1}{1 - b^{-\rho}} + \frac{1}{1 - b^{\rho-1}} \right) \lambda^{-\rho}, \end{aligned}$$

the first statement follows from Theorem 13.8.

5. If σ is an integer, B^σ may be wider than D^σ . The function $(1 - z)(\log(1 - z) - 1)$ belongs to $B^1(-id/ds)$ in H , but clearly it does not belong to $D^1(-id/ds)$.

6. Contrary to the case of normal operators, $D(A_+^\alpha)$ may be different from $D(A_+^\beta)$ even if $\text{Re } \alpha = \text{Re } \beta$.

Let S be the shift operator in c_0 :

$$(14.6) \quad S(\xi_1, \xi_2, \xi_3 \dots) = (0, \xi_1, \xi_2, \dots),$$

and let

$$(14.7) \quad \begin{aligned} A &= (1 - S)^{-1} = 1 + S + S^2 + S^3 + \dots, \\ A(\xi_1, \xi_2, \xi_3, \dots) &= (\xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \xi_3, \dots). \end{aligned}$$

The domain $D(A)$ is the set of all sequences (ξ_k) such that $\sum_{k=1}^{\infty} \xi_k = 0$, which forms a dense subspace. The resolvent is given by

$$(14.8) \quad (\lambda + A)^{-1} = \frac{1}{\lambda + 1} - \frac{1}{(\lambda + 1)^2} \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda + 1} \right)^{k-1} S^k, \quad \operatorname{Re} \lambda > 0.$$

Since the norm of an operator P of the form

$$P = \sum_{k=0}^{\infty} c_k S^k$$

with a dense domain is given by

$$(14.9) \quad \|P\| = \sum_{k=0}^{\infty} |c_k|,$$

the resolvent is estimated by

$$(14.10) \quad \begin{aligned} \|(\lambda + A)^{-1}\| &= \frac{1}{|\lambda + 1|} (1 + (|\lambda + 1| - |\lambda|)^{-1}) \\ &\leq 2 \left(\operatorname{Re} \lambda + \frac{1}{2} \right)^{-1}, \quad \operatorname{Re} \lambda > 0. \end{aligned}$$

Therefore A is an operator of type $(\pi/2, 2 \sec \theta)$.

By an easy calculation it is proved that

$$(14.11) \quad A^\alpha = 1 + \alpha S + \frac{\alpha(\alpha + 1)}{2} S^2 + \frac{\alpha(\alpha + 1)(\alpha + 2)}{3!} S^3 + \dots$$

for every complex number α .

Since A^{-1} is bounded, A^α is bounded for $\operatorname{Re} \alpha < 0$, and clearly $A^0 = 1$ is bounded. However, the pure imaginary power A^{it} is unbounded for any $t \neq 0$, because we have

$$\begin{aligned} \|A^{it}\| &= 1 + |it| + \left| \frac{it(it + 1)}{2} \right| + \left| \frac{it(it + 1)(it + 2)}{3!} \right| + \dots \\ &\geq 1 + |t| + \frac{1}{2}|t| + \frac{1}{3}|t| + \dots = \infty. \end{aligned}$$

More generally let us prove that

$$(14.12) \quad D(A^{\sigma+it}) \neq D(A^\sigma), \quad \sigma > 0, \quad t > 0.$$

If $D(A^{\sigma+it}) \supset D(A^\sigma)$, there would be a constant C such that

$$(14.13) \quad \| A^{\sigma+it}x \| \leq C(\| A^\sigma x \| + \| x \|), \quad x \in D(A^\sigma) .$$

Since A^{-1} is bounded and $D(A)$ is dense, it follows from Corollary 5.3 that $D(A^\sigma) = R(A^{-\sigma})$ and that there is a constant C_1 such that

$$(14.14) \quad \| A^\sigma x \| + \| x \| \leq C_1 \| y \|, \quad \text{for } x = A^{-\sigma}y .$$

Let y_j be a sequence in $D(A^{it})$ such that

$$(14.15) \quad \| y_j \| \leq 1, \quad \| A^{it}y_j \| \geq j, \quad j = 1, 2, \dots .$$

Theorem 7.3 proves that $A^{-\sigma}y_j \in D(A^{\sigma+it})$ and $A^{\sigma+it}A^{-\sigma}y_j = A^{it}y_j$. Thus we have by (14.13) and (14.14)

$$\| A^{it}y_j \| = \| A^{\sigma+it}A^{-\sigma}y_j \| \leq CC_1 \| y_j \| ,$$

which contradicts (14.15).

7. In the statement of Theorem 9.2 “ $\text{Re } \alpha > \theta \text{ Re } \beta$ ” cannot be replaced by “ $\text{Re } \alpha \geq \theta \text{ Re } \beta$ ”. In view of Theorem 8.1 we have

$$\| A^{\sigma+it}x \| \leq C \| Ax \|^\sigma \| x \|^{1-\sigma}, \quad x \in D(A), \quad 0 < \sigma < 1 .$$

But (14.13) does not necessarily hold.

8. Although the operator A in 6 is of type $(\pi/2, 2 \sec \theta)$, $-A$ is not an infinitesimal generator of any semi-group of class (A).

If $-A$ generates a semi-group $\exp(-tA)$, then we have

$$\exp(-tA)x = \frac{1}{2\pi i} (C, 1) - \int_{-i\infty}^{i\infty} e^{\lambda t} (\lambda + A)^{-1} x d\lambda, \quad x \in D(A^2) .$$

(See Hille-Phillips [7].) Since $D(A^2)$ is dense, and $(\lambda + A)^{-1}$ has expansion (14.8), it follows that $\exp(-tA)$ must have the form

$$\exp(-tA) = \exp(-t) + \sum_{k=1}^{\infty} f_k(t) S^k ,$$

where

$$\begin{aligned} f_k(t) &= \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\lambda t} \lambda^{k-1} (\lambda + 1)^{-k-1} d\lambda \\ &= \sum_{h=0}^{k-1} (-1)^{h+1} \binom{k-1}{h} \frac{t^{h+1}}{(h+1)!} \exp(-t) . \end{aligned}$$

If $\exp(-tA)$ is bounded, $\sum_{k=1}^{\infty} |f_k(t)|$ is finite, so that the function $\psi(z) = \sum_{k=1}^{\infty} f_k(t) z^k$ must be continuous on the disk $|z| \leq 1$. We have, however,

$$\begin{aligned}
\psi(z) &= \sum_{h=0}^{\infty} (-1)^{h+1} \frac{t^{h+1}}{(h+1)!} \exp(-t) \sum_{k=h+1}^{\infty} \binom{k-1}{h} z^k \\
&= \sum_{h=0}^{\infty} \left(\frac{-z}{1-z} \right)^{h+1} \frac{t^{h+1}}{(h+1)!} \exp(-t) \\
&= \left(\exp\left(\frac{-zt}{1-z}\right) - 1 \right) \exp(-t).
\end{aligned}$$

This shows that $\psi(z)$ is discontinuous at $z = 1$.

An example of operators of this kind has been given by Phillips (Hille-Phillips [6]), but this example is more interesting because $-A^{-1} = -1 + S$ generates a continuous contraction semi-group

$$\begin{aligned}
\exp(-tA^{-1}) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \exp(-t) S^k, \\
\|\exp(-tA^{-1})\| &= \exp(|t| - \operatorname{Re} t).
\end{aligned}$$

9. Even if $-A$ generates a contraction semi-group,

$$\|A_{\dagger}^{\gamma} x\| \leq \|Ax\|^{\gamma} \|x\|^{1-\gamma}, \quad 0 < \gamma < 1, \quad x \in D(A)$$

does not necessarily hold. Let $A = 1 - S$, $\gamma = 1/2$ and $x = (1, 2, 1, 0, 0, \dots)$. Then we have

$$\begin{aligned}
Ax &= (1, 1, -1, -1, 0, 0, \dots), \\
A^{1/2}x &= (1, 3/2, -1/8, \dots).
\end{aligned}$$

Thus

$$\|A^{1/2}x\| = 3/2 > \|Ax\|^{1/2} \|x\|^{1/2} = \sqrt{2}.$$

10. If X is not sequentially weakly complete, Theorem 11.5 may be false without assumption (11.11). The semi-group

$$T_t(\xi_k) = (e^{-kt}\xi_k), \quad (\xi_k) \in c_0$$

has the infinitesimal generator

$$-A(\xi_k) = (-k\xi_k), \quad D(A) = \{(\xi_k) \in c_0; A(\xi_k) \in c_0\}.$$

The element $x = (1, 1/2, 1/3, \dots)$ does not belong to $D(A)$, but

$$\langle T_s x, y' \rangle = \sum_{k=1}^{\infty} e^{-ks} \eta_k / k$$

is differentiable for every $y' = (\eta_k) \in l^1$ and

$$-\frac{d}{ds} \langle T_s x, y' \rangle = \sum_{k=1}^{\infty} e^{-ks} \eta_k \in B[0, \infty).$$

REMARKS ADDED ON REVISION. Suppose $\sigma > 0$ is not an integer. Let us denote by $D_*^\sigma(C_*^\sigma)$ the set of all $x \in D^\sigma(C^\sigma)$ which satisfies stricter estimate

$$R_n(\lambda, x) = o(\lambda^{-\sigma-1}), \lambda \rightarrow \infty$$

$$(T_t A^n x - A^n x = o(t^\rho), t \rightarrow 0).$$

Then we have

$$D_*^\sigma \supset D(A_+^\alpha), \operatorname{Re} \alpha = \sigma$$

and, if $-A$ generates a bounded semi-group,

$$D_*^\sigma \supset C_*^\sigma \supset D(A_+^\alpha), \operatorname{Re} \alpha = \sigma.$$

The inclusion $D_*^\sigma \supset C_*^\sigma$ is proved in the same way as in the proof of Proposition 11.2. Since $D_*^\sigma(C_*^\sigma)$ is a closed subspace of $D^\sigma(C^\sigma)$ with the norm $\|x\| + q_\sigma(x)$ ($\|x\| + p_\sigma(x)$) and contains every $D^\tau(C^\tau)$ with $\tau > \sigma$, the other inclusions follow from the estimates

$$\|x\| + q_\sigma(x) \leq C(\|x\| + \|A_+^\alpha x\|),$$

$$\|x\| + p_\sigma(x) \leq C(\|x\| + \|A_+^\alpha x\|)$$

and the definition of A_+^α .

$$D_*^{\operatorname{Re} \alpha} = C_*^{\operatorname{Re} \alpha} = D(A_+^\alpha)$$

holds for the operator A of Example 10 of Section 14.

In the space D_*^σ the approximation theorem (Lemma 2.4) holds in the norm topology. Thus if we used D_*^σ instead of D^σ we could avoid the strange topology of D^σ .

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