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A MODIFICATION OF MORITA'S CHARACTERIZATION OF DIMENSION

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Morita's characterization of dimension may be stated in the following form. Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-locally finite base $\mathscr{G}$ for the topology of R such that dim $(\bar{G} - G) \leqq n-1$ for all G in $\mathscr{G}$.

The main result of this paper is the following:

THEOREM. **Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-closure-preserving base $\mathscr{G}$ for the topology of R such that $\dim(\bar{G} - G) \leqq n-1$ for all G in $\mathscr{G}$.**

Thus the "locally finite" condition in Morita's characterization can be replaced by the weaker "closure-preserving" condition. A further result is that the "closure-preserving" condition can be replaced by the still weaker condition of "linearly-closure-preserving" provided the "base" condition is strengthened to a "star-base" condition.

Finally, several examples are given which show that the "linearly-closure-preserving" condition is weaker than the "closure-preserving" condition in important ways. In particular, the following is proved.

THEOREM. **There exists a nonmetric, regular T_1-space which has a σ-linearly-closure-preserving star-base.**

If the word "linearly" is deleted from the above theorem, the resulting statement is false since Bing has proved that a regular T_1-space with a σ-closure-preserving star-base is metrizable.

1. Introduction and results. Throughout this paper, dim R represents the usual covering dimension, and ind R represents the small inductive dimension for a topological space R. See [2; 3; 5].

Morita's well known characterization of dimension [5, Lemma 2.2, p. 351] states:

Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-locally finite base $\mathscr{G}$ for the topology of R such that $\dim(\bar{G} - G) \leqq n-1$ for all G in $\mathscr{G}$.

The main result of this paper is to modify Morita's result to:

THEOREM 1. *Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-closure-preserving base $\mathscr{G}$ for the topology of R such that* $\dim(\bar{G} - G) \leqq n - 1$ *for all G in $\mathscr{G}$.*

Following the terminology of Michael [4], we say that a collection $\mathscr{G}$ of subsets of a topological space is *closure-preserving* provided that for every subcollection $\mathscr{B} \subset \mathscr{G}$ it is true that

$$\cup\{\bar{B}: B \in \mathscr{B}\} = \overline{\cup\{B \in \mathscr{B}\}}$$

A collection $\mathscr{G}$ of subsets is called *σ-closure-preserving* provided

$$\mathscr{G} = \cup\{\mathscr{G}_i: i = 1, 2, \cdots\}$$

with each $\mathscr{G}_i$ closure-preserving.

Instead of proving Theorem 1 directly, we shall prove a similar result, Theorem 2, which has a weaker condition, but from which Theorem 1 can be proven easily. To facilitate the discussion of this and further results, we first make the following definitions.

DEFINITION. A collection $\mathscr{G}$ of subsets of a topological space is called *linearly-closure-preserving* provided that there exists a well ordering of $\mathscr{G} = \{G_0, G_1, \cdots, G_\alpha, \cdots: \alpha < \eta\}$ such that

$$\cup\{\bar{G}_\beta: \beta < \alpha\} = \overline{\cup\{G_\beta: \beta < \alpha\}}$$

for all $\alpha \leqq \eta$. A collection $\mathscr{G}$ of subsets of a topological space is called *σ-linearly-closure-preserving* provided $\mathscr{G} = \cup\{\mathscr{G}_i: i = 1, 2, \cdots\}$ with each $\mathscr{G}_i$ linearly-closure-preserving.

DEFINITION. A collection $\mathscr{G}$ of open subsets of a topological space R is called a *σ-closure-preserving* (respectively *σ-linearly-closure-preserving*) *star-base* for R provided $\mathscr{G} = \cup\{\mathscr{G}_i: i = 1, 2, \cdots\}$ is a σ-closure-preserving (respectively σ-linearly-closure-preserving) collection such that for every point x in R and for every open set D containing x there exists a positive integer $k = k(x, D)$ such that

$$\phi \neq S(x, \mathscr{G}_k) \subset D,$$

where $S(x, \mathscr{G}_k) = \cup\{G \in \mathscr{G}_k: x \in G\}$.

THEOREM 2. *Let R be a metric space. A necessary and sufficient condition that* dim $R \leqq n$ *is that there exists a σ-linearly-closure-preserving star-base $\mathscr{G}$ for the topology of R such that*

$$\dim(\bar{G} - G) \leqq n - 1$$

for all G in $\mathscr{G}$.

The Nagata-Smirnov [7; 9] characterization of metrizability for regular spaces (i.e., there exists a σ-locally finite base for the topology of the space) shows that Morita's result above can be modified to the following form:

Let R be a regular T_1-space. A necessary and sufficient condition that R be metrizable with $\dim R \leqq n$ is that there exists a σ-locally finite base $\mathscr{G}$ for the topology of R such that

$$\dim(\bar{G} - G) \leqq n - 1$$

for all G in $\mathscr{G}$.

A similar modification of Theorem 1 is not possible. Bing has given [1, Example C, p. 180] a nonmetric, regular T_1-space which has a σ-closure-preserving base. Bing has proven, however, [1, Theorem 4, p. 179] that a necessary and sufficient condition for a regular T_1-space to be metrizable is that there exists a σ-closure-preserving star-base for the topology of the space. Thus, as a direct result of Bing's Theorem and Theorem 1, we have:

THEOREM 3. *Let R be a regular T_1-space. A necessary and sufficient condition that R be metrizable with* $\dim R \leqq n$ *is that there exists a σ-closure-preserving star-base $\mathscr{G}$ for the topology of R such that* $\dim(\bar{G} - G) \leqq n - 1$ *for all G in $\mathscr{G}$.*

Theorem 3 raises the question of whether one can replace "σ-closure-preserving" by "σ-linearly-closure-preserving" in Theorem 3. This question is equivalent to the following one. Suppose a regular T_1-space R has a σ-linearly-closure-preserving star-base; does this imply that R is metrizable? The answer is in the negative as can be seen from the following example.

EXAMPLE. A nonmetric, regular T_1-space which has a σ-linearly-closure-preserving star-base. Let C denote the usual "middle third" Cantor set in $[0,1]$, and let Q denote the set of all rational points in $[0,1]$. The space R, which is to be the example, is the set of points of $C \cup Q$ with the following topology: V is open in $R = C \cup Q$ if and only if $V = U \cup W$, where U is open in the usual subspace topology of R, and W is any set of irrational points in R. In this topology the irrational points of R are discrete, and the topology induced on Q is the usual subspace topology of Q. Now, R is regular and T_1, but R is not metrizable.

To construct a σ-linearly-closure-preserving star-base for R, we first enumerate the rational points of R by $r_1, r_2, \cdots, r_k, \cdots$; and define

$$\mathscr{G}_{i,j} = \{(r_i - 1/j, r_i + 1/j) \cap R\}$$

for all $i, j \in N$ (where N is the set of natural numbers). Since each $\mathscr{G}_{i,j}$ contains only one open set, it is trivially linearly-closure-preserving. We define one additional collection $\mathscr{G}_0 = \{G_0, G_1, \cdots, G_\alpha, \cdots\}$ where $G_0 = R - C$, and $\{G_1, G_2, \cdots, G_\alpha, \cdots\}$ is the set of irrational points in R with any well ordering. Now G_0 is an open set in R such that $G_0 \cap C = \phi$ and $\bar{G}_0 \cap C \supset Q$. From this it follows that the collection $\mathscr{G}_0$ is a linearly-closure-preserving collection of open sets. It is easily verified that the collections

$$\mathscr{G}_0 \cup (\cup\{\mathscr{G}_{i,j}: i, j \in N\})$$

can be ordered into a single countable sequence of collections, and as such form a σ-linearly-closure-preserving star-base for R.

Theorem 2 raises the question of whether one can replace "star-base" by "base" in Theorem 2. This question is easily answered in the negative as we now show. Roy [8] has defined a metric space Δ which has the property that $\dim \Delta = 1$ and $\operatorname{ind} \Delta = 0$. Since ind $\Delta = 0$, there exists a base $\mathscr{G}$ for Δ such that $\dim(\bar{G} - G) = -1$ for all G in $\mathscr{G}$. If $\mathscr{G}$ is given any well ordering, and if the whole space Δ is added to the collection $\mathscr{G}$ as its first element, then $\mathscr{G}$ becomes a linearly-closure-preserving base for Δ such that $\dim(\bar{G} - \mathrm{G}) = -1$ for all G in $\mathscr{G}$. Since $\dim \Delta = 1$, it is clear that "star-base" cannot be replaced by "base" in Theorem 2.

2. Proof of Theorem 2. To prove the necessity of the condition, we note by Morita's result mentioned above that $\dim R \leqq \mathrm{n}$ implies that there exists a σ-locally finite base $\mathscr{G} = \cup\{\mathscr{G}_i: i \in N\}$ for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}$. Since R is a metric space, we may define

$$\mathscr{G}_{i,k} = \{G \in \mathscr{G}_i: \text{diameter of } G < 1/k\}$$

for all $i, k \in N$. Each $\mathscr{G}_{i,k}$ is locally finite (hence, linearly-closure-preserving), and $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}_{i,k}$ since $\mathscr{G}_{i,k} \subset \mathscr{G}_i$ for all k. By well ordering $\mathscr{G}' = \cup\{\mathscr{G}_{i,k}: i,k \in N\}$ into a single countable sequence of collections, we have that $\mathscr{G}'$ is a σ-linearly-closure-preserving star-base for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}'$.

The proof of the sufficiency will be broken up into several assertions. Each assertion will be assumed to have as hypothesis the condition of Theorem 2, i.e., $\mathscr{G} = \cup\{\mathscr{G}_i: i \in N\}$ is a σ-linearly-closure-preserving star-base for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G

in $\mathscr{G}$. The following notation and definitions will be used in the assertions.

For any subset S of a topological space R, the *boundary* of S is defined to be $\bar{S} \cap \overline{(R - S)}$, and is denoted by Bdry (S).

Since each collection $\mathscr{G}_i$ is linearly-closure-preserving, we may write $\mathscr{G}_i = \{G_{i0}, G_{i1}, \cdots, G_{i\alpha}, \cdots : \alpha < \eta_i\}$ and define a collection of open sets by

$$\left\{H_{i\alpha} = (G_{i\alpha} - \bigcup_{\beta<\alpha} \bar{G}_{i\beta}) : \alpha < \eta_i\right\},$$

and a collection of closed sets by

$$\left\{F_{i\alpha} = \left(\bar{G}_{i\alpha} - \bigcup_{\beta\leqq\alpha} G_{i\beta}\right) : \alpha < \eta_i\right\},$$

and let

$$\mathscr{H}_i = \{H_{i\alpha} \cap (R - F) : \alpha < \eta_i\}$$

for all $i \in N$, where F is defined below.

2.1. ASSERTION. For all $i \in N$, $\bigcup \{F_{i\beta} : \beta < \alpha\}$ is a closed set in R for every $\alpha \leqq \eta_i$.

Proof. Let i be arbitrary, but fixed. Let $\alpha \leqq \eta_i$ and let x be a limit point of $\cup\{F_{i\beta} : \beta < \alpha\}$. Then

$$x \in \overline{\bigcup_{\beta<\alpha} F_{i\beta}} = \overline{\bigcup_{\beta<\alpha} (\bar{G}_{i\beta} - \bigcup_{\delta\leqq\beta} G_{i\delta})} \subset \overline{\bigcup_{\beta<\alpha} \bar{G}_{i\beta}}.$$

Since the collection $\mathscr{G}_i$ is linearly-closure-preserving by hypothesis, $x \in \bigcup \{\bar{G}_{i\beta} : \beta < \alpha\}$. Let $\sigma < \alpha$ be the first index such that $x \in \bar{G}_{i\delta}$. It is easy to see that $x \notin G_{i\sigma}$, for $G_{i\sigma}$ is an open set which does not intersect $\bigcup\{F_{i\beta} : \sigma \leqq \beta < \alpha\}$. Hence, $x \in G_{i\sigma}$ would imply that x is a limit point of $\bigcup \{F_{i\beta} : \beta < \sigma\}$. But this would imply that

$$x \in \overline{\bigcup_{\beta<\sigma} F_{i\beta}} \subset \bigcup_{\beta<\sigma} \bar{G}_{i\beta},$$

and this would mean that there exists $\delta < \sigma$ such that $x \in \bar{G}_{i\delta}$ which is impossible by the definition of σ. Hence, $x \notin G_{i\sigma}$. Thus, we have that

$$x \in \left(\bar{G}_{i\sigma} - \bigcup_{\beta\leqq\sigma} G_{i\beta}\right) = F_{i\sigma},$$

and the assertion is proven.

The following notation will be used in the succeeding assertions. Let $F_i = \bigcup \{F_{i\beta} : \beta < \eta_i\}$, and let $F = \bigcup \{F_i : i \in N\}$.

2.2. ASSERTION. Dim $F \leqq n-1$.

Proof. By Assertion 2.1, F_i is closed for all $i \in N$. Hence, it suffices by the usual sum theorem [5, Theorem 5.2, p. 355] to prove that $\dim F_i \leqq n-1$ for all i. Let i be arbitrary, but fixed. Then by the subset theorem [5, Theorem 5.1, p. 355] we have that $\dim F_{i\alpha} \leqq n-1$ because

$$F_{i\alpha} \subset (\bar{G}_{i\alpha} - G_{i\alpha})$$

and $\dim(\bar{G}_{i\alpha} - G_{i\alpha}) \leqq n-1$ by hypothesis. By Assertion 2.1

$$\{F_{i\alpha}: \alpha < \eta_i\}$$

is a linearly-closure-preserving collection such that $\dim F_{i\alpha} \leqq n-1$ for all $\alpha < \eta_i$. Hence, the collection $\{F_{i\alpha}: \alpha < \eta_i\}$ satisfys the hypothesis of a sum theorem of Nagami [6, Theorem 1, p. 82]. Thus,

$$\dim\left(\bigcup \{F_{i\alpha}: \alpha < \eta_i\}\right) \leqq n-1$$

and the assertion is proven.

To complete the proof of Theorem 2, we need only prove that $\dim(R-F) \leqq 0$ by [5, Theorem 5.4, p. 355]. To prove that

$$\dim(R-F) \leqq 0$$

it suffices by Morita's characterization of dimension to demonstrate a σ-discrete base for $R-F$ each member of which has an empty boundary in $R-F$.

2.3. ASSERTION. The collections $\mathscr{H}_i$ are discrete in the subspace $R-F$ for all $i \in N$.

Proof. Let i be arbitrary, but fixed. We shall show that for every x in $R-F$ there exists an open neighborhood of x in $R-F$ which intersects at most one of the sets $H_{i\alpha} \cap (R-F)$. Let $x \in R-F$. If $x \notin \bigcup \{\bar{G}_{i\alpha}: \alpha < \eta_i\}$ then $R - \bigcup \{\bar{G}_{i\alpha}: \alpha < \eta_i\}$ is an open neighborhood of x in R which intersects none of the $H_{i\alpha}$, hence, none of the

$$H_{i\alpha} \cap (R-F)\,.$$

If, in the other case, $x \in \bigcup \{\bar{G}_{i\alpha}: \alpha < \eta_i\}$ let $\sigma < \eta_i$ denote the first index such that $x \in \bar{G}_{i\sigma}$. We may assume that $x \in G_{i\sigma}$, for otherwise,

$$x \in \left(\bar{G}_{i\sigma} - \bigcup_{\beta \leqq \sigma} G_{i\beta}\right) = F_{i\sigma} \subset F\,,$$

which is impossible because $x \in R-F$. By the definition of σ we see that

$$x \in \Bigl(G_{i\sigma} - \bigcup_{\beta<\sigma} \bar{G}_{i\beta}\Bigr) \subset H_{i\sigma} .$$

Clearly, $H_{i\sigma}$ is an open neighborhood of x which does not intersect any $H_{i\alpha}$ for $\alpha \neq \sigma$. Hence, $H_{i\sigma} \cap (R - F)$ is the required neighborhood of x. This completes the proof of Assertion 2.3.

2.4. ASSERTION. The collection $\bigcup \{\mathscr{H}_i : i \in N\}$ is a base for the subspace $R - F$.

Proof. Let $x \in R - F$. Let D be any open set in $R - F$ which contains x. Let D' be an open set in R such that $D = D' \cap (R - F)$. By hypothesis there exists an integer k such that $\phi \neq S(x, \mathscr{G}_k) \subset D'$. Let $\sigma < \eta_k$ be the first index such that $x \in G_{k\sigma}$, then $G_{k\sigma} \subset D'$. Now, $x \notin \bigcup \{\bar{G}_{k\beta} : \beta < \sigma\}$ for otherwise, $x \in \bigcup \{\bar{G}_{k\beta} : \beta < \sigma\}$ would imply that there exists an index $\delta < \sigma$ such that $x \in \bar{G}_{k\delta}$. Since $\delta < \sigma$, we would have that

$$x \in \Bigl(\bar{G}_{k\delta} - \bigcup_{\beta\leqq\delta} G_{k\beta}\Bigr) = F_{k\delta} \subset F .$$

This is impossible since $x \in R - F$. Thus

$$x \in \Bigl(G_{k\sigma} - \bigcup_{\beta<\sigma} \bar{G}_{k\beta}\Bigr) = H_{k\sigma} .$$

Hence, $x \in H_{k\sigma} \cap (R - F)$, which is an open neighborhood of x in $R - F$ and a subset of D. Assertion 2.4 is, therefore, proven.

2.5. ASSERTION. For each i, Bdry $(H_{i\alpha}) \subset F_i$ for all $\alpha < \eta_i$.

Proof. Let i be fixed, and let $\alpha < \eta_i$. Since $\mathscr{G}_i$ is a linearly-closure-preserving collection of open sets,

$$\text{Bdry}\,(H_{i\alpha}) = \text{Bdry}\Bigl(G_{i\alpha} - \bigcup_{\beta<\alpha} \bar{G}_{i\beta}\Bigr) \subset \bigcup \{\text{Bdry}\,(G_{i\beta}) : \beta \leqq \alpha\} .$$

Let $x \in \text{Bdry}\,(H_{i\alpha})$. Since $\bigcup \{G_{i\beta} : \beta < \alpha\}$ is an open set which does not intersect $H_{i\alpha}$, we have that $x \notin \bigcup \{G_{i\beta} : \beta < \alpha\}$. Let $\delta \leqq \alpha$ be the first index such that $x \in \text{Bdry}\,(G_{i\delta})$. Then

$$x \in \Bigl(\bar{G}_{i\delta} - \bigcup_{\beta\leqq\delta} G_{i\beta}\Bigr) = F_{i\delta} \subset F_i .$$

2.6. ASSERTION. Bdry $(H_{i\alpha} \cap (R - F)) = \phi$ in the subspace $R - F$ for all $i \in N$, and for all $\alpha < \eta_i$.

Proof. This assertion follows from Assertion 2.5 and the fact that the boundary of $(H_{i\alpha} \cap (R - F))$ with respect to the subspace $R - F$ is a subset of the boundary of $H_{i\alpha}$ with respect to the space R.

By Assertions 2.3, 2.4, and 2.6 we have shown that

$$\mathscr{H} = \bigcup \{\mathscr{H}_i : i \in N\}$$

is a σ-discrete base for $R - F$ such that $\dim(\bar{H} - H) = -1$ for all H in $\mathscr{H}$. Hence, $\dim(R - F) \leqq 0$, and Theorem 3 is completely proven.

3. Proof of Theorem 1. The proof of the necessity of the condition is trivial.

To prove the sufficiency, let $\mathscr{G}$ be the σ-closure-preserving base for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}$. By the same method as was used in the proof of the necessity of Theorem 2, $\mathscr{G}$ may be "rearranged" into a σ-closure-preserving star-base. Thus the condition of Theorem 2 is satisfied. We may, therefore, conclude that $\dim R \leqq n$, and Theorem 1 is proven.

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