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TWO THEOREMS ON METRIZABILITY OF MOORE SPACES

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One of the outstanding questions in point set topology is whether each normal Moore space is metrizable. The primary result of this paper is to reduce that question to the problem of deciding whether each normal Moore space is locally metrizable at some point. Also, the question of metrizability for normal, separable spaces is reduced to that for normal, separable, locally compact spaces.

The question as to whether every normal Moore space is metrizable has received considerable attention [1], [2]. In this note it is proved that if every normal Moore space is locally metrizable at some point, then every normal Moore space is metrizable. Also, the question of metrizability for normal, separable spaces is reduced to that for normal, separable locally compact spaces. This complements Jones' result (2, Theorem 5) that every normal separable Moore space is metrizable, provided $2^{\aleph_0} < 2^{\aleph_1}$. In the proof of the first theorem, a construction device similar to one described by Roy [4] is used.

By a development is meant a sequence of collections of regions satisfying Axiom 0 and the first three parts of Axiom 1 of [3].

THEOREM 1. *If there is a normal Moore space which is not metrizable, there is one which is not locally metrizable at any point.*

Proof. Suppose that S^0 is a nonmetrizable, normal Moore space, M^0 is a dense subset of S^0 which is of minimal cardinality, and $\{G_n^0\}$ is a development of S^0. (By taking M^0 to be of minimal cardinality, a property such as separability of S^0 is preserved.) There exist sequences $w = S^0, S^1, S^2, \cdots, u = M^0, M^1, M^2, \cdots,$ and $v = \{G_n^0\}, \{G_n^1\}, \{G_n^2\}, \cdots$ such that for each integer $j > 0$

(i) M^j is a dense subset of $S^j - S^{j-1}$ of minimal cardinality,

(ii) for each point P of M^j and each positive integer k,

$$S_{P,k}^{j+1} \text{ is } S^0 \times (P) \times (k) \text{ and } S^{j+1} \text{ is } S^j + \sum_{P \in M^j} \sum_{k=1}^{\infty} S_{P,k}^{j+1} ,$$

(iii) the statement that R^{j+1} is a region of G^{j+1} means that either

(1) for some point P of M^j and some region R^0 of G_n^0 and some positive integer i, R^{j+1} is $R^0 \times (P) \times (i)$, or

(2) for some point P of M^j and some region R^j of G_n^j containing P and some positive integer $i \geqq n$,

$$R^{j+1} = R^j + \sum_{q \in R^j \cdot M^j} (S_{q,i}^{j+1} + S_{q,i+1}^{j+1} + \cdots).$$

It is clear that for each positive integer j

(1) S^j is a Moore space with development $\{G_n^j\}$,

(2) S^{j+1}-S^j is a dense open set in S^{j+1},

(3) the cardinality of M^{j+1} is the same as that of M^j, (again, this preserves separability in case S^0 is separable)

(4) if R is a region of G_n^{j+1} intersecting S^j, then there is a region g of G_n^j such that $R \cdot S^j$ is g,

(5) if R^j is a region of G_n^j, there is a region R^{j+1} of G_n^{j+1} containing R^j,

(6) if each of i and j is a positive integer, P is a point in M^j and k is some positive integer greater than or equal to j such that R^{j+1} intersects $S_{P,i}^k$, then $S_{P,i}^k$ is a subset of R^{j+1}.

Moreover, each S^j is a normal Moore space. Since S^0 is normal, assume that S^{j-1} is normal and consider S^j. If H and K are mutually exclusive, closed subsets of S^j, there exist domains D_H^{j-1} and D_K^{j-1}, in S^{j-1}, containing $H \cdot S^{j-1}$ and $K \cdot S^{j-1}$ respectively. Since S^j is a Moore space, it follows that if P is in $H \cdot S^{j-1}$, there is an integer n_P such that if R is a region of $G_{n_P}^j$ containing P then $R \cdot S^{j-1}$ is a subset of D_H^{j-1} and $\bar{R}$ does not intersect K. If P is in $H \cdot S^{j-1}$, denote by R_P some region of $G_{n_p}^j$ containing P and similarly for Q a point of $K \cdot S^{j-1}$. If $D_H^j = \sum_{P \in H \cdot S^{j-1}} R_p$ and $D_K^j = \sum_{Q \in K \cdot S^{j-1}} R_Q$, then it is obvious that D_H^j and D_K^j are mutually exclusive domains in S^j containing $H \cdot S^{j-1}$ and $K \cdot S^{j-1}$ respectively. Now no point of K is a limit point of D_H^j and no point of H is a limit point of D_K^j. For suppose that Q belongs to K and is not in S^{j-1}. Then there is a point Y of M^{j-1} and a positive integer i such that Q is a point of $S^0 \times (Y) \times (i)$. But each region of G_{i+1}^j containing Q is a subset of $S^0 \times (Y) \times (i)$ and no region of G_{i+1}^j containing a point of S^{j-1} intersects $S^0 \times (Y) \times (i)$. It is clear then that no region of G_{i+1}^j contains Q and intersects a region R_p of D_H^j, else R_p contains Q and this is impossible. If i is a positive integer and Y is a point of M^{j-1} such that neither D_H^j nor D_K^j intersects $S^0 \times (Y) \times (i)$, then $S^0 \times (Y) \times (i)$ is normal and there exist mutually exclusive domains $D_{H,Y,i}$ and $D_{K,Y,i}$ containing $H \cdot (S^0 \times (Y) \times (i))$ and $K \cdot (S^0 \times (Y) \times (i))$ respectively. Then $D_H = D_H^j + \sum D_{H,Y,i}$ and $D_K = D_K^j + \sum D_{K,Y,i}$ are mutually exclusive domains, in S^j, containing H and K respectively. Thus S^j is normal.

Denote by S^w the space in which P is a point if and only if P is a point of some S^j and suppose that H_n^w is defined as follows:

R^w belongs to H_n^w if and only if it is true that there are an integer k, an integer $m \geqq n$, and a sequence $R_m^k, R_m^{k+1}, R_m^{k+2}, \cdots$ such that

(1) R_m^k is a region of G_m^k,

(2) $S^{k+j-1}\cdot R_m^{k+j} = R_m^{k+j-1}$, $j = 1, 2, 3, \cdots$

(3) Subject to condition (2), R_m^{k+j-1} is the maximal element in G_m^{k+j-1} containing R_m^{k+j}, for $j = 0, 1, 2, \cdots$. (That is, the positive integer i of part (2) of the definition (iii) given in the proof of Theorem 1 is chosen to be minimal for all possible points P of $M^j \cdot R^{k+j}$),

(4) $R_m^k \subset R_m^{k+1} \subset R_m^{k+2} \subset \cdots$,

(5) $R_m^{k+i} \subset S^{k+i}$ and $R_m^{k+i} \not\subset S^{k+i-1}$,

(6) $R^w = \sum_{i=0}^{\infty} R_m^{k+1}$. For each positive integer t, denote by G_t^w the collection to which R belongs if and only if there is an integer $s \geqq t$ and an element R^w of H_s^w such that R is R^w. Note that if $R^w = \sum_{i=0}^{\infty} R_m^{k+i}$ is a region then the boundary of R^w is the boundary of R_m^k and is a subset of S^k. It is clear that for each positive integer n, G_n^w is a covering of the space and G_{n+1}^w is a subcollection of G_n^w. To establish that S^w is a Moore space, it will suffice to prove that the third part of Axiom 1 is satisfied. To this end, suppose that each of A and B is a point of S^w and R is a region containing A and B.

Case 1. There are a positive integer j, a point P of M^j, and a positive integer k such that each of A and B belongs to $S^0 \times (P) \times (k)$. But S^{j+1} is a Moore space so there is an integer n such that no region of G_n^{j+1} contains both A and B, but the closure of each region of G_n^{j+1} is a subset of $R\cdot(S^0 \times (P) \times (k))$, if it contains A. Hence, by the definition of region in S^w, it follows immediately that each region of G_n^w containing A does not contain B, but the closure of it is a subset of R.

Case 2. The conditions of Case 1 are not satisfied. Denote by j the least integer such that for some point P of M^j and for some integer k, A is a point of $S^0 \times (P) \times (k)$, and denote by t the least integer such that for some point Q of M^t and for some integer e, B is a point of $S^0 \times (Q) \times (e)$. If t is j, it is clear that there is a positive integer n such that the closure of no region of G_n^{j+1} intersects both $S^0 \times (Q) \times (e)$ and $S^0 \times (P) \times (k)$. If t is not j, assume t is less than j. Then there must be a finite sequence of positive integers, $n_1, n_2, \cdots, n_{j-t}$, and a finite sequence of points $P_1, P_2, \cdots, P_{j-t}$ such that for each i, P_i is a point of S^{t+i-1}, P_{i+1} is a point of $S^0 \times (P_i) \times (n_i)$, and B is in $S^0 \times (P_{j-t}) \times (n_{j-t})$. (This essentially traces the construction of S^w as regards A and B). But again, if A is not P_1, there is a positive integer n such that the closure of no region of G_n^t intersects A and P_1, and the closure of each region of G_n^t containing A is a subset of $R\cdot S^t$. It follows as in Case 1 that no region of G_n^w contains both A and B, but the closure of each region of G_n^w containing A is a subset of R. If A is P_1, no region of $G_{n_1+1}^w$ contains both A and B. Thus S^w is a Moore space.

To prove that S^w is normal, suppose that H and K are mutually exclusive, closed point sets. Then $H \cdot S^0$ and $K \cdot S^0$ are mutually exclusive, closed point sets, if each exists. In S^0 there are mutually exclusive domains $D_{H \cdot S^0}$ and $D_{K \cdot S^0}$ containing $H \cdot S^0$ and $K \cdot S^0$, respectively. For each point P of $H \cdot S^0$, denote by R_P a region in S^w containing P such that $R_P \cdot S^0$ is a subset of $D_{H \cdot S^0}$ and $\bar{R}_P \cdot K$ does not exist, and similarly for P a point of $K \cdot S^0$. Denote by D_0 the domain to which x belongs if and only if there is a point P of $H \cdot S^0$ such that x is point of R_P, and denote by D_0' the domain to which x belongs if and only if there is a point P of $K \cdot S^0$ such that x is a point of R_P. Then $\bar{D}_0$ does not intersect K and $\bar{D}_0'$ does not intersect H. For suppose that Q is a limit point of D_0 which is in K. Then Q is in some S^j where j is positive. Indeed, if R_m^j is a region of G_n^j containing Q, then there is a point P of $H \cdot S^0$ such that R_m^j intersects R_p. But since j is positive, this means that R_P contains R_m^j and this is a contradiction to Q belonging to K.

Similary, if H_1 is $H \cdot S^1$ and K_1 is $K \cdot S^1$, there exist mutually exclusive domains D_1 and D_1' in S^w such that D_1 contains H_1, D_1' contains K_1, $\bar{D}_1$ does not intersect $\overline{(D_0 + D_0')}$, and $\overline{D_1'}$ does not intersect $\overline{(D_0 + D_0')}$. In general, there exist sequences $\alpha = D_0, D_1, D_2, \cdots$ and $\beta = D_1', D_2', D_3', \cdots$ such for each n, D_n contains $H \cdot S^n$, D_n' contains $K \cdot S^n$, D_n does not intersect D_n', $\overline{D_{n+1}}$ does not intersect $\overline{(\sum_{i=0}^n D_i + \sum_{i=0}^n D_1')}$, and $\overline{D_{n+1}'}$ does not intersects $\overline{(\sum_{i=0}^n D_i + \sum_{i=0}^n D_i')}$. It follows that $D_H = \sum_{n=0}^\infty D_n$ and $D_K = \sum_{n=0}^\infty D_n'$ are mutually exclusive domains such that D_H contains H and D_K contains K. Thus S^w is normal.

That S^w is not locally metrizable at any point follows from the property that each region in the development of S^w contains a non-metrizable, normal Moore space.

COROLLARY 1. *If there is a normal, separable Moore space which is not metrizable, then there is one which is not locally metrizable at any point.*

Proof. The cardinality of each M^j of Theorem 1 is $\aleph_0$ and $\sum_{j=1}^\infty M^j$ is countable.

COROLLARY 2. *There exists a Moore space which is not locally metrizable at any point.*

Proof. Remove the condition of normality from the hypothesis of Theorem 1.

THEOREM 2. *If there is a normal, separable, nonmetrizable Moore space, then there is one which is also locally compact.*

Proof. Suppose (S, Ω) is normal, separable Moore space which is not metrizable. There exist [2], in S, an uncountable point set M with no limit point and a countable subset K of $S \cdot M$ such that every point of M is a limit point of K. Denote the subspace $K + M$ with the relative topology by (S_1, Ω_1). This space is normal, separable, nonmetrizable and is a Moore space.

Enumerate $K: A_1, A_2, A_3, \cdots$. For each point x of M denote by $\{n_i(x)\}_{i=1}^{\infty}$ an increasing sequence of positive integers such that $\text{limit}_{i \to \infty} A_{n_i}(x) = x$, according to the topology Ω_1. Now consider the space (S_1, Ω_2), where Ω_2 is the topology induced by the following definition of region:

The point set R is a region if and only if either (1) for some point P of K, R is the degenerate set whose only element is P, or (2) some point x of M and some positive integer i, R is the set to which P belongs if and only if $P = x$ or $P = A_{n_j}(x)$ for some j greater than or equal to i. Such a region is denoted by $R(x, i)$. Note that if the point x of M is a limit point of the subset K' of K according to the topology Ω_2, then it is according to Ω_1.

Now (S_1, Ω_2) is clearly locally compact, separable, but not completely separable, and thus not metrizable. Also, it is a Moore space, for let G'_n be the collection to which the region R belongs if and only if R is a degenerate region, or, for some point x of M and some positive integer $i \geqq n$, $R = R(x, i)$. Then $G'_1, G'_2, G'_3, \cdots$ gives a development for (S_1, Ω_2). It remains to be shown that (S_1, Ω_2) is normal. To this end, suppose that I and J are mutually exclusive closed point sets in S_1 (according to Ω_2). There exist mutually exclusive domains U and V (in Ω_1) containing $I \cdot M$ and $J \cdot M$ respectively. Then U and V are open according to Ω_2, and $(U - U \cdot J + I \cdot K)$ and $(V - V \cdot I + J \cdot K)$ are mutually exclusive domains in Ω_2 containing I and J respectively. Thus (S_1, Ω_2) is normal.

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